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Research Article

A Comprehensive Overview on the Formation of Homomorphic Copies in Coset Graphs for the Modular Group

Hanan Alolaiyan,[1] **Abdul Razaq**,[2] **Awais Yousaf**,[3] **and Rida Zahra**[4]

[1]*Department of Mathematics, King Saud University, Riyadh, Saudi Arabia*
[2]*Department of Mathematics, Division of Science and Technology, University of Education Lahore, Lahore, Pakistan*
[3]*Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan*
[4]*Department of Mathematics, Govt. Girls Elementary School Narri, Khushab, Pakistan*

Correspondence should be addressed to Abdul Razaq; abdul.razaq@ue.edu.pk

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This work deals with the well-known group-theoretic graphs called coset graphs for the modular group G and its applications. The group action of G on real quadratic fields forms infinite coset graphs. These graphs are made up of closed paths. When M acts on the finite field Z_p, the coset graph appears through the contraction of the vertices of these infinite graphs. Thus, finite coset graphs are composed of homomorphic copies of closed paths in infinite coset graphs. In this work, we have presented a comprehensive overview of the formation of homomorphic copies.

1. Introduction

The study of groups via their actions has been a central theme in group theory, since the completion of the classification of simple groups in the 1980s. This most commonly takes the form of actions on vector spaces and similar commutative objects or on more elementary combinatorial objects. It is not an exaggeration to say that the modular group G (see [1–4]) is the single most important infinite discrete group, through its myriad connections with number theory, geometry, and topology. There is a long and venerable history of studying its actions, particularly on finite sets, which goes back to before the turn of the twentieth century. The modular group G has two generators f and g, where f: $x \longrightarrow -1/x$ and g: $x \longrightarrow x - 1/x$ are linear fractional transformations. The finite presentation of G is $<f, g: f^2 = g^3 = 1>$. It means that it is a free product of C_2 and C_3. The linear fractional transformation h: $x \longrightarrow 1/x$ extends G to G' because it inverts f and g; that is, $h^2 = (fh)^2 = (gh)^2 = 1$. Thus, extended modular group G' has three generators f, g, and h and its finite presentation is $<f, g, h: f^2 = g^3 = h^2 = (fh)^2 = (gh)^2 = 1>$.

Graph theory has applications in various branches of mathematics [5, 6]. Several topological and algebraic structures can be studied in a more effective way by using graphs. Graphical techniques are specifically utilized to investigate the finitely generated groups. The graphs prove to be an effective and simple method to solve many mathematical problems [7–9].

The use of graphs to represent group actions has a venerable history. Cayley [8] published the first work on this topic. Mathematicians like Coxeter [9], Burnside [10], Stothers [11], Everitt [12], Conder [13], Whitehead [14], and others provided pioneering works on graphical representations of groups. The action of a modular group on certain objects can be represented by a certain type of graphs, called coset graphs. These were introduced by Higman in 1978. Later, in 1983, Mushtaq [15] laid their foundation. These graphs consist of triangles connected to each other. The edges of triangles are permuted anticlockwise to represent by g. Each vertex of a triangle is connected by f to another vertex of the triangle (which may be the same triangle).

Moreover, the vertices of the coset graph that are fixed by f and g are represented by heavy dots. Since $(gh)^2 = 1$ implies

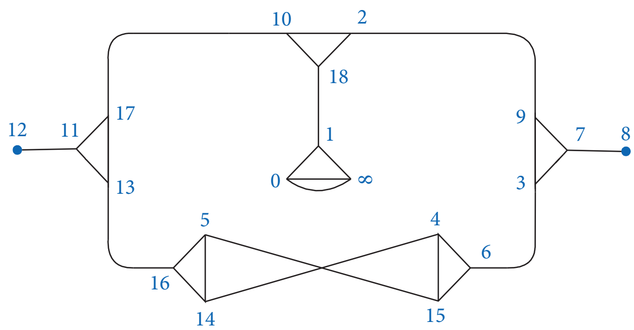

FIGURE 1: The coset graph of $Z_{19} \cup \{\infty\}$.

$hgh^{-1} = g^{-1-1}$, h turns around the direction of the triangles like reflection. Thus, we do not introduce h-edges in coset graphs, so that they remain simple.

The action of G' on finite field Z_p is not possible because f maps 0 to ∞. Thus, we add ∞ to Z_p in order to make the action possible.

Example 1. Let us consider the action of modular group on $Z_{19} \cup \{\infty\}$. The permutation representations f, g, and h are

$$\begin{aligned} f&: (0,\infty)(13,16)(1,18)(11,12)(2,9)(10,17)(3,6)(7,8)(4,14)(15,5),\\ g&: (\infty,1,0)(10,18,2)(7,9,3)(15,6,4)(16,14,5)(17,11,13)(8)(12),\\ h&: (0,\infty)(10,2)(13,3)(5,4)(16,6)(11,7)(12,8)(17,9)(15,14)(1)(18). \end{aligned} \tag{1}$$

The corresponding coset graph is shown in Figure 1.

Definition 1. Let Γ and Γ' be two coset graphs; then Γ' is a homomorphic copy of Γ if

(i) order of Γ' is less than the order of Γ, that is, $|\Gamma'| < |\Gamma|$,

(ii) u is a vertex in Γ such that $(u)\,x = u$ for some $x \in G$, and then there exists some vertex $u' \in \Gamma'$ such that $(u')x = u$.

Let q' be a real quadratic irrational number; then $q' = ((\alpha + \sqrt{m})/\beta)$, where m is a square-free natural number and $(\alpha, ((\alpha^2 + m)/\beta), \beta) = 1$. In [16], Mushtaq studied the group action of G on real quadratic fields and showed that the corresponding coset graphs are infinite. Figure 2 shows a small patch of these graphs.

Due to the emergence of infinite graphs, the action of G on $q' = ((\alpha + \sqrt{m})/\beta)$ through coset graphs is not easy to study. Therefore, the action of G on $Z_p \cup \{\infty\}$ becomes important. The coset graphs for $Z_p \cup \{\infty\}$ are homomorphic copies of the infinite graphs for $q' = ((\alpha + \sqrt{m})/\beta)$, where $m \equiv n^2 \mathrm{mod} p$, for any natural number n. For example, the coset graph shown in Figure 1 is the homomorphic copy of the coset graph for $((\alpha + \sqrt{17})/\beta)$ because $17 \equiv 6^2 \mathrm{mod} 19$.

For further details about coset graphs, we refer the readers to [17–22].

The main contributions of this paper are as follows:

(1) A thorough study on the formation of homomorphic copies of coset graphs is presented

(2) We have developed a formula to compute all homomorphic copies of the closed path of rank 4

2. Closed Paths in Coset Graph

Definition 2. A closed path in a coset graph containing a vertex fixed by $(fg)^{m_1}(fg^{-1})^{m_2}, \ldots, (fg)^{m_{k-1}}(fg^{-1})^{m_k} \in \mathrm{PSL}(2, Z)$, where $m_1, m_2, \ldots, m_k \in N$ is called a closed path of rank k. It is denoted by $(m_1, m_2, \ldots, m_k)$. In [23], it has been proved that the rank of closed paths is always even.

Remark 1. Let $x, y \in G$ and $(u)x = u$. Then vertex $(u)y$ is fixed by $y^{-1}xy$.

