Research Article

The Locating Chromatic Number of Book Graph

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Let $G = (V(G), E(G))$ be a connected graph and $c: V(G) \rightarrow \{1, 2, \dots, k\}$ be a proper k -coloring of G . Let Π be a partition of vertices of G induced by the coloring c . We define the color code $c_{\Pi}(v)$ of a vertex $v \in V(G)$ as an ordered k -tuple that contains the distance between each partition to the vertex v . If distinct vertices have distinct color code, then c is called a *locating k -coloring* of G . The locating chromatic number of G is the smallest k such that G has a locating k -coloring. In this paper, we determine the locating chromatic number of book graph.

1. Introduction

All graphs considered in this paper are assumed to be simple, connected, and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. By P_n and S_n , we denote a path on n vertices and a star on $n + 1$ vertices, respectively. The distance, $d(u, v)$, between two vertices u and v is defined as the length of the shortest path connecting them in G .

For a graph G and a positive integer k , a coloring $c: V(G) \rightarrow \{1, 2, \dots, k\}$ with $c(u) \neq c(v)$ for every two adjacent vertices u and v is called a proper k -coloring of G . Let $\Pi = \{C_1, C_2, \dots, C_k\}$ be a partition of vertices of G induced by the coloring c , where C_i denotes the set of vertices receiving color i . A color code, $c_{\Pi}(v)$, of a vertex v with respect to Π is an ordered k -tuple $(d(u, C_1), d(u, C_2), \dots, d(u, C_k))$, where $d(u, C_i) = \min\{d(u, x) : x \in C_i\}$ for every $i = 1, 2, \dots, k$. If distinct vertices have distinct color codes, then c is called a locating k -coloring of G . The locating chromatic number of G , denoted by $\chi_L(G)$, is the smallest k for which G has a locating k -coloring.

The concept of locating chromatic number of a graph was introduced in 2002 by Chartrand et al. [1]. This concept is derived from the graph partition dimension and graph coloring. The study about locating chromatic number of a graph has grown constantly. Chartrand et al. [1] determined the locating chromatic number of some well-known graph classes such as paths with $\chi_L(P_n) = 3$ for $n \geq 3$, cycles

with $\chi_L(C_n) = 3$ for odd $n \geq 3$ and $\chi_L(C_n) = 4$ for even $n \geq 4$, and double stars with $\chi_L(S_{m,n}) = n + 1$ for $1 \leq m \leq n$ and $n \geq 2$. In [2], Behtoei and Omoomi investigated the locating chromatic number for Cartesian product of graphs. Welyyanti et al. [3] discussed the locating chromatic number for disconnected graphs. Furthermore, the locating chromatic number of spinner graph was proved by Rianti and Narwen [4].

Let G and H be two given graphs. The Cartesian product of G and H , denoted by $G \times H$, is a graph having the vertex set $V(G \times H) = V(G) \times V(H)$ such that two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.

For a positive integer $n \geq 1$, a book graph, denoted by B_n , is a graph obtained from n copies of a cycle of order 4 sharing one common edge. The book graph B_n is also called the Cartesian product $P_2 \times S_n$. In this paper, we investigate the locating chromatic number of book graph.

The following theorem proved by Chartrand et al. [1] is useful in determining the locating chromatic number of a graph.

Theorem 1. *Let c be a locating coloring in a connected graph G and $N(v)$ be the set of vertices adjacent to v . If u and v are distinct vertices of G such that $d(u, w) = d(v, w)$ for all $w \in V(G) - \{u, v\}$, then $c(u) \neq c(v)$. In particular, if u and v are nonadjacent vertices of G such that $N(u) = N(v)$, then $c(u) \neq c(v)$.*

2. Result and Discussion

In this section, we determine the exact value for the locating chromatic number of book graphs. Let B_n be a book graph with the vertex set $V(B_n) = \{u_i, v_i: i = 0, 1, \dots, n\}$ and the edge set $E(B_n) = \{u_i v_i: i = 0, 1, \dots, n\} \cup \{u_0 u_i, v_0 v_i: i = 1, 2, \dots, n\}$.

Two observations below follow from definition of proper coloring.

Observation 1. Let c be a locating coloring of the book graph B_n , $n \geq 1$. Then, $c(u_i) \neq c(v_i)$ for $i = 0, 1, \dots, n$.

Observation 2. Let c be a locating coloring of the book graph B_n , $n \geq 1$. Then, $c(u_0) \neq c(u_i)$ and $c(v_0) \neq c(v_i)$ for $i = 1, 2, \dots, n$.

Lemma 1. Let c be a locating coloring of the book graph B_n , $n \geq 1$. Then, $(c(u_i), c(v_i)) \neq (c(u_j), c(v_j))$ for $1 \leq i \neq j \leq n$.

