Research Article

Novel Investigation of Multivariable Conformable Calculus for Modeling Scientific Phenomena

Mohammed K. A. Kaabar,1,2 Francisco Martínez,3 Inmaculada Martínez,3 Zailan Siri,1 and Silvestre Paredes3

1Institute of Mathematical Sciences, Faculty of Science, University of Malaya, Kuala Lumpur 50603, Malaysia
2Gofa Camp, Near Gofa Industrial College and German Adebabay, Nifas Silk-Lafto 26649, Addis Ababa, Ethiopia
3Department of Applied Mathematics and Statistics, Technological University of Cartagena, Cartagena 30203, Spain

Correspondence should be addressed to Mohammed K. A. Kaabar; mohammed.kaabar@wsu.edu and Zailan Siri; zailansiri@um.edu.my

Received 11 September 2021; Accepted 29 October 2021; Published 23 November 2021

Academic Editor: Antonio Di Crescenzo

Copyright © 2021 Mohammed K. A. Kaabar et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

New investigation on the conformable version (CoV) of multivariable calculus is proposed. The conformable derivative (CoD) of a real-valued function (RVF) of several variables (SVs) and all related properties are investigated. An extension to vector-valued functions (VVFs) of several real variables (SRVs) is studied in this work. The CoV of chain rule (CR) for functions of SVs is also introduced. At the end, the CoV of implicit function theorem (IFTm) for SVs is established. All results in this work can be potentially applied in studying various modeling scenarios in physical oceanography such as Stommel’s box model of thermohaline circulation and other related models where all our results can provide a new analysis and computational tool to investigate these models or their modified formulations.

1. Introduction

Many definitions of derivative have been proposed based on two categorizations: global (nonlocal) and local types. In the first one, the nonlocal fractional derivative (FrDr) is represented via integral transformation or any other related transformations where nonlocality is seen in this type along with a memory. The Riemann–Liouville and Caputo fractional definitions are considered as the most commonly known fractional definitions. Several approximate analytical and numerical methods have been recently developed to solve nonlinear fractional and local differential equations that are encountered while modeling various scientific phenomena such as the generalized Riccati expansion for solving the nonlinear KPP equation via FrDr [1] and the trigonometric quintic B-spline method for solving the nonlinear telegraph equation constructed via CoV [2]. The nonlinear Klein–Fock–Gordon equation has been analytically solved via two novel techniques such as the methods of generalized Riccati expansion and generalized exponential function [3]. Fractional vector (FV) calculus has been recently introduced in [4] to represent the spherical coordinates framework using the fractional derivative and integral of Caputo and Riemann–Liouville senses, respectively. FV calculus can be used effectively in modeling processes in fractal media and other fractional dynamical systems such as hydrodynamical and electrodynamical systems [4]. Fractional calculus is a powerful tool in modeling the potential flow past a sphere of an inviscid fluid [5] which is a very important research problem because this problem particularly investigates the Laplacian in three-dimensional space (spherical coordinates) which is used in governing various physical and mechanical systems in heat conduction, elasticity, Newtonian gravitational potential, ideal fluid flow, and electrostatics [6]. In addition, this problem has been applied in finding the stream function solutions for the stokes flow inside viscous sphere in an inviscid extensional flow [7]. The basics of these essential notions are mentioned in [8, 9]. The local definition is based on certain incremental ratios. A new local derivative definition was initially
formulated by Khalil et al. [10, 11], which is called conformable derivative (CoD). The main aim of CoD is to avoid some obstacles of solving nonlocal fractional differential equation where the analytical solutions can be very complicated to obtain. Consequently, several research works have mathematically analyzed some essential notions [10, 12–16]. The Schrodinger–Hirota equation and modified KdV–Zakhvor–Kuznetsov equation have been solved with the help of generalized CoDs [17]. CoDs have been employed in investigating the wick-type stochastic nonlinear evolution equations via the improved technique of Kudryashov [18] (see also the wick-type stochastic KdV equation formulated in the context of generalized CoDs [19]). According to the mathematical investigation of CoD in [20], it is clear that CoD cannot be addressed as a fractional derivative. Therefore, in our study, we have addressed CoD as a modified form of usual derivative which has some applications in physics and engineering due to the fact that the measurements in physics are local. Therefore, this definition is highly applicable in theoretical physics. CoD can still be helpful in many related modeling scenarios.

The CoV of analytic functions’ theory has been proposed in [21]. In addition, new results are investigated on the contour conformable integral in [22, 23]. Thus, the definition of the contour conformable integral has been utilized in [23].

Studying conformable derivative and integral is essential in various fields of natural sciences and engineering. The need for local derivative is highly appreciated in multidisciplinary sciences. While the nonlocal fractional derivatives can provide a good explanation to the dynamics of certain systems, particularly in modeling epidemic diseases, the difficulty of obtaining exact or analytical solutions to the problems formulated in the sense of nonlocal fractional derivatives can make the investigation of fractional-order systems a real challenge for researchers. Thus, researchers have paid more attention to conformable derivative and other related local derivatives in modeling scientific phenomena. While there are some recent studies concerning the mathematical analysis of conformable calculus such as the multivariable conformable calculus [15] that was introduced in 2018, the behavior of conformable derivatives of functions in arbitrary Banach spaces [24] that was investigated in 2021, the differential geometry of curves [25] that was investigated in 2019 in the senses of conformable derivatives and integrals, and the behavioral framework for the conformable linear differential systems’ stability [26] that was carefully studied in 2020 to utilize the importance of CoV in modeling scenarios of control theory and power electronics, our results in this work provide a comprehensive investigation of α-derivative of a function of SVs and all related properties, the CoV of CR for functions of SVs, and the CoV of IFThm involving many numerical examples to validate our obtained results. According to the best of our knowledge, our original investigation in this article provides an essential mathematical analysis tool for researchers working on modeling phenomena in physics and engineering in the sense of conformable calculus because all theorems and properties in this work will be needed in such modeling scenarios.

The article consists of the following sections: essential notions of the CoV of calculus are mentioned in Section 2. Then, the α-derivative of RVF of SVs is investigated, and all its main properties are established in Section 3. Furthermore, these results are extended to the VVFs of SRVs. In addition, the CR for functions of SVs is introduced in two particular cases in Section 4. In the last part, the CoV of IFThm for SVs is obtained in Section 5 by first establishing the conformable theorem of existence and regularity of the implicit function for single equation. Second, this result is extended to a system of several equations and SRVs. Some concluding remarks are specified in Section 6.