Suppose that u_1 and u_2 are any two vertices in a closed path C, such that $(u_1)x_1 = u_1$ and $(u_2)x_2 = u_2$. Let $(u_1)x_3 = u_2$; then $x_1^{-1}x_3$ also maps u_1 to u_2. Clearly, x_3 and $x_1^{-1}x_3$ are the only possible paths to travel from u_1 to u_2. By contraction of vertices u_1 and u_2, we mean u_1 and u_2 merge to form a node $u = u_1 = u_2$ such that $u = u_1 = u_2$ is fixed by both x_3 and $x_1^{-1}x_3$. This can be done by making a closed path C' containing u such that $(u)x_3 = u$ and then by applying $x_1^{-1}x_3$ on u such that $x_1^{-1}x_3$ ends at u. Consequently, a graph γ is evolved, which is a homomorphic copy of C. Note that, in addition to u_1 and u_2, there are some other pairs of vertices in C, which also compose γ by contraction. In fact, during the formation of γ by contracting u_1 and u_2, some more pairs also get contracted. How many are they? The following theorems help to calculate this number.

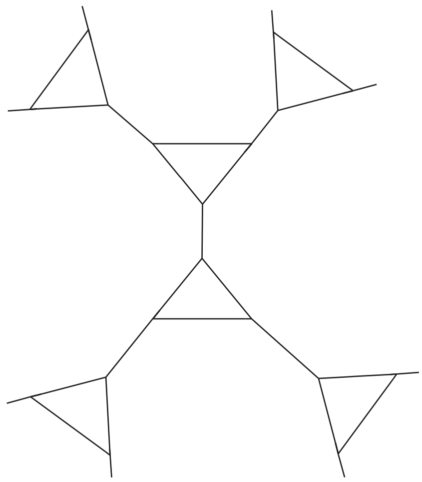

Figure 2: An overview of the small portion of infinite graphs.

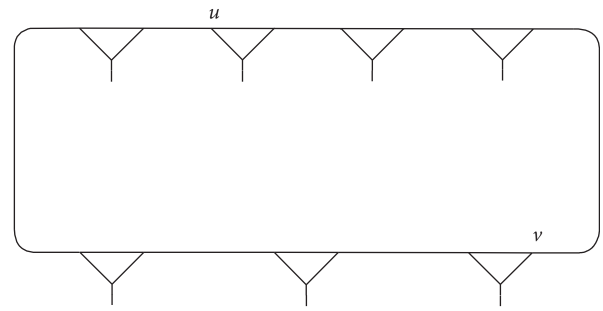

Figure 3: The closed path (4, 3).

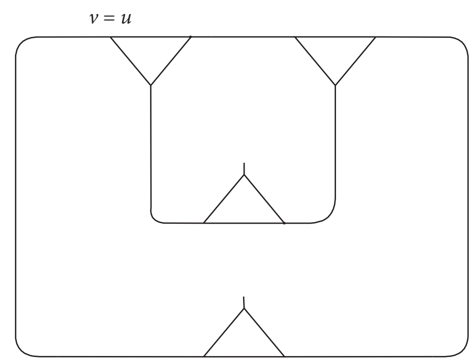

Figure 4: A homomorphic copy of (4, 3).

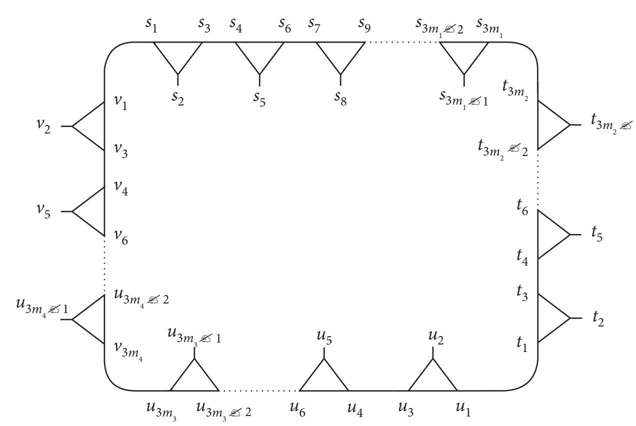

Figure 5: The closed path (m_1, m_2, m_3, m_4), where $m_1 > m_2 > m_3$ and $m_2 = m_4$.

Theorem 1 (see [24]). *Let a homomorphic copy γ of C be formed by contracting its vertices u_1 and u_2. Then γ is obtainable also by contracting the pair $(u_1)x$, $(u_2)x$ for some $x \in G$.*

Theorem 2 (see [24]). *The number of pairs to obtain γ is equal to the number of elements $x \in G$, such that $(u_1)x$ and $(u_2)x$ lie in C.*

Example 2. Consider a closed path (see Figure 3) containing a vertex v which is a fixed point of $(fg)^4(fg^{-1})^3$. Thus, it is a rank two closed path, denoted by $(4,3)$.

Figure 4 represents the homomorphic copy of $(4,3)$ generated through contraction of vertices u and v.

3. Formation of Homomorphic Copies through Contraction of Vertices

The coset graphs are made up of closed paths. The vertices of infinite graphs are contracted in a specific manner to evolve finite coset graphs. Therefore, a question arises: how many distinct homomorphic copies can be created by contracting all pairs in a closed path? In this work, we have developed a technique to find all homomorphic copies of the closed paths (m_1, m_2, m_3, m_4), where $m_1 > m_2 > m_3$ and $m_2 = m_4$, in coset graphs. Diagrammatically (m_1, m_2, m_3, m_4), where $m_1 > m_2 > m_3$ and $m_2 = m_4$, is shown in Figure 5.

In the remaining part of the paper, we denote the closed path (m_1, m_2, m_3, m_4), where $m_1 > m_2 > m_3$ and $m_2 = m_4$, by ψ. Throughout this paper, the mirror image of any homomorphic copy γ is denoted by γ^*. If $x = fg^{\kappa_1} fg^{\kappa_2}, \ldots, fg^{\kappa_n} \in \mathrm{PSL}(2, Z)$, where $\kappa_i = 1$ or -1, then let $x^* = fg^{-\kappa_1} fg^{-\kappa_2}, \ldots, fg^{-\kappa_n}$. If x fixes any vertex u, then the vertex fixed by x^* is u^*.

Remark 2. Since $(gh)^2 = 1$ implies $hgh^{-1} = g^{-1}$, h turns around the direction of the triangles like reflection. If γ is obtained by contracting vertices u_1 and u_2 of any closed path C, then the mirror image γ^* of γ can be created by contracting u_1^* and u_2^*. It should be noted that u_1 and u_2 do not need to lie in the same closed path C. From Figure 5, we have

$$\begin{aligned} s_i^* &= s_{3m_1-(i-1)}, \\ t_j^* &= v_{3m_2-(j-1)}, \\ u_k^* &= u_{3m_3-(k-1)}, \\ v_l^* &= t_{3m_4-(l-1)}, \end{aligned} \tag{2}$$

for $i = 1, 2, 3, \ldots, 3m_1$, $j = 1, 2, 3, \ldots, 3m_2$, $k = 1, 2, 3, \ldots, 3m_3$, and $l = 1, 2, 3, \ldots, 3m_4$. Thus, for each vertex u in ψ, there exists a vertex u^* in ψ.

Remark 3. Some homomorphic copies have symmetry about the vertical axis; that is, they have the same orientations as those of their mirror images. In other words, they are mirror images of themselves. The homomorphic copy γ of any circuit C having a vertex u fixed by x_1, x_2 has a symmetry about vertical axis if and only if γ contains a vertex u^* fixed by x_1^*, x_2^*.

