Hindawi
Journal of Mathematics
Volume 2021, Article ID 3980250, 7 pages
https://doi.org/10.1155/2021/3980250

*Research Article*

# Oscillatory Properties for Second-Order Impulsive Neutral Dynamic Equations with Positive and Negative Coefficients on Time Scales

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Received 2 June 2020; Revised 6 August 2020; Accepted 20 January 2021; Published 4 February 2021

Academic Editor: Ali Jaballah



We study oscillatory properties for second-order impulsive neutral dynamic equations with positive and negative coefficients on time scales. By using variable substitution, we obtain sufficient conditions for several dynamic equations to be oscillatory.

## 1. Introduction

In this paper, we are concerned with the oscillation of the following second-order impulsive neutral dynamic equations with positive and negative coefficients:

$$\begin{cases} \left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)f(x(\tau(t))) - q(t)f(x(\zeta(t))) = 0, & t \in \mathbb{J}_{\mathbb{T}}\backslash\{t_k\}, \\ z(t_k^+) = M_k(z(t_k)), & k \in N, \\ z^{\Delta}(t_k^+) = N_k\left(z^{\Delta}(t_k)\right), & k \in N, \end{cases} \tag{1}$$

where $z(t) = x(t) + b(t)x(\theta(t))$ and $\mathbb{J}_{\mathbb{T}}\colon = [t_0, \infty) \cap \mathbb{T}$, $\mathbb{T}$ is unbounded above time scale with $t_k \in \mathbb{T}$, $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots \lim_{k\longrightarrow\infty} t_k = \infty$, and $x(t_k^+) = \lim_{h\longrightarrow 0^+} x(t_k + h)$ and $x^{\Delta}(t_k^+) = \lim_{h\longrightarrow 0^+} x^{\Delta}(t_k + h)$, which represent right limits of $x(t)$ and $x^{\Delta}(t)$ at $t = t_k$ in the sense of time scales. Furthermore, if $t_k$ is right-scattered, then $x(t_k^+) = x(t_k)$ and $x^{\Delta}(t_k^+) = x^{\Delta}(t_k)$. $x(t_k^-)$ and $x^{\Delta}(t_k^-)$ can be defined similarly. Throughout this paper, we assume that $f$ is continuous on $\mathbb{T}$, all the impulsive points $t_k$ are right-dense, and $a(t)$, $p(t)$, and $q(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^+)$, where $C_{\mathrm{rd}}$ denotes the set of rd-continuous functions. There exist positive constants $\alpha_k$, $\beta_k$, $c_k$, and $d_k$ such that $\alpha_k \le (M_k(x)/x) \le \beta_k$ and $c_k \le (N_k(x)/x) \le d_k$ for $x \ne 0$ and $k \in \mathbb{N}$.

During the past decades, the oscillation of impulsive differential equations and impulsive difference equations has been investigated by many authors [1–3]. In recent years, many researchers focus their attention on the oscillation of dynamic equations on time scales [4–9]. The theory of impulsive dynamic equations has received considerable attention [10]. Moreover, dynamic equations with positive and negative coefficients have been of great interest. Many results on the oscillatory properties for dynamic equations with positive and negative coefficients have been obtained

[11, 12]. However, fewer papers are on the oscillation of impulsive neutral dynamic equations with positive and negative coefficients.

For example, Huang and Feng [13] have considered the following equation:

$$\begin{cases} x^{\Delta\Delta}(t) + f(t, x^{\sigma}(t)) = 0, \quad \mathbb{J}_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}, t \neq t_k, k = 1, 2, \ldots, \\ x(t_k^+) = g_k(x(t_k)), \quad k = 1, 2, \ldots, \\ x^{\Delta}(t_k^+) = h_k(x^{\Delta}(t_k)), \quad k = 1, 2, \ldots, \\ x(t_0^+) = x(t_0), \quad k = 1, 2, \ldots, \\ x^{\Delta}(t_0^+) = x^{\Delta}(t_0), \quad k = 1, 2, \ldots. \end{cases} \tag{2}$$

By using Riccati transformation techniques, they obtained sufficient conditions for oscillation of all solutions.

The rest of this paper is organized as follows: In Section 2, we present some basic definitions and preliminary results. In Section 3, we state and prove several oscillatory results. Finally, two examples are given to illustrate our obtained results.

## 2. Preliminaries

In this section, we introduce definitions and preliminary results which are used in this paper. Throughout the paper, we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T}: a \leq t \leq b\}. \tag{3}$$

Open intervals and half-open intervals etc. are defined accordingly.

*Definition 1.* For $t \in \mathbb{T}$, we define the forward jump operator $\sigma$: $\mathbb{T} \longrightarrow \mathbb{T}$ by $\sigma(t)$: $=$ in$\{s \in \mathbb{T}: s > t\}$. If $\sigma(t) > t$, then $t$ is called right-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then $t$ is called right-dense.

*Definition 2.* A function $f$: $\mathbb{T} \longrightarrow \mathbb{R}$ is rd-continuous if it is continuous at all right-dense points and its left-sided limit exists (and is finite) at a left-dense point. We denote the set of rd-continuous functions by $C_{\text{rd}}(\mathbb{T}, \mathbb{R})$.

*Definition 3.* Assume $f$: $\mathbb{T} \longrightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then, we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|, \tag{4}$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of $f$ at $t$.