2. Fundamental Notions

Definition 1 (See [10]). For a function \( f: [0, \infty) \to R \), the \( \alpha \)th order CoD can be written as

\[
(T_a f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{-\alpha}) - f(t)}{\epsilon},
\]

\( \forall t > 0, \alpha \in (0, 1] \). If \( f \) is \( \alpha \)-differentiable function (\( \alpha \) DF) in some \( (0,a), a > 0 \), and \( \lim_{\epsilon \to 0^+} (T_a f)(t) \) exists; then, it is expressed as

\[
(T_a f)(0) = \lim_{\epsilon \to 0^+} (T_a f)(t).
\]

Theorem 1 (See [10]). If \( f: [0, \infty) \to R \) is \( \alpha \) DF at \( t_0 > 0 \), \( \alpha \in (0,1] \), then \( f \) is continuous function (CF) at \( t_0 \).

Theorem 2 (See [10]). Assuming that \( \alpha \in (0,1] \) and \( f, g \) are \( \alpha \) DF s at a point \( t > 0 \), we have

(i) \( T_a(af + bg) = a(T_a f) + b(T_a g) \), \( \forall a, b \in R \).
(ii) \( T_a(t^p) = pt^{\alpha-1}, \forall p \in R \).
(iii) \( T_a(\lambda) = 0, \forall \) constant functions \( f(t) = \lambda \).
(iv) \( T_a(fg) = f(T_a g) + g(T_a f) \).
(v) \( T_a(f/g) = g(T_a f) - f(T_a g)/g^2 \).
(vi) If we suppose that \( f \) is differentiable, then \( (T_a f)(t) = t^{1-\alpha}d/dt (t) \).

From Definition 1, the CoD of some functions are expressed as

(i) \( T_a(1) = 0 \).
(ii) \( T_a(\sin(at)) = at^{1-\alpha} \cos(at) \).
(iii) \( T_a(\cos(at)) = -at^{1-\alpha} \sin(at) \).
(iv) \( T_a(e^{at}) = at^{1-\alpha}e^{at} \), \( a \in R \).

Definition 2 (See [11]). The left CoD beginning from \( a \) of function \( f: [a, \infty) \to R \) of order \( \alpha \in (0,1] \) is expressed as

\[
(T_a^\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon}, \ t > a.
\]

For \( a = 0 \), it is expressed as \( (T_a f)(t) \). If \( f \) is \( \alpha \) DF in some \( (a, b) \), then we set
(4) \[ (T^a_{a}f)(t) = \lim_{t \to a^+} (T^a_{a}f)(t). \]

**Theorem 3** (See [12]). Suppose that \( f, g : (a, \infty) \to R \) are left \( \alpha \) DFs, where \( \alpha \in (0, 1] \). Let us assume that \( h(t) = f(g(t)), h(t) \) is a DF \( \forall t \neq a \) and \( g(t) \neq 0 \), and we get

(5) \[ (T^\alpha_{a}h)(t) = (T^\alpha_{a}f)(g(t)) \cdot (T^\alpha_{a}g)(t) \cdot (g(t))^\alpha - 1. \]

If \( t = a \), then we obtain

(6) \[ (T^\alpha_{a}h)(a) = \lim_{t \to a^+} (T^\alpha_{a}f)(g(t)) \cdot (T^\alpha_{a}g)(t) \cdot (g(t))^\alpha - 1. \]

**Theorem 4** (Rolle’s theorem (RT) [10]). Suppose that \( a > 0, \alpha \in (0, 1] \), and function \( f : [a, b] \to R \) satisfies

- \( i \) \( f \) is CF on \([a, b]\)
- \( ii \) \( f \) is a DF on \([a, b]\)
- \( iii \) \( f(a) = f(b) \)

Then, \( \exists c \in (a, b), \exists (T^\alpha_{a}f)(c) = 0. \)

**Theorem 5** (Mean value theorem (MVT) [10]). Assume that \( a > 0, \alpha \in (0, 1] \), and function \( f : [a, b] \to R \) satisfies

- \( i \) \( f \) is CF on \([a, b]\)
- \( ii \) \( f \) is a DF on \([a, b]\)

Then, \( \exists c \in (a, b), \exists \) we have

(7) \[ (T^\alpha_{a}f)(c) = \frac{f(b) - f(a)}{b^\alpha/a^\alpha - a^\alpha/a^\alpha}. \]

**Theorem 6** (Modified mean value theorem (MMVT) [27]). Assume that \( a > 0, \alpha \in (0, 1] \), and function \( f : [a, b] \to R \) satisfies

- \( i \) \( f \) is CF on \([a, b]\)
- \( ii \) \( f \) is a DF on \([a, b]\)

Then, \( \exists c \in (a, b), \exists \) we have

(8) \[ \frac{(T^\alpha_{a}f)(c)}{c^\alpha/a} = \frac{f(b) - f(a)}{(b/a) - (a/a)}. \]

**Theorem 7** (See [13]). Suppose that \( a > 0, \alpha \in (0, 1] \), and function \( f : [a, b] \to R \) satisfies

- \( i \) \( f \) is CF on \([a, b]\)
- \( ii \) \( f \) is a DF on \([a, b]\)

Then, we get

- \( i \) If \( (T^\alpha_{a}f)(t) > 0 \) \( \forall t \in (a, b) \), then \( f \) is increasing on \([a, b]\)
- \( ii \) If \( (T^\alpha_{a}f)(t) < 0 \) \( \forall t \in (a, b) \), then \( f \) is decreasing on \([a, b]\)

Let us express the CoV of partial derivative (PDr) of a real-valued function (RVF) with SVs as follows.

**Definition 3** (See [14, 15]). Assume that \( f \) is a RVF with \( n \) variables, and there is a point \( \mathbf{a} = (a_1, \ldots, a_n) \in R^n \), where its \( i \)th component is positive. Then, the limit is written as

(9) \[ \lim_{\epsilon \to 0} \frac{f(a_1, \ldots, a_i + \epsilon a_i^{1-a}, \ldots, a_n) - f(a_1, \ldots, a_n)}{\epsilon} \]

If the above limit exists, the \( i \)th CoV of PDr of \( f \) of the order \( \alpha \in (0, 1] \) at \( \mathbf{a} \), represented by \( \frac{\partial^\alpha f}{\partial x_i^\alpha} \).