3.1. Proposed Scheme. Since ψ has $3(m_1+m_2+m_3+m_4)$ number of vertices, the total number of pairs in ψ is $\binom{3(m_1+m_2+m_3+m_4)}{2}$. We contract a pair of vertices u_1, u_2 of ψ such that a homomorphic copy γ is obtained. By using Theorem 2, we find all pairs of vertices in ψ, which form γ; let those be n in numbers. Now, we have two possibilities:

(i) If by contracting u_1 and u_2 vertices u_1^* and u_2^* are not contracted, then γ does not possess a vertical symmetry. Therefore, γ^* consumes n more pairs of vertices of ψ.

(ii) If u_1, u_2 and u_1^*, u_2^* are contracted all together, then γ has a symmetry about vertical axis. Therefore, γ^* does not consume any pair. Thus, γ has n pairs of vertices.

Next, we contract ψ at one of the remaining pairs and the process continues until all $\binom{3(m_1+m_2+m_3+m_4)}{2}$ pairs are exhausted.

Let $i_1 = 0, 1, 2, \ldots, m_2 - 1$. First, we contract vertex s_{3m_1} with vertices s_{3i_1+1} and the following result is obtained.

Theorem 3. *If vertex s_{3m_1} is contracted with vertices s_{3i_1+1}, then m_2 distinct homomorphic copies α_{i_1} of ψ are obtained. Furthermore, there are $3(m_2^2+3m_2-2)$ pairs of vertices for these homomorphic copies.*

Proof. Let $D_1 = \{\alpha_{i_1}\colon i_1 = 0, 1, 2, \ldots, m_2 - 1\}$ be the set of homomorphic copies of ψ obtained by contracting s_{3m_1} with s_{3i_1+1}. In Figure 5, it can be seen that $(fg)^{m_2}(fg^{-1})^{m_3}(fg)^{m_4}(fg^{-1})^{i_1}f$ and $g^{-1}(fg^{-1})^{m_1-i_1-1}$ are the possible paths between s_{3m_1} and s_{3i_1+1}. Therefore, vertex $s_{3m_1} = s_{3i_1+1}$ in D_1 is fixed by $(fg)^{m_2}(fg^{-1})^{m_3}(fg)^{m_4}(fg^{-1})^{i_1}f$ and $g^{-1}(fg^{-1})^{m_1-i_1-1}$. It is also clear from Figure 5 that $E_1 = \{f, fg^{-1}, fg, fgf, fgfg^{-1}, (fg)^2, \ldots, (fg)^{i_1}, (fg)^{i_1}f, (fg)^{i_1}fg^{-1}, (fg)^{i_1+1}, g, g^{-1}, e\}$ is the set of elements x of the modular group such that, for all $x \in E_1$, both $(s_{3m_1})x$ and $(s_{3i_1+1})x$ lie in ψ. Since $|E_1| = 3(i_1+2)$, by Theorem 2, there are $3(i_1+2)$ pairs in ψ to form α_{i_1}.

Next, we show that all homomorphic copies α_{i_1} of ψ in D_1 are different and no copy of these is a mirror image of another.

Let $\alpha_j, \alpha_k \in D_1$; then α_j is evolved by contracting s_{3m_1} and s_{3j+1}, whereas α_k is obtained by contracting s_{3m_1} and s_{3k+1}. Now $\alpha_j = \alpha_k$ if and only if there exists an element x in E_1 such that $(s_{3m_1})x = s_{3m_1}$ and $(s_{3j+1})x = s_{3k+1}$. One can see that only $e \in E_1$ maps s_{3m_1} to itself, but $(s_{3j+1})e \neq s_{3k+1}$.

Now suppose that $\alpha_j = \alpha_k^*$; then there must exist some x in E_1 which sends s_{3m_1} to $s_{3m_1}^* = s_1$ and s_{3j+1} to $s_{3k+1}^* = s_{3(m_1-k)}$. But E_1 does not contain such element. This means that all diagrams in D_1 are distinct. Thus, $|D_1| = m_2$ and there are $3\sum_{i_1=0}^{m_2-1}(i_1+2)$ pairs of vertices to create α_{i_1}.

Now we check how many diagrams in D_1 have a symmetry about vertical axis. For this, let $\alpha_j = \alpha_j^*$; then E_1 contains an element x such that

$$\begin{aligned}(s_{3m_1})x &= s_{3m_1}^* = s_1,\\ (s_{3j+1})x &= s_{3j+1}^* = s_{3(m_1-j)},\\ \text{or } (s_{3m_1})x &= s_{3j+1}^* = s_{3(m_1-j)},\\ (s_{3j+1})x &= s_{3m_1}^* = s_1.\end{aligned} \tag{3}$$

This is possible only if $i_1 = 0$; in this case, we have $e \in E_1$ such that $(s_{3m_1})e = s_1^*$ and $(s_1)e = s_{3m_1}^* = s_1$. So, we conclude that only α_0 has a symmetry about vertical axis; that is, α_0 and its mirror image α_0^* have the same orientations, and all other $m_2 - 1$ homomorphic copies in ψ do not possess a vertical symmetry. Hence there are $6\sum_{i_2=1}^{m_2-1}(i_2+2) + 6 = 3(m_2^2 + 3m_2 - 2)$ pairs to form $D_1 = \{\alpha_{i_1}\colon i_1 = 0, 1, 2, \ldots, m_2 - 1\}$.

We obtain all the results by using the same technique, so, from now onwards, instead of providing proofs of the theorems, we will present tables, which give the necessary information of the family of homomorphic copies evolved.

Let $i_2 = 1, 2, \ldots, m_2 - 1$. We contract the pair t_{3m_2} and t_{3i_2+1} and obtain the following result. By using Theorem 1, one can see that these vertices are not contracted in Theorem 3. □

Theorem 4. *If vertex t_{3m_2} is contracted with vertices t_{3i_2+1}, then $m_2 - 1$ distinct homomorphic copies α_{i_2}' of ψ are obtained. Furthermore, there are $3(m_2^2+3m_2-4)$ pairs of vertices for these homomorphic copies.*

Table 1 shows the complete information of the family of homomorphic copies α_{i_2}'. The information provided in Table 1 can be verified by the same technique used in the proof of Theorem 3.

Let us contract u_{3m_3} with u_{3i_3+1}, where $i_3 = 0, 1, 2, \ldots, m_3 - 1$. By using Theorem 1, it can be easily verified that these pairs of vertices are not utilized in the previous theorems.

Theorem 5. *If vertex u_{3m_3} is contracted with vertices u_{3i_3+1}, then m_3 distinct homomorphic copies α_{i_3}'' of ψ are obtained. Furthermore, there are $3(m_3^2+3m_3-2)$ pairs of vertices for these homomorphic copies.*

All details of the family of homomorphic copies α_{i_3}'' have been provided in Table 2.

Next, we contract s_{3m_1} with t_{3i_1+1} and formulate the following Theorem.

Theorem 6. *If vertex s_{3m_1} is contracted with vertices t_{3i_1+1}, then m_2 distinct homomorphic copies β_{i_1} of ψ are obtained. Furthermore, there are $3(m_2^2+3m_2)$ pairs of vertices for these homomorphic copies.*

Table 1: Necessary information regarding homomorphic copies $\{\alpha'_{i_2}\}$.