Proof. Assume that $(c(u_i), c(v_i)) = (c(u_j), c(v_j))$ for some i and j , $i \neq j$. This means that $c(u_i) = c(u_j)$ and $c(v_i) = c(v_j)$. We know that $d(u_i, x) = d(u_j, x)$ for every $x \in V(B_n) - \{u_i, u_j, v_i, v_j\}$. As $c(v_i) = c(v_j)$ and $d(u_i, v_i) = d(u_j, v_j) = 1$, regardless the color of x , we have $c_\Pi(u_i) = c_\Pi(u_j)$, a contradiction. $\square$

Lemma 2. Let c be a locating k -coloring of the book graph B_n , for $k \geq 4$ and $n \geq 1$. Then,

$$n \leq 2 \binom{k-2}{2} + 2(k-1) - 1. \quad (1)$$

Proof. Start coloring the graph by giving color C_1 and C_2 for the vertices in the middle, that is, $c(u_0) = C_1$ and $c(v_0) = C_2$. From Observation 2, we know that we should have $c(u_i) \neq C_1$ and $c(v_i) \neq C_2$ for each $i = 1, 2, \dots, n$. $\square$

Case 1. Assume $\{c(u_a), c(v_a)\}$ does not contain either C_1 or C_2 . Thus, the total color that we can choose is $\binom{k-2}{2} \times 2$ because every outer part of sheets requires 2 different colors from observation 1, which means there are $\binom{k-2}{2}$ possibilities. But, since $c(u_a)$ and $c(v_a)$ are reversible without any consequences, the total that can be obtained is $\binom{k-2}{2} \times 2$.

Case 2. If $c(u_a) = 2$, the total possibility of the colors being used is $(k-1)$ because $c(v_a)$ can only be given with $1, 3, 4, \dots, k$, where only one color can be used by book graph B_n for $n \geq 2$. The illustration is shown in Figure 1.

Case 3. If $c(v_a) = 1$, the total colors can be used are $(k-1)$ because $c(u_a)$ can only be given by color $2, 3, 4, \dots, k$, where only two colors can be used on book graph B_n for $n \geq 2$. The illustration is shown in Figure 2.

FIGURE 1: Illustration of Case 2.

FIGURE 2: Illustration of Case 3.

Since the coloring $c(v_a) = 1$ and $c(u_a) = 2$ have been used in Case 2, then the coloring is not reusable, and thus, the total coloring is $(k-1) - 1$. If uses k colors, then $\chi_L(B_n)$ must satisfy the following

$$\begin{aligned} n &\leq 2 \binom{k-2}{2} + (k-1) + (k-1) - 1, \\ n &\leq 2 \binom{k-2}{2} + 2(k-1) - 1. \end{aligned} \quad (2)$$

Theorem 2. Locating chromatic number of book graph B_n is

$$\chi_L(B_n) = \min \left\{ k: n \leq 2 \binom{k-2}{2} + 2(k-1) - 1 \right\}, \quad (3)$$

for $k \geq 4$.

Proof. Firstly, we will prove the existence of $\chi_L(B_n)$. We can clearly see that $\left\{ k: n \leq 2 \binom{k-2}{2} + 2(k-1) - 1 \right\}$ is the subset from a set of real numbers. So from the well-ordering principle, the set must have the smallest element, say

$$k^* := \min \left\{ k: n \leq 2 \binom{k-2}{2} + 2(k-1) - 1 \right\}, \quad (4)$$

that yields

$$(k^* - 1) \geq \min \left\{ k: n \leq 2 \binom{k-2}{2} + 2(k-1) - 1 \right\} = k^*. \quad (5)$$

It was contradiction, which means it is impossible that the locating chromatic of book graph has a coloring for $(k^* - 1)$ colors. It should be

$$\chi_L(B_n) = k^* = \min \left\{ k: n \leq 2 \binom{k-2}{2} + 2(k-1) - 1 \right\}, \quad \text{for } k \geq 4. \quad (6)$$

□

Corollary 1. Locating chromatic number of book graph B_n is

$$\chi_L(B_n) = \left\lceil \sqrt{n - \frac{3}{4}} + \frac{3}{2} \right\rceil. \quad (7)$$

Proof. Let k be the color needed for locating chromatic of book graph B_n , then

$$\begin{aligned} n &\leq 2 \binom{k-2}{2} + 2(k-1) - 1, \\ n &\leq \frac{(k-2)(k-3)}{2} \times 2 + 2k - 2 - 1, \\ (k^2 - 5k + 6 + 2k - 3) &\geq n, \\ k^2 - 3k + 3 &\geq n, \\ \left(k - \frac{3}{2}\right)^2 - \frac{9}{4} + 3 &\geq n, \\ \left(k - \frac{3}{2}\right)^2 + \frac{3}{4} &\geq n, \\ \left(k - \frac{3}{2}\right)^2 &\geq n - \frac{3}{4}, \\ k &\geq \sqrt{n - \frac{3}{4}} + \frac{3}{2}. \end{aligned} \quad (8)$$

Then, the smallest possible k is

$$k_{\min} = \left\lceil \sqrt{n - \frac{3}{4}} + \frac{3}{2} \right\rceil. \quad (9)$$

□

3. Conclusion

The locating chromatic number of book graph that has been found is

$$\chi_L(B_n) = k^* = \min \left\{ k: n \leq \binom{k-2}{2} + 2(k-1) - 1 \right\}, \quad \text{for } k \geq 4, \quad (10)$$

or

$$k_{\min} = \left\lceil \sqrt{n - \frac{3}{4}} + \frac{3}{2} \right\rceil. \quad (11)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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