**3. \(\alpha\)-Derivative of a RVF of SVs**

**Definition 4.** Suppose that \( f \) is a RVF with \( n \) variables \( x_1, \ldots, x_n \), and \( \alpha \in (0, 1] \). Then, we say that \( f \) is a DF at \( \mathbf{a} = (a_1, \ldots, a_n) \in R^n \), each \( a_i > 0 \), if any of the three conditions which are equivalent to each other is verified:

- \( i \) There is a linear transformation \( L : R^n \to R \), such that

(10) \[ \lim_{h \to 0} \frac{f(a_1 + h_1 a_1^{1-a}, \ldots, a_n + h_n a_n^{1-a}) - f(a_1, \ldots, a_n) - L(h)}{\|h\|} = 0, \]

Where \( \mathbf{h} = (h_1, \ldots, h_n), \|\mathbf{h}\| = \sqrt{h_1^2 + \cdots + h_n^2} \), and \( \alpha \in (0, 1] \).

- \( ii \) There is a linear transformation \( L : R^n \to R \) and a function \( \epsilon : h \to \epsilon(h) \), such that

(11) \[ f(a_1 + h_1 a_1^{1-a}, \ldots, a_n + h_n a_n^{1-a}) - f(a_1, \ldots, a_n) = L(h) + \epsilon(h)\|h\|. \]

And \( \lim_{h \to 0} \epsilon(h) = 0. \)
(iii) There is a linear transformation \( L : R^n \to R \) and \( n \)
functions \( \varepsilon_i : h \to \varepsilon_i(h) \) \( \forall \)

\[
\begin{align*}
  \forall i = 1, 2, \ldots n, \text{ \it \theexist} \\
  f(a_1 + h_1 a_1^{-1-a}, \ldots, a_n + h_n a_n^{-1-a}) - f(a_1, \ldots, a_n) = L(h) + \sum_{i=1}^n \varepsilon_i(h) h_i,
\end{align*}
\]  

(12)

And \( \lim_{\|h\| \to 0} \varepsilon_i(h) = 0 \) for \( i = 1, 2, \ldots, n. \)

The linear transformation \( L : R^n \to R \) is defined by \( L(h) = \sum_{i=1}^n a_i h_i \), with \( h = (h_1, \ldots, h_n) \) and \( a_1, \ldots, a_n \in R \).

This linear transformation is denoted by \( D^a f(a) \), which is called CoD of \( f \) of the order \( a \in (0,1] \) at \( a \).

**Remark 1.** The equivalence of conditions (i) and (ii) is immediate, since

\[
\begin{align*}
  \lim_{h \to 0} \varepsilon(h) &= 0 \Leftrightarrow \varepsilon(h) \|h\| = o(\|h\|).
\end{align*}
\]  

(13)

To see the equivalence between conditions (ii) and (iii), we take

\[
\begin{align*}
  \varepsilon_i &= \varepsilon(h) \frac{h_i}{\|h\|} \\
  \varepsilon(h) &= \frac{1}{\|h\|} \sum_{i=1}^n \varepsilon_i(h) h_i.
\end{align*}
\]  

(14)

As \( \|h_i / \|h\| \| \leq 1 \), then we have the following:

(i) If \( \lim_{h \to 0} \varepsilon_i(h) = 0 \), then \( \lim_{h \to 0} \varepsilon_i(h) = 0 \)

(ii) If \( \lim_{h \to 0} \varepsilon_i(h) = 0 \) for \( i = 1, \ldots, n \), then we obtain

\[
\lim_{h \to 0} \|\varepsilon(h)\| \leq \lim_{h \to 0} \frac{1}{\|h\|} \sum_{i=1}^n \|\varepsilon_i(h)\| \leq \lim_{h \to 0} \sum_{i=1}^n \|\varepsilon_i(h)\| = 0.
\]  

(15)

i.e., \( \lim_{h \to 0} \varepsilon(h) = 0 \). Hence, the conditions (ii) and (iii) are equivalent.

**Example 1.** Consider a function \( f \) defined by \( f(x,y) = e^x - 2 \cos y \) and a point \( (a,b) \in R^2 \), with \( a > 0 \) and \( b > 0 \); then, \( D^a f(a,b) (h_1, h_2) = h_1 a^{1-a} e^a + 2 h_1 b^{1-a} \sin b \).

Solution: to prove this, let us note that

\[
\begin{align*}
  \lim_{(h_1, h_2) \to (0,0)} \frac{f(a + a^{1-a} h_1, b + b^{1-a} h_2) - f(a, b) - L(h_1, h_2)}{\| (h_1, h_2) \|} & = \frac{e^{a^{1-a} h_1} - 2 \cos (b + b^{1-a} h_2) - (e^a - 2 \cos b) - (h_1 a^{1-a} e^a + 2 h_1 b^{1-a} \sin b)}{\sqrt{h_1^2 + h_2^2}} \\
  & \leq \lim_{h_1 \to 0} \frac{e^{a^{1-a} h_1} - e^a - h_1 a^{1-a} e^a}{h_1} - 2 \lim_{h_2 \to 0} \frac{\cos (b + b^{1-a} h_2) - \cos b + b^{1-a} \sin b}{h_2} \\
  & = \lim_{h_1 \to 0} \left( \frac{e^{a^{1-a} h_1} - e^a - a^{1-a} e^a}{h_1} - 2 \lim_{h_2 \to 0} \left( \frac{\cos (b + b^{1-a} h_2) - \cos b + b^{1-a} \sin b}{h_2} \right) \right) \\
  & = (a^{1-a} e^a - a^{1-a} e^a) - 2(b^{1-a} \sin b + b^{1-a} \sin b) = 0.
\end{align*}
\]  

(16)

**Theorem 8.** If \( a \) RVF \( f \) with \( n \) variables \( a \) DF at \( a = (a_1, \ldots, a_n) \in R^n \), each \( a_i > 0 \), then \( f \) is CF at \( a \in R^n \).

**Proof.** Since \( f \) is a DF at \( a \), we can write the following:

\[
\begin{align*}
  f(a_1 + h_1 a_1^{-1-a}, \ldots, a_n + h_n a_n^{-1-a}) - f(a_1, \ldots, a_n) = \sum_{i=1}^n a_i h_i + o(\|h\|).
\end{align*}
\]  

(17)
By taking the limits of the two sides of the equality as $h \to 0$, we have
\[
\lim_{h \to 0} f(a + h_1 a_1^{-\alpha} + \ldots + a_n h_n a_n^{-\alpha}) = f(a_1, \ldots, a_n).
\] (18)

Hence, $f$ is CF at $a \in \mathbb{R}^n$.