The pairs that are contracted	(t_{3m_2}, t_{3i_2+1})
Family of homomorphic copies evolved	$D_2 = \{\alpha'_{i_2}\}$
$\lvert D_2\rvert$	$m_2 - 1$
Number of pairs for each α'_{i_2}	$3(i_2 + 2)$
Number of homomorphic copies that have vertical axis of symmetry	0
Total number of pairs for D_2	$3(m_2^2 + 3m_2 - 4)$

Table 2: Necessary information regarding homomorphic copies $\{\alpha''_{i_3}\}$.

The pairs that are contracted	(u_{3m_3}, u_{3i_3+1})
Family of homomorphic copies evolved	$D_3 = \{\alpha''_{i_3}\}$
$\lvert D_3\rvert$	m_3
Number of pairs for each α''_{i_3}	$3(i_3 + 2)$
Homomorphic copies possessing vertical axis of symmetry	Only α''_0
Total number of pairs D_3	$3(m_3^2 + 3m_3 - 2)$

For complete information of β_{i_1}, see Table 3.

Now we contract vertex s_{3m_1} with vertices u_{3i_3+1}, where $i_3 = 0, 1, 2, \ldots, m_3 - 1$, and acquire the following result.

Theorem 7. *If vertex s_{3m_1} is contracted with vertices u_{3i_3+1}, then m_3 distinct homomorphic copies β'_{i_3} of ψ are obtained. Furthermore, there are $3(m_3^2 + 3m_3)$ pairs of vertices for these homomorphic copies.*

Table 4 provides all information regarding homomorphic copies β'_{i_3}, evolved in Theorem 7.

The following theorem is evolved by contracting s_{3m_1} with v_{3i_4+1}, where $i_4 = 0, 1, 2, \ldots, m_4 - 1$.

Theorem 8. *If vertex s_{3m_1} is contracted with vertices v_{3i_4+1}, then m_4 distinct homomorphic copies β''_{i_4} of ψ are obtained. Furthermore, there are $(3/2)(m_4^2 + 3m_4)$ pairs of vertices for these homomorphic copies.*

The complete information of the family of homomorphic copies β''_{i_4} obtained in Theorem 8 is given in Table 5.

The next theorem is obtained by contracting t_{3m_2} with s_{3i_5+1}, where $i_5 = 1, 2, \ldots, m_1 - 1$.

Theorem 9. *If vertex t_{3m_2} is contracted with vertices s_{3i_5+1}, then $m_1 - 1$ distinct homomorphic copies β''_{i_5} of ψ are obtained. Furthermore, there are $(3/2)(m_1^2 + 3m_1 - 4)$ pairs of vertices for these homomorphic copies.*

All details of the generated homomorphic copies are given in Table 6.

Now we contract vertex t_{3m_2} with vertices u_{3i_6+1} for all $i_6 = 1, 2, \ldots, m_3 - 1$.

Theorem 10. *If vertex t_{3m_2} is contracted with vertices u_{3i_6+1}, then $m_3 - 1$ distinct homomorphic copies $\beta^{iv}_{i_6}$ of ψ are obtained. Furthermore, there are $3(m_3^2 + 3m_3 - 4)$ pairs of vertices for these homomorphic copies.*

Table 3: Necessary information regarding homomorphic copies $\{\beta_{i_1}\}$.

The pairs that are contracted	(s_{3m_1}, t_{3i_1+1})
Family of homomorphic copies evolved	$D_4 = \{\beta_{i_1}\}$
$\lvert D_4\rvert$	m_2
Number of pairs for each β_{i_1}	m_2
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_4	$3(m_2^2 + 3m_2)$

Table 4: Necessary information regarding homomorphic copies $\{\beta'_{i_3}\}$.

The pairs that are contracted	(s_{3m_1}, u_{3i_3+1})
Family of homomorphic copies evolved	$D_5 = \{\beta'_{i_3}\}$
$\lvert D_5\rvert$	m_3
Number of pairs for each β'_{i_3}	$3(i_3 + 2)$
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_5	$3(m_3^2 + 3m_3)$

Table 5: Necessary information regarding homomorphic copies $\{\beta''_{i_4}\}$.

The pairs that are contracted	(s_{3m_1}, v_{3i_4+1})
Family of homomorphic copies evolved	$D_6 = \{\beta''_{i_4}\}$
$\lvert D_6\rvert$	m_4
Number of pairs for each β''_{i_4}	$3(i_4 + 2)$
Homomorphic copies possessing vertical axis of symmetry	All β''_{i_4}
Total number of pairs for D_6	$(3/2)(m_4^2 + 3m_4)$

Table 6: Necessary information regarding homomorphic copies $\{\beta''_{i_5}\}$.

The pairs that are contracted	(t_{3m_2}, s_{3i_5+1})
Family of homomorphic copies evolved	$D_7 = \{\beta''_{i_5}\}$
$\lvert D_7\rvert$	$m_1 - 1$
Number of pairs for each β''_{i_5}	$3(i_5 + 2)$
Homomorphic copies possessing vertical axis of symmetry	All β''_{i_5}
Total number of pairs for D_7	$(3/2)(m_1^2 + 3m_1 - 4)$

The complete information of the family of homomorphic copies $\beta^{iv}_{i_6}$ is provided in Table 7.

In the following theorem, u_{3m_3} is contracted with t_{3i_1+1}.

Theorem 11. *If vertex u_{3m_3} is contracted with vertices t_{3i_1+1}, then m_2 distinct homomorphic copies $\beta^{v}_{i_1}$ of ψ are obtained. Furthermore, there are $(3/2)(m_2^2 + 3m_2)$ pairs of vertices for these homomorphic copies.*

All details of homomorphic copies $\beta^{v}_{i_1}$ can be found in Table 8.

Let us now contract vertex v_{3m_4} with vertices u_{3i_6+1}, where $i_6 = 1, 2, \ldots, m_3 - 1$.

Table 7: Necessary information regarding homomorphic copies $\{\beta^{iv}_{i_6}\}$.

The pairs that are contracted	(t_{3m_2}, u_{3i_6+1})
Family of homomorphic copies evolved	$D_8 = \{\beta^{iv}_{i_6}\}$
$\lvert D_8\rvert$	$m_3 - 1$
Number of pairs for each $\beta^{iv}_{i_6}$	$3(i_6 + 2)$
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_8	$3(m_3^2 + 3m_3 - 4)$

Table 8: Necessary information regarding homomorphic copies $\{\beta^{v}_{i_1}\}$.

The pairs that are contracted	(u_{3m_3}, t_{3i_1+1})
Family of homomorphic copies evolved	$D_9 = \{\beta^{v}_{i_1}\}$
$\lvert D_9\rvert$	m_2
Number of pairs for each $\beta^{v}_{i_1}$	$3(i_1 + 2)$
Homomorphic copies possessing vertical axis of symmetry	All $\beta^{v}_{i_1}$
Total number of pairs for D_9	$(3/2)(m_2^2 + 3m_2)$

Table 9: Necessary information regarding homomorphic copies $\{\beta^{vi}_{i_6}\}$.

The pairs that are contracted	(v_{3m_4}, u_{3i_6+1})
Family of homomorphic copies evolved	$D_{10} = \{\beta^{vi}_{i_6}\}$
$\lvert D_{10}\rvert$	$m_3 - 1$
Number of pairs for each $\beta^{vi}_{i_6}$	$3(i_6 + 2)$
Homomorphic copies possessing vertical axis of symmetry	All $\beta^{vi}_{i_6}$
Total number of pairs for D_{10}	$(3/2)(m_3^2 + 3m_3 - 4)$

Theorem 12. *If vertex v_{3m_4} is contracted with vertices u_{3i_6+1}, then $m_3 - 1$ distinct homomorphic copies $\beta^{vi}_{i_6}$ of ψ are obtained. Furthermore, there are $(3/2)(m_3^2 + 3m_3 - 4)$ pairs of vertices for these homomorphic copies.*

All information related to the homomorphic copies created in Theorem 12 is given in Table 9.

Let us contract vertex s_{3m_1} with vertices s_{3i_7+1}, where $i_7 = m_2 + 1, m_2 + 2, \ldots, m_1 - 1$. Consequently, we have the following theorem.

Theorem 13. *If vertex s_{3m_1} is contracted with vertices s_{3i_7+1}, then $m_1 - m_2 - 1$ distinct homomorphic copies γ_{i_7} of ψ are obtained. Furthermore, there are $6(m_2 + 2)(m_1 - m_2 - 1)$ pairs of vertices for these homomorphic copies.*

Table 10 completely describes the family of homomorphic copies evolved in Theorem 11.

Suppose that $\varepsilon_1 = \begin{cases} 0, & \text{if } m_1 + m_3 \text{ is even} \\ 1, & \text{if } m_1 + m_3 \text{ is odd} \end{cases}$ and $i_8 = m_3 + 1, m_3 + 2, \ldots, ((m_1 + m_3 - \varepsilon_1)/2)$. Let us contract v_{3m_4} with s_{3i_8+1} to obtain Theorem 14.

Theorem 14. *If vertex v_{3m_4} is contracted with vertices s_{3i_8+1}, then $(1/2)(m_1 - m_3 - \varepsilon_1)$ distinct homomorphic copies γ_{i_8}' of ψ are obtained. Furthermore, there are $3(m_3 + 2)(m_1 - m_3 - 1)$ pairs of vertices for these homomorphic copies.*

Table 10: Necessary information regarding homomorphic copies $\{\gamma_{i_7}\}$.

The pairs that are contracted	(s_{3m_1}, s_{3i_7+1})
Family of homomorphic copies evolved	$D_{11} = \{\gamma_{i_7}\}$
$\lvert D_{11}\rvert$	$m_1 - m_2 - 1$
Number of pairs for each γ_{i_7}	$3(m_2 + 2)$
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_{11}	$6(m_2 + 2)(m_1 - m_2 - 1)$

Table 11: Necessary information regarding homomorphic copies $\{\gamma_{i_8}'\}$.

The pairs that are contracted	(v_{3m_4}, s_{3i_8+1})
Family of homomorphic copies evolved	$D_{12} = \{\gamma_{i_8}'\}$
$\lvert D_{12}\rvert$	$(1/2)(m_1 - m_3 - \varepsilon_1)$
Number of pairs for each γ_{i_8}'	$3(m_3 + 2)$
Homomorphic copies possessing vertical axis of symmetry	No one if $m_1 + m_3$ is odd and $\gamma_{((m_1+m_3)/2)}'$ if $m_1 + m_3$ is even
Total number of pairs for D_{12}	$3(m_3 + 2)(m_1 - m_3 - 1)$

All information of the homomorphic copies evolved in Theorem 14 is provided in Table 11.

The following theorem emerges as a result of contracting v_{3m_4} with t_{3i_9+1}, where $i_9 = m_3 + 1, m_3 + 2, \ldots, m_2 - 1$.

Theorem 15. *If vertex v_{3m_4} is contracted with vertices t_{3i_9+1}, then $m_2 - m_3 - 1$ distinct homomorphic copies γ_{i_9}'' of ψ are obtained. Furthermore, there are $6(m_3 + 2)(m_2 - m_3 - 1)$ pairs of vertices for these homomorphic copies.*

Table 12 shows the complete information of all homomorphic copies created in Theorem 15.

Suppose that $i_{10} = 1, 2, \ldots, m_1 - 1$ and $i_{11} = 1, 2, \ldots, m_2 - 1$. Let us contract $s_{3i_{10}+1}$ with $t_{3i_{11}}$and obtain the following results.

Theorem 16. *If vertices $s_{3i_{10}+1}$ are contracted with vertices $t_{3i_{11}}$, then $(m_1 - 1)(m_2 - 1)$ distinct homomorphic copies $\delta_{(i_{10}, i_{11})}$ of ψ are obtained. Furthermore, there are $12(m_1 - 1)(m_2 - 1)$ pairs of vertices for these homomorphic copies.*

All necessary details of the family of homomorphic copies evolved in this process are provided in Table 13.

Let $i_{12} = 1, 2, 3, \ldots, ((m_3 - \varepsilon_2 - 2)/2)$, where $\varepsilon_2 = \begin{cases} 0, & \text{if } m_3 \text{ is even} \\ 1, & \text{if } m_3 \text{ is odd} \end{cases}$. The following results have been constructed by contracting $s_{3i_{10}+1}$ with $u_{3i_{12}}$.

Theorem 17. *If vertices $s_{3i_{10}+1}$ are contracted with vertices $u_{3i_{12}}$, then $(1/2)(m_1 - 1)(m_3 - \varepsilon_2 - 2)$ distinct homomorphic*

Table 12: Necessary information regarding homomorphic copies $\{\gamma_{i_9}''\}$.

The pairs that are contracted	(v_{3m_4}, t_{3i_9+1})
Family of homomorphic copies evolved	$D_{13} = \{\gamma_{i_9}''\}$
$\vert D_{13}\vert$	$m_2 - m_3 - 1$
Number of pairs for each γ_{i_9}''	$3(m_3 + 2)$
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_{13}	$6(m_3 + 2)(m_2 - m_3 - 1)$

Table 13: Necessary information regarding homomorphic copies $\{\delta_{(i_{10},i_{11})}\}$.

The pairs that are contracted	$(s_{3i_{10}+1}, t_{3i_{11}})$
Family of homomorphic copies evolved	$D_{14} = \{\delta_{(i_{10},i_{11})}\}$
$\vert D_{14}\vert$	$(m_1 - 1)(m_2 - 1)$
Number of pairs for each $\delta_{(i_{10},i_{11})}$	6
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_{14}	$12(m_1 - 1)(m_2 - 1)$

Table 14: Necessary information regarding homomorphic copies $\{\delta'_{(i_{10},i_{12})}\}$.

The pairs that are contracted	$(s_{3i_{10}+1}, u_{3i_{12}})$
Family of homomorphic copies evolved	$D_{15} = \{\delta_{(i_{10},i_{12})}'\}$
$\vert D_{15}\vert$	$(1/2)(m_1 - 1)(m_3 - \varepsilon_2 - 2)$
Number of pairs for each $\delta_{(i_{10},i_{12})}'$	6
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_{15}	$6(m_1 - 1)(m_3 - \varepsilon_2 - 2)$

copies $\delta'_{(i_{10},i_{12})}$ of ψ are obtained. Furthermore, there are $6(m_1 - 1)(m_3 - \varepsilon_2 - 2)$ pairs of vertices for these homomorphic copies.

See Table 14 for complete information of the homomorphic copies.