**Theorem 9.** If a RVF $f$ with $n$ variables is a DF at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, each $a_i > 0$, then $\partial^\alpha / \partial x_i^\alpha f(a)$ exists for $1 \leq i \leq n$ and the CoD of $f$ of the order $\alpha \in (0, 1]$ is expressed as
\[
D^\alpha f(a)(h) = \sum_{i=1}^n \frac{\partial^\alpha}{\partial x_i^\alpha} f(a)h_i,
\] (19)
where $h = (h_1, \ldots, h_n)$.

**Proof.** By setting $h_j = 0, \forall j \neq i$ in the formula (12), we have
\[
f(a_1, \ldots, a_i + h_i a_i^{-\alpha} + \ldots + a_n) - f(a_1, \ldots, a_n) = \alpha_i h_i + \epsilon_i(h).
\] (20)

By multiplying by $1/h_i$, we can write
\[
\frac{f(a_1, \ldots, a_i + h_i a_i^{-\alpha} + \ldots + a_n) - f(a_1, \ldots, a_n)}{h_i} = \alpha_i + \epsilon_i(h).
\] (21)

By taking the limits of the two sides of the equality as $h_i \to 0$, we have
\[
\alpha_i = \frac{\partial^\alpha}{\partial x_i^\alpha} f(a), \quad \forall i = 1, 2, \ldots, n.
\] (22)

Finally, by substituting the values above $\alpha_i$ in the formula
\[
D^\alpha f(a)(h) = \sum_{i=1}^n \alpha_i h_i,
\] the result is followed.

**Corollary 1.** If a RVF $f$ with $n$ variables is a DF at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, each $a_i > 0$, then $D^\alpha f(a)$ is unique.

**Remark 2.** If a RVF $f$ with $n$ variables is a DF at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, each $a_i > 0$, then the CoV of gradient of $f$ of the order $\alpha \in (0, 1]$ at $a$ is
\[
\nabla^\alpha f(a) = \left(\frac{\partial^\alpha}{\partial x_1^\alpha} f(a), \ldots, \frac{\partial^\alpha}{\partial x_n^\alpha} f(a)\right).
\] (23)

Also, the matrix form (MF) of equation (19) is given as follows:
\[
D^\alpha f(a)(h) = \nabla^\alpha f(a) \cdot h = \left(\frac{\partial^\alpha}{\partial x_1^\alpha} f(a), \ldots, \frac{\partial^\alpha}{\partial x_n^\alpha} f(a)\right) \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}
\] (24)

**Theorem 10.** Let $\alpha \in (0, 1], f, g: X \to \mathbb{R}$ be a RVF defined in an open set (OS): $X \subset \mathbb{R}^n, \forall (x_1, \ldots, x_n) \in X$, each $x_i > 0$, and a point $a = (a_1, \ldots, a_n) \in X$. If $f, g$ are DF at $a$, then we have
\begin{align*}
(i) & \quad D^\alpha (\lambda f + \mu g)(a) = \lambda D^\alpha f(a) + \mu D^\alpha g(a), \\
(ii) & \quad D^\alpha (f \cdot g)(a) = D^\alpha f(a) \cdot g(a) + f(a) D^\alpha g(a).
\end{align*}

**Proof.** (i) follows from Definition 4; thus, it follows the proof of (i).

For (ii), let $A = (a_1 + h_1 a_1^{-\alpha} + \ldots + a_n + h_n a_n^{-\alpha})$, and then, we have
\[
\begin{aligned}
\lim_{h \to 0} & \left(\frac{(f \cdot g)(A) - (f \cdot g)(a)}{h} - (D^\alpha f(a) \cdot g(a) + f(a) \cdot D^\alpha g(a)(h)) \right) \\
= & \lim_{h \to 0} \left(\frac{f(A) - f(a)}{h} D^\alpha f(a) \cdot g(a) + f(a) \cdot \frac{(g(A) - g(a)) - D^\alpha g(a) h}{h} \right) + \lim_{h \to 0} \left(\frac{f(A) - f(a)}{h} \cdot (g(A) - g(a))\right) \\
= & 0 + \lim_{h \to 0} \left(\frac{D^\alpha f(a)(h) \cdot D^\alpha g(a)(h)}{h} \right) = \lim_{h \to 0} \left(\frac{D^\alpha f(a)}{h} \right) \cdot \left(\frac{D^\alpha g(a)}{h} \right) \cdot h = 0.
\end{aligned}
\] (25)

**Theorem 11.** Let $\alpha \in (0, 1], f: X \to \mathbb{R}$ be a RVF defined in an OS: $X \subset \mathbb{R}^n, \forall x = (x_1, \ldots, x_n) \in X$, each $x_i > 0$, and a point $a = (a_1, \ldots, a_n) \in X$. If the function $f$ has all CoV of PDVs of the order $\alpha$ at each point of a neighbourhood of the point $a$, $U(a)$, with $U(a) \subset X$, and they are continuous at $a$, then $f$ is a DF at $a$.

**Proof.** See Theorem 2.1 proof in [27].
\[ x_i > 0, \text{ and the point } a = (a_1, \ldots, a_n) \in X. \text{ The function } f \text{ is a } DF \text{ at } a \text{ iff its components are a } DF \text{ at } a, \text{ and if these components are } f_1, f_2, \ldots, f_m, \text{ then the components of } D^\alpha f(a) \text{ are the } \alpha\text{-derivatives } D^\alpha f_j(a), \text{ for } j = 1, 2, \ldots, m, \text{ i.e.,}
\]
\[ f = (f_1, f_2, \ldots, f_m) \Rightarrow D^\alpha f(a) = (D^\alpha f_1(a), D^\alpha f_2(a), \ldots, D^\alpha f_m(a)). \quad (26) \]

**Proof.** It is similarly proven as same as traditional calculus by applying \( D^\alpha \) instead of derivative.

**Remark 4.** Assume that \( \alpha \in (0, 1], f : X \rightarrow R^m \) is a VVF defined in an OS: \( X \subset R^n, \forall x = (x_1, \ldots, x_n) \in X, \text{ each } x_i > 0, \text{ and a point } a = (a_1, \ldots, a_n) \in X. \) If function \( f \) is \( \alpha \)-differentiable at \( a \), then \( \partial^\alpha / \partial x_i^\alpha f_j(a) \) exists for \( i = 1, 2, \ldots, n, \text{ and } j = 1, 2, \ldots, m, \text{ and the CoV of } \alpha-Jacobian \text{ of } f \text{ of order } \alpha \in (0, 1] \text{ at } a \text{ is expressed as}
\]
\[
\begin{pmatrix}
\frac{\partial^\alpha}{\partial x_1^\alpha} f_1(a) & \ldots & \frac{\partial^\alpha}{\partial x_n^\alpha} f_1(a) \\
\vdots & \ddots & \vdots \\
\frac{\partial^\alpha}{\partial x_1^\alpha} f_m(a) & \ldots & \frac{\partial^\alpha}{\partial x_n^\alpha} f_m(a)
\end{pmatrix}
\quad (27)
\]

### 4. The Chain Rule

Let us prove the CR for the functions of SVs in 2 particular cases. For the proof’s purpose, the continuity’s hypothesis of PDrs is given [28].