Recall that $i_{13} = 1, 2, \ldots, m_1 - 1$ and $\varepsilon_2 = \begin{cases} 0 & \text{if } m_3 \text{ is even} \\ 1 & \text{if } m_3 \text{ is odd} \end{cases}$. Let $\varepsilon_3 = \begin{cases} 0 & \text{if } m_1 \text{ is even} \\ 1 & \text{if } m_1 \text{ is odd} \end{cases}$, and we obtain the following theorem by contracting $s_{3i_{13}+1}$ with $u_{3((m_3-\varepsilon_2)/2)}$.

Theorem 18. *If vertices $s_{3i_{13}+1}$ are contracted with vertex $u_{3((m_3-\varepsilon_2)/2)}$, then $\begin{cases} ((m_1 - \varepsilon_2)/2) & \text{if } m_3 \text{ is even} \\ m_1 - 1 & \text{if } m_3 \text{ is odd} \end{cases}$ distinct homomorphic copies $\delta_{(i_{13},((m_3-i_3)/2))}''$ of ψ are obtained. Furthermore, there are $\begin{cases} 6(m_1 - 1) & \text{if } m_3 \text{ is even} \\ 12(m_1 - 1) & \text{if } m_3 \text{ is odd} \end{cases}$ pairs of vertices for these homomorphic copies.*

All information related to the homomorphic copies created in Theorem 18 is provided in Table 15.

Now we contract $t_{3i_{14}+1}$ with $u_{3i_{15}}$, where $i_{14} = 1, 2, \ldots, m_2 - 1$ and $i_{15} = 1, 2, \ldots, m_3 - 1$, so that Theorem 19 is evolved.

Theorem 19. *If vertices $t_{3i_{14}+1}$ are contracted with vertices $u_{3i_{15}}$, then $(m_2 - 1)(m_3 - 1)$ distinct homomorphic copies $\delta_{(i_{14},i_{15})}$ of ψ are obtained. Furthermore, there are $12(m_2 - 1)(m_3 - 1)$ pairs of vertices for these homomorphic copies.*

For complete details about the homomorphic copies evolved in Theorem 19, see Table 16.

Let $i_{16} = 1, 2, 3, \ldots, m_4 - i_{14}$. The following theorem is obtained by contracting $t_{3i_{14}+1}$ with $v_{3i_{16}}$.

Theorem 20. *If vertices $t_{3i_{14}+1}$ are contracted with vertices $v_{3i_{16}}$, then $(1/2)m_2(m_2 - 1)$ distinct homomorphic copies $\delta^{iv}_{(i_{14},i_{16})}$ of ψ are obtained. Furthermore, there are $6(m_2 - 1)^2$ pairs of vertices for these homomorphic copies.*

All important information regarding $\delta^{iv}_{(i_{14},i_{16})}$ is provided in Table 17.

Recall that $\varepsilon_3 = \begin{cases} 0 & \text{if } m_1 \text{ is even} \\ 1 & \text{if } m_1 \text{ is odd} \end{cases}$, and let $i_{17} = 1, 2, 3, \ldots, ((m_1 - 2 - \varepsilon_3)/2)$ and $i_{18} = i_{17} + 1, i_{17} + 2, \ldots, m_1 - i_{17}$. In Theorem 21, we contract vertices $s_{3i_{17}+1}$ with vertices $s_{3i_{18}}$ to create homomorphic copies $\mu_{(i_{17},i_{18})}$.

Theorem 21. *If vertices $s_{3i_{17}+1}$ are contracted with vertices $s_{3i_{18}}$, then $(1/4)(m_1^2 - 2m_1 + \varepsilon_3)$ distinct homomorphic copies $\mu_{(i_{17},i_{18})}$ of ψ are obtained. Furthermore, there are $3(m_1^2 - 3m_1 + 2)$ pairs of vertices for these homomorphic copies.*

All information about the family of homomorphic copies evolved in Theorem 21 is given in Table 18.

Now, let us contract vertices $t_{3i_{19}+1}$ with $t_{3i_{20}}$, where $i_{19} = 1, 2, 3, \ldots, m_2 - 2$ and $i_{20} = i_{19} + 1, i_{19} + 2, i_{19} + 3, \ldots, m_2 - 1$.

Theorem 22. *If vertices $t_{3i_{19}+1}$ are contracted with vertices $t_{3i_{20}}$, then $(1/2)(m_2 - 2)(m_2 - 1)$ distinct homomorphic copies $\mu_{(i_{19},i_{20})}'$ of ψ are obtained. Furthermore, there are $6(m_2 - 2)(m_2 - 1)$ pairs of vertices for these homomorphic copies.*

Table 19 provides all information regarding $\mu'_{(i_{19},i_{20})}$.

Recall that $\varepsilon_2 = \begin{cases} 0 & \text{if } m_3 \text{ is even} \\ 1 & \text{if } m_3 \text{ is odd} \end{cases}$ and let $i_{21} = 1, 2, 3, \ldots, ((m_3 - 2 + \varepsilon_2)/2)$ and $i_{22} = i_{21} + 1, i_{21} + 2, i_{21} + 3, \ldots, m_3 - i_{21}$. The next theorem is obtained by contracting $u_{3i_{21}+1}$ with $u_{3i_{22}}$

Theorem 23. *If vertices $u_{3i_{21}+1}$ are contracted with vertices $u_{3i_{22}}$, then $(1/4)(m_3^2 - 2m_3 + \varepsilon_2)$ distinct homomorphic copies $\mu_{(i_{21},i_{22})}''$ of ψ are obtained. Furthermore, there are $3(m_3^2 - 3m_3 + 2)$ pairs of vertices for these homomorphic copies.*

Table 15: Necessary information regarding homomorphic copies $\{\delta''_{(i_{13},((m_3-i_3)/2))}\}$.

The pairs that are contracted	$(s_{3i_{13}+1}, u_{3((m_3-\varepsilon_2)/2)})$
Family of homomorphic copies evolved	$D_{16} = \{\delta''_{(i_{13},((m_3-i_3)/2))}\}$
$\|D_{16}\|$	$\begin{cases} ((m_1-\varepsilon_2)/2) & \text{if } m_3 \text{ is even} \\ m_1 - 1 & \text{if } m_3 \text{ is odd} \end{cases}$
Number of pairs for each $\delta''_{(i_{13},((m_3-i_3)/2))}$	6
Homomorphic copies possessing vertical axis of symmetry	$\delta''_{((m_1/2),(m_3/2))}$ if m_1 and m_3 are even and no one otherwise
Total number of pairs for D_{16}	$\begin{cases} 6(m_1-1) & \text{if } m_3 \text{ is even} \\ 12(m_1-1) & \text{if } m_3 \text{ is odd} \end{cases}$

Table 16: Necessary information regarding homomorphic copies $\{\delta''_{(i_{14},i_{15})}\}$.

The pairs that are contracted	$(t_{3i_{14}+1}, u_{3i_{15}})$
Family of homomorphic copies evolved	$D_{17} = \{\delta''_{(i_{14},i_{15})}\}$
$\|D_{17}\|$	$(m_2-1)(m_3-1)$
Number of pairs for each $\delta''_{(i_{14},i_{15})}$	6
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_{17}	$12(m_2-1)(m_3-1)$

Table 17: Necessary information regarding homomorphic copies $\{\delta^{iv}_{(i_{14},i_{16})}\}$.

The pairs that are contracted	$(t_{3i_{14}+1}, v_{3i_{16}})$
Family of homomorphic copies evolved	$D_{18} = \{\delta^{iv}_{(i_{14},i_{16})}\}$
$\|D_{18}\|$	$(1/2)m_2(m_2-1)$
Number of pairs for each $\delta^{iv}_{(i_{14},i_{16})}$	6
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_{18}	$6(m_2-1)^2$

Table 18: Necessary information regarding homomorphic copies $\{\mu_{(i_{17},i_{18})}\}$.

The pairs that are contracted	$(s_{3i_{17}+1}, s_{3i_{18}})$
Family of homomorphic copies evolved	$D_{19} = \{\mu_{(i_{17},i_{18})}\}$
$\|D_{19}\|$	$(1/4)(m_1^2 - 2m_1 + \varepsilon_3)$
Number of pairs for each $\mu_{(i_{17},i_{18})}$	6
Homomorphic copies possessing vertical axis of symmetry	$(1/2)(m_1 - 2 + \varepsilon_3)$
Total number of pairs for D_{19}	$3(m_1^2 - 3m_1 + 2)$

Table 19: Necessary information regarding homomorphic copies $\{\mu'_{(i_{19},i_{20})}\}$.

The pairs that are contracted	$(t_{3i_{19}+1}, t_{3i_{20}})$
Family of homomorphic copies evolved	$D_{20} = \{\mu'_{(i_{19},i_{20})}\}$
$\|D_{20}\|$	$(1/2)(m_2-2)(m_2-1)$
Number of pairs for each $\mu'_{(i_{19},i_{20})}$	6
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_{20}	$6(m_2-2)(m_2-1)$

All details of the generated homomorphic copies are given in Table 20.

Now suppose that $i_{23} = 1, 2, 3, \ldots, ((m_1 - \varepsilon_3)/2)$. The following results have been constructed by contracting $s_{3i_{23}+1}$ with $s_{3i_{23}}$.

Theorem 24. *If vertices $s_{3i_{23}+1}$ are contracted with vertices $s_{3i_{23}}$, then $(1/2)(m_1 - \varepsilon_3)$ distinct homomorphic copies $\nu_{(i_{23},i_{23})}$ of ψ are obtained. Furthermore, there are $3(m_1 - 1)$ pairs of vertices for these homomorphic copies.*

See Table 21 for complete details of the evolved homomorphic copies.

Now, we contract $t_{3i_{24}+1}$ with $t_{3i_{24}}$, where $i_{24} = 1, 2, 3, \ldots, m_2 - 1$.

Theorem 25. *If vertices $t_{3i_{24}+1}$ are contracted with vertices $t_{3i_{24}}$, then $m_2 - 1$ distinct homomorphic copies $\nu'_{(i_{24},i_{24})}$ of ψ are obtained. Furthermore, there are $6(m_2 - 1)$ pairs of vertices for these homomorphic copies.*

The complete information of the family of homomorphic copies $\nu'_{(i_{24},i_{24})}$ obtained in Theorem 25 is given in Table 22.

Suppose that $i_{25} = 1, 2, 3, \ldots, ((m_3 - \varepsilon_2)/2)$. Let us contract $u_{3i_{25}+1}$ and $u_{3i_{25}}$ and obtain the following results.

Theorem 26. *If vertices $u_{3i_{25}+1}$ are contracted with vertices $u_{3i_{25}}$, then $(1/2)(m_3 - \varepsilon_2)$ distinct homomorphic copies $\nu''_{(i_{25},i_{25})}$ of ψ are evolved. Furthermore, there are $3(m_3 - 1)$ pairs of vertices for these homomorphic copies.*

All information of the family of homomorphic copies $\nu''_{(i_{25},i_{25})}$ evolved in Theorem 26 is provided in Table 23.

Lastly, we contract 4 pairs of vertices to evolve 4 homomorphic copies.

Theorem 27

(i) Let η_1 be the homomorphic copy of ψ obtained by contracting s_1 and t_1. Then there are $6(m_2 + m_3 + 2)$ pairs of vertices for η_1.

(ii) Let η_2 be the homomorphic copy obtained by contracting s_1 and u_1. Then there are $3(m_2 + 2m_3 + 2)$ pairs of vertices for η_2.

(iii)) Let η_3 be the homomorphic copy obtained by contracting s_1 and v_1. Then there are $6(m_2 + 1)$ pairs of vertices to generate η_3.

(iv) Let η_4 be the homomorphic copy obtained by contracting t_1 and u_1. Then there are $6(m_3 + 1)$ pairs of vertices for η_4.

$$\text{Let} \quad \Gamma = \begin{cases} 0 & \text{if } m_1 \text{ and } m_3 \text{ are odd} \\ 4 & \text{if } m_1 \text{ and } m_3 \text{ are even} \\ -1 & \text{if } m_1 \text{ is even and } m_3 \text{ is odd or vice versa} \end{cases}.$$

We are now in a position to prove the main theorem of this paper.

Table 20: Necessary information regarding homomorphic copies $\{\mu''_{(i_{21},i_{22})}\}$.

The pairs that are contracted	$(u_{3i_{21}+1}, u_{3i_{22}})$: $i_{21} = 1, 2, 3, \ldots, ((m_3 - 2 + \varepsilon_2)/2)$ and $i_{22} = i_{21} + 1, i_{21} + 2, i_{21} + 3, \ldots, m_3 - i_{21}$
Family of homomorphic copies evolved	$D_{21} = \{\mu''_{(i_{21},i_{22})}\}$
$\lvert D_{21}\rvert$	$(1/4)(m_3^2 - 2m_3 + \varepsilon_2)$
Number of pairs for each $\mu''_{(i_{21},i_{22})}$	6
Homomorphic copies possessing vertical axis of symmetry	$(1/2)(m_3 - 2 + \varepsilon_2)$
Total number of pairs for D_{21}	$3(m_3^2 - 3m_3 + 2)$

Table 21: Necessary information regarding homomorphic copies $\{\nu_{(i_{23},i_{23})}\}$.

The pairs that are contracted	$(s_{3i_{23}+1}, s_{3i_{23}})$
Family of homomorphic copies evolved	$D_{22} = \{\nu_{(i_{23},i_{23})}\}$
$\lvert D_{22}\rvert$	$(1/2)(m_1 - \varepsilon_3)$
Number of pairs for each $\nu_{(i_{23},i_{23})}$	3
Homomorphic copies possessing vertical axis of symmetry	$\nu_{((m_1/2),(m_1/2))}$
Total number of pairs for D_{22}	$3(m_1 - 1)$

Table 22: Necessary information regarding homomorphic copies $\{\nu'_{(i_{24},i_{24})}\}$.

The pairs that are contracted	$(t_{3i_{24}+1}, t_{3i_{24}})$
Family of homomorphic copies evolved	$D_{23} = \{\nu_{(i_{24},i_{24}')}\}$
$\lvert D_{23}\rvert$	$m_2 - 1$
Number of pairs for each $\nu'_{(i_{24},i_{24})}$	3
Homomorphic copies possessing vertical axis of symmetry	No one
Total number of pairs for D_{23}	$6(m_2 - 1)$

Table 23: Necessary information regarding homomorphic copies $\{\nu''_{(i_{25},i_{25})}\}$.