**Theorem 13 (CR).** Suppose that \( t \in R \) and \( x = (x_1, \ldots, x_n) \in R^n. \) If \( f(t) = (f_1(t), \ldots, f_n(t)) \) is a DF at \( a > 0 \Rightarrow \alpha \in (0, 1] \text{ and a RVF } g, \text{ with } n \text{ variables } x_1, \ldots, x_m \text{ is a } DF \text{ at } f(a) \in R^m, \text{ all } f_i(a) > 0 \alpha \in (0, 1]. \text{ Then, the composition } g \circ f \text{ is } \alpha\text{-differentiable at } a \text{ and}
\]
\[
(T\alpha g \circ f)(a) = \sum_{i=1}^{n} \frac{\partial^\alpha g}{\partial x_i^\alpha} (f(a)) \cdot (f_i(a))^{\alpha-1} \cdot (T\alpha f_j)(a).
\quad (28)
\]

**Proof.** Assume \( \in C^1(U(f(a)), R), \exists U(f(a)) \text{ is the point } f(a) \text{ neighborhood}. \) Suppose that \( h(t) = (g \circ f)(t) = g(f(t)). \) By setting \( u = a + \epsilon t \) in Definition 1, we see that
\[
(T\alpha h)(a) = \lim_{t \to a} \frac{h(t) - h(a)}{t - a} \alpha = \lim_{t \to a} \frac{g(f(t)) - g(f(a))}{t - a} \alpha.
\quad (29)
\]

Without loss of generality (WLOG), \( U(f(a)) \) is assumed to be an open ball (OB), represented by \( B(f(a), r). \) Since \( f \) is a CF, then along with points \( (f_1(a), \ldots, f_n(a)) \) and \( (f_1(t), \ldots, f_n(t)), \) the points \( (f_1(a), f_2(t), \ldots, f_n(t)), \ldots, (f_1(a), f_2(a), \ldots, f_n(t)) \) and lines connecting them must also be the ball \( B(f(a), r). \) In fact, using the classical MVT for differentiable functions (DFs) of one variable is in the following computation [28]:

\[
\begin{align*}
\frac{h(t) - h(a)}{t - a} \alpha &= \frac{g(f(t)) - g(f(a))}{t - a} \alpha \\
&= \frac{g(f_1(t), \ldots, f_n(t)) - g(f_1(a), f_2(t), \ldots, f_n(t))}{t - a} \alpha \\
&\quad + \frac{g(f_1(a), f_2(t), \ldots, f_n(t)) - g(f_1(a), f_2(a), f_3(t), \ldots, f_n(t))}{t - a} \alpha + \cdots \\
&\quad + \frac{g(f_1(a), f_2(a), \ldots, f_n(t)) - g(f_1(a), \ldots, f_n(a))}{t - a} \alpha \\
&= \frac{\partial}{\partial x_1} g(c_1, f_2(t), \ldots, f_n(t)) \left(\frac{f_1(t) - f_1(a)}{t - a}\right)^{\alpha-1} \\
&\quad + \frac{\partial}{\partial x_2} g(f_1(a), c_2, \ldots, f_n(t)) \left(\frac{f_2(t) - f_2(a)}{t - a}\right)^{\alpha-1} + \cdots \\
&\quad + \frac{\partial}{\partial x_n} g(f_1(a), f_2(a), \ldots, c_n) \left(\frac{f_n(t) - f_n(a)}{t - a}\right)^{\alpha-1}, \quad (30)
\end{align*}
\]
where $c_i$ is between $f_i(a)$ and $f_i(t)$ $\forall i = 1, 2, \ldots, n.$

By taking limits as $t \to a$ and using the continuity of PDrs of $g$ as well as the fact that $c_i \to f_i(a)$ $\forall i = 1, 2, \ldots, n,$ equation (29) is expressed as

\[
(T_a h)(a) = \lim_{t \to a} \frac{h(t) - h(a)}{t - a} a^{1-a} = \lim_{t \to a} \frac{g(f(t)) - g(f(a))}{t - a} a^{1-a}
\]

\[
= \lim_{t \to a} \left( \frac{\partial}{\partial x_1} g(c_1, f_2(t)) \frac{f_1(t) - f_1(a)}{t - a} a^{1-a} + \ldots \right)
\]

\[+ \frac{\partial}{\partial x_2} g(f_1(t), f_2(t)) \frac{f_1(t) - f_1(a)}{t - a} a^{1-a} + \ldots
\]

\[+ \frac{\partial}{\partial x_n} g(f_1(t), f_2(t), \ldots, f_n(t)) \frac{f_n(t) - f_n(a)}{t - a} a^{1-a} + \frac{\partial}{\partial x_1} g(f(a)) \cdot f_1'(a) \cdot a^{1-a}
\]

\[+ \frac{\partial}{\partial x_2} g(f(a)) \cdot f_2'(a) \cdot a^{1-a} \cdot f_1(a) \cdot a^{1-a} + \frac{\partial}{\partial x_2} g(f(a)) \cdot f_2'(a) \cdot a^{1-a} \cdot f_1(a) \cdot a^{1-a}
\]

\[= \frac{\partial}{\partial x_1} g(f(a)) \cdot f_1(a) \cdot (T_{a} f_1)(a) + \frac{\partial}{\partial x_2} g(f(a)) \cdot f_2(a) \cdot (T_{a} f_2)(a) + \ldots
\]

\[+ \frac{\partial}{\partial x_n} g(f(a)) \cdot f_n(a) \cdot (T_{a} f_n)(a).
\]

Our proof is completely done.

Remark 5. The MF of equation (28) is expressed as

\[
D^a (g \cdot f) (a) (h) = \left[ \frac{\partial}{\partial x_1} g(f(a)), \ldots, \frac{\partial}{\partial x_n} g(f(a)) \right]
\]

\[\cdot \left( \begin{array}{ccc}
    f_1(a)^{a-1} & 0 & 0 \\
    0 & \ddots & 0 \\
    0 & 0 & f_n(a)^{a-1}
\end{array} \right) \cdot \left( \begin{array}{c}
    (T_{a} f_1)(a) \\
    \vdots \\
    (T_{a} f_n)(a)
\end{array} \right) \cdot h.
\]

\[
\frac{\partial}{\partial x_i} (g \cdot f) (a) = \sum_{j=1}^{m} \frac{\partial}{\partial y_j} g(f(a)) \cdot f_j(a)^{a-1} \cdot \frac{\partial}{\partial x_i} f_j(a)
\]

\[\forall i = 1, 2, \ldots, n.
\]

Theorem 14 (CR). Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$. If $f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is a $\alpha$ DF at $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, each $a_i > 0 \Rightarrow \alpha \in (0, 1]$, and a RVF $g$ with variables $y_1, \ldots, y_m$ is a $\alpha$ DF at $f(a) \in \mathbb{R}^m$, where all $f_j(a) > 0 \Rightarrow \alpha \in (0, 1]$. Then, the composition $g \circ f$ is $\alpha$-differentiable at $a$, and we have

\[
\frac{\partial}{\partial x_i} (g \circ f) (a) = \sum_{j=1}^{m} \frac{\partial}{\partial y_j} g(f(a)) \cdot f_j(a)^{a-1} \cdot \frac{\partial}{\partial x_i} f_j(a)
\]

\[\forall i = 1, 2, \ldots, n.
\]
Proof. Proof. According to PDr’s definition and Theorem 14, the following is implied:

\[ D^\alpha (g \circ F)(a)(h) = \left( \frac{\partial^\alpha}{\partial y_1} g(f(a)), \ldots, \frac{\partial^\alpha}{\partial y_m} g(f(a)) \right) \cdot \begin{pmatrix} f_1(a)^{\alpha - 1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & f_m(a)^{\alpha - 1} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^\alpha}{\partial x_1} f_1(a) \\ \vdots \\ \frac{\partial^\alpha}{\partial x_n} f_n(a) \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}. \] (34)

5. Conformable Implicit Function Theorem

In this section, the CoV of IFThm for SVs is obtained. First, we establish the conformable theorem of existence and regularity of the implicit function for the case of a single equation.

Theorem 15 (Conformable implicit function theorem for the case of a single equation). Suppose that \( a \in (0, 1], F : X \rightarrow R \) is a RVF defined in an OS: \( X \subseteq R^{n+1}, \exists \forall (x_1, \ldots, x_n, y) \in X, \) each \( x_i, y > 0, \) and point \( (a_1, \ldots, a_n, b) \in X. \) Suppose that

\[ \frac{\partial\alpha}{\partial x_i} g(x_1, \ldots, x_n) = -\frac{\partial\alpha/\partial x_i}{\partial y}=F((x_1, \ldots, x_n), g((x_1, \ldots, x_n))) \cdot g((x_1, \ldots, x_n))^{\alpha-1} \] (35)

Proof. WLOG, \( X \) is assumed to be an OB, represented by \( B((a_1, \ldots, a_n, b), \epsilon_0). \) Let \( \rho \in (0, \epsilon_0). \) If we call \( \delta = \sqrt{\epsilon_0^2 - \rho^2}, \) it is verified that

\[ \| (x_1, \ldots, x_n) - (a_1, \ldots, a_n) \| < \delta \text{ and } |y - b| < \rho \Rightarrow (x_1, \ldots, x_n, y) \in B((a_1, \ldots, a_n, b), \epsilon_0). \] (36)

\[ F(a_1, \ldots, a_n, b - \rho) > 0, \]

\[ F(a_1, \ldots, a_n, b + \rho) < 0. \] (37)

By the continuity of \( F \) at \((a_1, \ldots, a_n, b - \rho)\) and \((a_1, \ldots, a_n, b + \rho),\) there exists \( \delta' \in (0, \delta), \) such that

\[ \| (x_1, \ldots, x_n) - (a_1, \ldots, a_n) \| < \delta' \Rightarrow [F(x_1, \ldots, x_n, b - \rho) > 0 \text{ and } F(x_1, \ldots, x_n, b + \rho) < 0]. \] (38)

Remark 6. The MF of equation (33) is expressed as

- (i) \( F(a_1, \ldots, a_n, b) = 0 \)
- (ii) \( F \in C_0(X, R) \)
- (iii) \( \frac{\partial\alpha/\partial y} \neq 0 \)

Then, there is a neighborhood \( U \subseteq R^n \) of \((a_1, \ldots, a_n, b), \) such that there is a unique function \( (UF) \) \( y = g(x_1, \ldots, x_n) \) that satisfies the following:

\[ g(a_1, \ldots, a_n) = b \text{ and } F(x_1, \ldots, x_n, g(x_1, \ldots, x_n)) = 0, \forall (x_1, \ldots, x_n) \in U. \]

Finally, \( y = g(x_1, \ldots, x_n) \) is \( C_0 \) in \( U, \) and for every \( i = 1, 2, \ldots, n, \) we have

Since the function \( y \mapsto F(x_1, \ldots, x_n, y) \) is CF on the interval \([b - \rho, b + \rho], \forall (x_1, \ldots, x_n) \in B((a_1, \ldots, a_n), \delta'), \) and via the classical Bolzano’s theorem (BThm), it implies that \( \exists y \in (b - \rho, b + \rho) \exists F(x_1, \ldots, x_n, y) = 0, \) for each \( x = (x_1, \ldots, x_n). \) Then, \( y \)’s value is unique, since a function, whose derivative is positive, has more than zero. On the other hand, by having \( U = B((a_1, \ldots, a_n), \delta'), \) for each \((x_1, \ldots, x_n) \in U, \) there exists a unique \( y \) such that \( y = g(x_1, \ldots, x_n) \), and then, this function will be proven to be CF on \( B((a_1, \ldots, a_n), \delta'). \) The continuity of the function \( g \) at the point \((a_1, \ldots, a_n)\) is clear since for each \( \rho > 0, \) \( \exists \) a value \( \delta' > 0 \) \( \forall\)
\[ \| (x_1, \ldots, x_n) - (a_1, \ldots, a_n) \| < \delta \Rightarrow \| b - y \| < \rho \]  

The function \( g \) continuity will be proven at any point \((x_1, \ldots, x_n) \in B((a_1, \ldots, a_n), \delta')\) by simply substituting \( B((a_1, \ldots, a_n), \delta')\) for an OB: \( B((x_1, \ldots, x_n)) \) is contained in \( B((a_1, \ldots, a_n), \delta')\).