The pairs that are contracted	$(u_{3i_{25}+1}, u_{3i_{25}})$
Family of homomorphic copies evolved	$D_{24} = \{\nu''_{(i_{25},i_{25})}\}$
$\lvert D_{24}\rvert$	$(1/2)(m_3 - \varepsilon_2)$
Number of pairs for each $\nu''_{(i_{25},i_{25})}$	3
Homomorphic copies possessing vertical axis of symmetry	$\nu''_{((m_3/2),m_3/2)}$
Total number of pairs for D_{24}	$3(m_3 - 1)$

Theorem 28. *There are* $(1/4)\{m_1^2 + 4m_2^2 + m_3^2 + 4(m_1m_2 + m_2m_3) + 2m_1m_2 + 8m_2 + 4m_3 + \Gamma\}$ *numbers of distinct homomorphic copies of* ψ *by contracting all the pairs in* ψ.

Proof. Let us contract the following pairs of vertices:

(i) s_{3m_1} and s_{3i_1+1}
(ii) t_{3m_2} and t_{3i_2+1}
(iii) u_{3m_3} and u_{3i_3+1}
(iv) s_{3m_1} and t_{3i_1+1}
(v) s_{3m_1} and u_{3i_3+1}
(vi) s_{3m_1} and v_{3i_4+1}
(vii) t_{3m_2} and s_{3i_5+1}
(viii) t_{3m_2} and u_{3i_6+1}
(ix) u_{3m_3} and t_{3i_1+1}
(x) v_{3m_4} and u_{3i_6+1}
(xi) s_{3m_1} and s_{3i_7+1}
(xii) v_{3m_4} and s_{3i_8+1}
(xiii) v_{3m_4} and t_{3i_9+1}
(xiv) $s_{3i_{10}+1}$ and $t_{3i_{11}}$
(xv) $s_{3i_{10}+1}$ and $u_{3i_{12}}$
(xvi) $s_{3i_{13}+1}$ and $u_{3((m_3-\varepsilon_2)/2)}$
(xvii) $t_{3i_{14}+1}$ and $u_{3i_{15}}$
(xviii) $t_{3i_{14}+1}$ and $v_{3i_{16}}$
(xix) $s_{3i_{17}+1}$ and $s_{3i_{18}}$
(xx) $t_{3i_{19}+1}$ and $t_{3i_{20}}$
(xxi) $u_{3i_{21}+1}$ and $u_{3i_{22}}$
(xxii) $s_{3i_{23}+1}$ and $s_{3i_{23}}$
(xxiii) $t_{3i_{24}+1}$ and $t_{3i_{24}}$
(xxiv) $u_{3i_{25}+1}$ and $u_{3i_{25}}$
(xxv) s_1 and t_1
(xxvi) s_1 and u_1
(xxvii) s_1 and v_1
(xxviii) t_1 and u_1

Then, by Theorems 3 to 27, we obtain the set

$$F = \left\{ \begin{array}{c} \alpha_{i_1}, \alpha'_{i_2}, \alpha''_{i_3}, \beta_{i_1}, \beta'_{i_3}, \beta''_{i_4}, \beta''_{i_5}, \beta^{iv}_{i_6}, \beta^{v}_{i_1}, \beta^{vi}_{i_6}, \\ \gamma_{i_7}, \gamma'_{i_8}, \gamma''_{i_9}, \delta_{(i_{10}, i_{11})}, \delta'_{(i_{10}, i_{12})}, \delta''_{(i_{13}, ((m_3 - i_3)/2))}, \\ \delta''_{(i_{14}, i_{15})}, \delta^{iv}_{(i_{14}, i_{16})}, \mu_{(i_{17}, i_{18})}, \mu'_{(i_{19}, i_{20})}, \mu''_{(i_{21}, i_{22})}, \nu_{(i_{23}, i_{23})}, \nu'_{(i_{24}, i_{24})}, \nu''_{(i_{25}, i_{25})}, \\ \eta_1, \eta_2, \eta_3, \eta_4 \end{array} \right\}, \quad (4)$$

of homomorphic copies of ψ and there are

$$\begin{aligned} S = {} & 3\left(m_2^2 + 3m_2 - 2\right) + 3\left(m_2^2 + 3m_2 - 4\right) + 3\left(m_3^2 + 3m_3 - 2\right) + 3\left(m_2^2 + 3m_2\right) \\ & + 3\left(m_3^2 + 3m_3\right) + \frac{3}{2}\left(m_4^2 + 3m_4\right) + \frac{3}{2}\left(m_1^2 + 3m_1 - 4\right) + 3\left(m_3^2 + 3m_3 - 4\right) \\ & + \frac{3}{2}\left(m_2^2 + 3m_2\right) + \frac{3}{2}\left(m_3^2 + 3m_3 - 4\right) + 6\left(m_2 + 2\right)\left(m_1 - m_2 - 1\right) \\ & + 3\left(m_3 + 2\right)\left(m_1 - m_3 - 1\right) + 6\left(m_3 + 2\right)\left(m_2 - m_3 - 1\right) \\ & + 12\left(m_1 - 1\right)\left(m_2 - 1\right) + 6\left(m_1 - 1\right)\left(m_3 - \varepsilon_2 - 2\right) \\ & + \begin{cases} 6\left(m_1 - 1\right) & \text{if } m_3 \text{ is even} \\ 12\left(m_1 - 1\right) & \text{if } m_3 \text{ is odd} \end{cases} + 12\left(m_2 - 1\right)\left(m_3 - 1\right) + 6\left(m_2 - 1\right)^2 \\ & + 3\left(m_1^2 + 3m_1 - 2\right) + 6\left(m_2 - 2\right)\left(m_2 - 1\right) + 3\left(m_3^2 - 3m_3 - 2\right) \\ & + 3\left(m_1 - 1\right) + 6\left(m_3 - 1\right) + 3\left(m_3 - 1\right) + 6\left(m_2 + m_3 + 2\right) \\ & + 3\left(m_2 + 2m_3 + 2\right) + 6\left(m_2 + 1\right) + 6\left(m_3 - 1\right), \end{aligned} \quad (5)$$

pairs to form F. $S = \begin{pmatrix} 3\left(m_1 + m_2 + m_3 + m_4\right) \\ 2 \end{pmatrix}$ is the total number of pairs in ψ and

$$|F| = \frac{1}{4}\left\{m_1^2 + 4m_2^2 + m_3^2 + 4\left(m_1 m_2 + m_2 m_3\right) + 2m_1 m_2 + 8m_2 + 4m_3 + \Gamma\right\}. \quad (6)$$

Thus, there are

$$\frac{1}{4}\left\{m_1^2 + 4m_2^2 + m_3^2 + 4\left(m_1 m_2 + m_2 m_3\right) + 2m_1 m_2 + 8m_2 + 4m_3 + \Gamma\right\}, \quad (7)$$

numbers of distinct homomorphic copies obtained by contracting all pairs in ψ. □

4. Conclusion

In this paper, we have highlighted the significance of homomorphic copies of coset graphs. We show how the bigger coset graphs get contracted to transform into their homomorphic copies. As a sample, we consider the closed path ψ of rank 4; that is, $\psi = (m_1, m_2, m_3, m_4)$, where $m_1 > m_2 > m_3$ and $m_2 = m_4$, in coset graphs. We prove that the total numbers of homomorphic copies of ψ are

$$\frac{1}{4}\left\{m_1^2 + 4m_2^2 + m_3^2 + 4\left(m_1 m_2 + m_2 m_3\right) + 2m_1 m_2 + 8m_2 + 4m_3 + \Gamma\right\}. \quad (8)$$

It is hoped that the ideas presented in this study will also help to generate the homomorphic copies of other types of graphs for various purposes.

Data Availability

No data were used to support this study.

Conflicts of Interest

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