Finally, let us now show formula (35). By applying Theorem 14 to equation,

\[ F(x_1, \ldots, x_n, y) = 0, \]  

we have

\[ \frac{\partial^a}{\partial x^a_i} F(x, g(x)) + \frac{\partial^a}{\partial y^a} F(x, g(x)) \cdot g(x)^{a - 1} \cdot \frac{\partial^a}{\partial x^a_i} g(x) = 0. \]  

\[ \forall i = 1, 2, \ldots, n, \exists x = (x_1, \ldots, x_n). \]  

Solving \( \frac{\partial^a}{\partial x^a_i} g(x), \) we get (35). In addition, the formula (35) on the right side is continuous, so the continuity of \( \text{CoV of PDs} \) \( \frac{\partial^a}{\partial x^a_i} g(x) \) \( \forall, \) \( i = 1, 2, \ldots, n, \) follows.

Theorem 15 will provide us a help in computing \( \text{CoV of PDs} \) of implicit function of \( SVs. \)

**Example 2. Consider**

\[ F(x, y, z) = x^3 + 3y^2 + 4xz^2 - 3y^2 - 5 = 0. \]  

This equation’s one solution is \((1, 1, 1), F\) is obviously in \( C_a \) which is OB, represented by \( B((1, 1, 1), \varepsilon_0), \) with \( x, y, z > 0, \) for some \( \alpha \in (0, 1], \) since

\[ \frac{\partial^a}{\partial x^a} F(1, 1, 1) = \left[ 8x^2 - 6yz^2 \right]_{(1, 1, 1)} = 2 \neq 0. \]  

Theorem 15 implies that there is a neighbourhood, \( U \subset R^2, \) of \((1, 1), \exists \) a UF: \( z = g(x, y) \) satisfies the following: \( g(1, 1) = 1 \) and \( F(x, y, g(x, y)) = 0, \forall (x, y) \in U. \)

Moreover, \( z = g(x, y) \) is \( C_a \) in \( U, \) and we have

\[ \frac{\partial^a}{\partial x^a} g(x, y) = \left( \frac{3x + 4z^2}{8x - 6yz} \right) x^{1-a}, \]  

\[ \frac{\partial^a}{\partial y^a} g(x, y) = \left( \frac{6y - 3z^2}{8x - 6yz} \right) y^{1-a}. \]  

Finally, we obtain \( \frac{\partial^a}{\partial x^a} g(1, 1) = -(7/2) \) and \( \frac{\partial^a}{\partial y^a} g(1, 1) = -(3/2). \)

Finally, \( \text{CoV of IF-thm for a system of several equations and SRVs is obtained.} \)

**Theorem 16** (Conformable general implicit function theorem). Let \( \alpha \in (0, 1], F: X \rightarrow R^n \) be a VVF defined in an OS: \( X \subset R^{R_{m}^*}, \exists \forall (x; y) = (x_1, \ldots, x_n; y_1, \ldots, y_m) \in X, \) each \( x_i, y_j > 0, \) and point \( (a; b) = (a_1, \ldots, a_n; b_1, \ldots, b_m) \in X. \) Assume that

(i) \( F(a; b) = 0 \)

(ii) \( F \in C_a (X, R^m) \)

(iii) \( \det \left[ J^a_F(a; b) \right] \neq 0 \)

Then, there is a neighbourhood, \( U \subset R^m, \) of \( a \exists \exists U \) UF: \( f: U \rightarrow R^n, x \rightarrow y = f(x) \) that satisfies \( f(a) = b \) and \( F(x; f(x)) = 0, \forall x \in U. \)

Finally, \( y = f(x) \) is class \( C_a \) in \( U, \) and for every \( i = 1, 2, \ldots, n, \) we have

\[ \left[ \frac{\partial^a}{\partial x^a_i} F \right] = \left( \begin{array}{cc} f_1^{a - 1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_m^{a - 1} \end{array} \right), \]

\[ \frac{\partial^a}{\partial x^a_i} F = \left( \begin{array}{ccc} \frac{\partial^a}{\partial y^a_1} F_1 & \cdots & \frac{\partial^a}{\partial y^a_m} F_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial^a}{\partial y^a_m} F_m & \cdots & \frac{\partial^a}{\partial y^a_m} F_m \end{array} \right). \]  

Proof. The proof of existence and uniqueness of the implicit function is done similar to the traditional multivariable calculus by applying mathematical induction on \( q \) and using the conformable implicit function theorem for several variables [28].

Let us now show formula (44). Assume that a system with several equations and SRVs is expressed as

\[ F(x; y) = 0 \lor \left\{ \begin{array}{l} F_1(x_1, \ldots, x_n; y_1, \ldots, y_m) = 0 \\ \vdots \\ F_m(x_1, \ldots, x_n; y_1, \ldots, y_m) = 0 \end{array} \right., \]

which satisfies hypotheses (i)–(iii) of the theorem; then, this system is defined in a neighbourhood, \( U \subset R^m, \) of \( a, \) the implicit function \( y = f(x) \) class \( C_a \) in \( U, \) such that \( f(a) = b \) which satisfies equation (1), i.e.,

\[ F(x; f(x)) = 0 \lor \left\{ \begin{array}{l} F_1(x; f_1(x), \ldots, f_m(x)) = 0 \\ \vdots \\ F_m(x; f_1(x), \ldots, f_m(x)) = 0 \end{array} \right., \]

By applying \( \text{CoV of CR} \) to the above equation, we have
Consider a system of two equations and two real variables:

\[
\begin{align*}
\frac{\partial^a F_1}{\partial x_1^a} + \frac{\partial^a F_1}{\partial y_1^a} \cdot f_1^{a-1} + \frac{\partial^a F_1}{\partial y_m^a} \cdot f_m^{a-1} - \frac{\partial^a f_1}{\partial x_1^a} &= 0, \\
\vdots \\
\frac{\partial^a F_m}{\partial x_1^a} + \frac{\partial^a F_m}{\partial y_1^a} \cdot f_1^{a-1} + \frac{\partial^a F_m}{\partial y_m^a} \cdot f_m^{a-1} - \frac{\partial^a f_m}{\partial x_1^a} &= 0
\end{align*}
\]  

(48)

for all \(i = 1, 2, \ldots, n\).

In addition, the MF of equation (48) is given as follows:

\[
\begin{align*}
\begin{pmatrix}
\frac{\partial^a F_1}{\partial x_1^a} \\
\vdots \\
\frac{\partial^a F_m}{\partial x_1^a}
\end{pmatrix}
\end{align*}
= - \begin{pmatrix}
\frac{\partial^a F_1}{\partial y_1^a} & \ldots & \frac{\partial^a F_1}{\partial y_m^a} \\
\vdots & \ddots & \vdots \\
\frac{\partial^a F_m}{\partial y_1^a} & \ldots & \frac{\partial^a F_m}{\partial y_m^a}
\end{pmatrix}
\begin{pmatrix}
f_1^{a-1} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & f_m^{a-1}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1^a} \\
\vdots \\
\frac{\partial f_m}{\partial x_1^a}
\end{pmatrix}
\]  

(49)

Since \(J^a F\) and \(f^{a-1}\) are regular matrices by hypothesis, we have

\[
\begin{align*}
\begin{pmatrix}
\frac{\partial^a f_1}{\partial x_1^a} \\
\vdots \\
\frac{\partial^a f_m}{\partial x_1^a}
\end{pmatrix}
\end{align*}
= - \begin{pmatrix}
\frac{\partial^a F_1}{\partial y_1^a} & \ldots & \frac{\partial^a F_1}{\partial y_m^a} \\
\vdots & \ddots & \vdots \\
\frac{\partial^a F_m}{\partial y_1^a} & \ldots & \frac{\partial^a F_m}{\partial y_m^a}
\end{pmatrix}
\begin{pmatrix}
f_1^{a-1} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & f_m^{a-1}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1^a} \\
\vdots \\
\frac{\partial f_m}{\partial x_1^a}
\end{pmatrix}
\]  

(50)

which completes the proof.

We will now show that Theorem 16 can be used to compute CoV of PDRs of systems with several equations and SRVs.

**Example 3.** Consider a system of two equations and two real variables:

\[
\begin{align*}
F_1(x, y, z, w) &= x^2 + y^2 + z^2 + w^2 - 6 = 0, \\
F_2(x, y, z, w) &= x^2 - y^2 + z^2 - w^2 = 0.
\end{align*}
\]  

(51)

One solution of this equation is \((x, y, z, w) = (1, 1, \sqrt{2}, \sqrt{2})\). Clearly, \(F = (F_1, F_2)\) is in \(C_p\), which is OB: \(B((1, 1, \sqrt{2}, \sqrt{2}), \varepsilon_0)\), with \(x, y, z, w > 0\), for some \(\alpha \in (0, 1]\), since
\[
\det\left[f^a_{\alpha a} F((\sqrt{2}, \sqrt{2}, 1, 1))\right] = \det\left[\begin{array}{cc}
2z^{2-a} & 2w^{2-a} \\
2z^{2-a} & -2w^{2-a}
\end{array}\right]_{(1,1,\sqrt{2},\sqrt{2})} = \frac{-32}{2^a} \neq 0. \tag{52}
\]

Theorem 16 implies that there is a neighbourhood, \( U \subset \mathbb{R}^2 \), of \((\sqrt{2}, \sqrt{2}) \) \( \exists \) \( U \) \( F = (f_1, f_2) \) given by
\[
\begin{align*}
\begin{cases}
z = f_1(x, y), \\
w = f_2(x, y),
\end{cases}
\end{align*}
\tag{53}
\]
that satisfies
\[
\begin{align*}
\left( \frac{\partial^a f_1}{\partial x^a} \right) &= \left( \begin{array}{cc}
z^{a-1} & 0 \\
0 & w^{a-1}
\end{array} \right)^{-1} \left( \begin{array}{cc}
nz^{2-a} & 2nw^{2-a} \\
n2z^{2-a} & -2nw^{2-a}
\end{array} \right)^{-1} \left( \begin{array}{cc}
z^{2-a} & 0 \\
0 & w^{2-a}
\end{array} \right) = \frac{1}{4} \left( \begin{array}{cc}
z^{2-a} & 0 \\
0 & w^{2-a}
\end{array} \right), \\
\left( \frac{\partial^a f_2}{\partial y^a} \right) &= \left( \begin{array}{cc}
z^{a-1} & 0 \\
0 & w^{a-1}
\end{array} \right)^{-1} \left( \begin{array}{cc}
2z^{2-a} & 2nw^{2-a} \\
2z^{2-a} & -2nw^{2-a}
\end{array} \right)^{-1} \left( \begin{array}{cc}
z^{2-a} & 0 \\
0 & w^{2-a}
\end{array} \right) = \frac{1}{4} \left( \begin{array}{cc}
z^{2-a} & 0 \\
0 & w^{2-a}
\end{array} \right),
\end{align*}
\tag{55}
\]

Finally, we have
\[
\begin{align*}
\left( \frac{\partial^a f_1}{\partial x^a} \right)_{(1,1,\sqrt{2},\sqrt{2})} &= \left( \begin{array}{cc}
1 \\
0
\end{array} \right), \\
\left( \frac{\partial^a f_2}{\partial y^a} \right)_{(1,1,\sqrt{2},\sqrt{2})} &= \left( \begin{array}{cc}
0 \\
0
\end{array} \right).
\end{align*}
\tag{56}
\]

According to all of our numerical examples in this work, with the help of our investigation to all proposed theorems and properties in this article, the numerical examples show that all our obtained results are valid and efficient. As seen in all these examples, the computations are simple and cost-efficient, which are important when scientific phenomena are modelled in the sense of conformable calculus. All in all, the simplicity of computations is always needed in modeling applications in comparison to other complicated computations that are needed using other approaches in other fractional operators. All our obtained results in this work are accurate because our results coincide with the usual integer-order results.

### 6. Conclusion

A new investigation on the CoV of multivariable calculus has been discussed in this work in detail. The \( \alpha \)-derivative of a function of SVs and all related properties have been investigated. The CoV of CR for functions of SVs has also been studied. The CoV of IF\( \alpha \)m has been presented in the last part of our results, and numerical experiments have been conducted to support our theoretical results. The findings of this investigations show that all results formulated via CoD
coincide with the ones in the traditional integer case. All results in this work can be potentially applied in modeling oceanographic phenomena such as Stommel’s box model of thermohaline circulation and other related models where this study’s analysis can be further extended or generalized in future works for various related physical models.

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no conflicts of interest.

Authors’ Contributions

Mohammed K. A. Kaabar involved in actualization, developed methodology, performed formal analysis, validated and investigated the study, prepared the initial draft, and supervised and edited the original draft. Francisco Martínez, Inmaculada Martínez, and Silvestre Paredes involved in actualization, developed methodology, performed formal analysis, validated and investigated the study, and prepared the initial draft. Zailan Siri developed methodology, performed formal analysis, validated and investigated the study, and prepared the initial draft. All authors read and approved the final version.

References