*Definition 4.* A function $x$ is a solution of (1), if it satisfies $(a(t)z^{\Delta}(t))^{\Delta} + p(t)f(x(\tau(t))) - q(t)f(x(\zeta(t))) = 0$ a.e. on $\mathbb{J}_{\mathbb{T}} \backslash \{t_k\}$, $k = 1, 2, \ldots$, and for each $k = 1, 2, \ldots$, $x$ satisfies the impulsive conditions $z(t_k^+) = M_k(z(t_k))$ and $z^{\Delta}(t_k^+) = N_k(z^{\Delta}(t_k))$, where $z(t) = x(t) + b(t)x(\theta(t))$.

*Definition 5.* A solution $x$ of (1) is oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all solutions are oscillatory.

**Lemma 1** (see [14]). *Let the function $m \in PC^1(\mathbb{T}, \mathbb{R})$ satisfy the inequalities*

$$\begin{aligned} m^{\Delta}(t) &\leq p(t)m(t) + q(t), \quad t \neq t_k, \\ m(t_k^+) &\leq d_k m(t_k) + b_k, \quad k = 1, 2, \ldots, \end{aligned} \tag{5}$$

*where $p$ and $q \in PC^1(\mathbb{T}, \mathbb{R})$ and $d_k > 0$ and $b_k$ are constants; then*

$$\begin{aligned} m(t) \leq{}& m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left( \int_{t_0}^{t} p(s)\Delta s \right) \\ &+ \sum_{t_0 < t_k < t} \left( \prod_{t_k < t_j < t} d_j \exp\left( \int_{t_k}^{t} p(s)\Delta s \right) \right) b_k \\ &+ \int_{t_0}^{t} \prod_{s < t_k < t} d_k \exp\left( \int_{s}^{t} p(\sigma)\Delta\sigma \right) q(s)\Delta s, \quad t \geq t_0, \end{aligned} \tag{6}$$

*where $PC = \{x: \mathbb{J}_{\mathbb{T}} \longrightarrow \mathbb{R}$ which is rd – continuous except at $t_k, k = 1, 2, \ldots,$ for which $x(t_k^-), x(t_k^+), x^{\Delta}(t_k^-),$ and $x^{\Delta}(t_k^+)$ exist with $x(t_k^-) = x(t_k)$ and $x^{\Delta}(t_k^-) = x^{\Delta}(t_k)\}$.*

## 3. Main Results

In this section, we first consider the dynamic equation

$$\begin{cases} \left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)f(x(\tau(t))) - q(t)f(x(\zeta(t))) = 0, \\ z(t_k^+) = M_k(z(t_k)), \\ z^{\Delta}(t_k^+) = \eta_k z^{\Delta}(t_k), \end{cases} \tag{7}$$

that is, the case $c_k = d_k = \eta_k$ of equation (1).

We assume the following conditions hold:

(H1) $0 \leq b(t) < 1$

(H2) $\sum_{k=1}^{\infty} (\int_{t_k}^{t_{k+1}} (1/a(t))\Delta t \prod_{i=1}^{k} \eta_i) = \infty$

(H3) $f$ is nondecreasing, $(f(u)/u) > 0$ for $u \neq 0$, and $f(uv) \geq f(u)f(v)$ for $uv > 0$

(H4) $\theta(t)$, $\tau(t)$, and $\zeta(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $\theta(t) \leq t$, $\lim_{t\longrightarrow\infty}\theta(t) = \infty$, $\lim_{t\longrightarrow\infty}\tau(t) = \infty$, and $\lim_{t\longrightarrow\infty}\zeta(t) = \infty$

(H5) $\alpha_k \geq 1$, $\sum_{k=1}^{\infty}(\beta_k - 1) < \infty$, and $0 < \eta_k \leq 1$

(H6) $\theta(t) \neq t_k$, $\tau(t) \neq t_k$, and $\zeta(t) \neq t_k$ for $t \neq t_k$

**Theorem 1.** *Assume (H1)–(H6) are satisfied, if the inequality*

$$\begin{cases} y^{\Delta}(t) + p(t)f(y(\tau(t)))f(S(\tau(t))) \leq 0, \\ y(t_k^+) \leq \eta_k y(t_k), \end{cases} \tag{8}$$

*has no eventually positive solution, where*

$$S(t) = (1 - b(t)) \int_{T_1}^{t} \frac{1}{a(t)} \Delta t. \tag{9}$$

*Then, every bounded solution of (7) is oscillatory.*

*Proof.* Assume that $x(t)$ is an eventually bounded positive solution of (7); then, there exists $T \in \mathbb{T}$ large enough and $T \geq t_0$ such that $x(t) > 0$, $x(\theta(t)) > 0$, $x(\tau(t)) > 0$, and $x(\zeta(t)) > 0$ for all $t \in [T, \infty)$. Let $F(t) = z(t) + \int_t^{\infty}(1/a(s))\int_T^s q(u)f(x(\zeta(u)))\Delta u \Delta s, t \in (t_k, t_{k+1}]$; we find $(a(t)F^{\Delta}(t))^{\Delta} + p(t)f(x(\tau(t))) = 0$, which implies that

$$\left(a(t)F^{\Delta}(t)\right)^{\Delta} = -p(t)f(x(\tau(t))) < 0. \tag{10}$$

Then, $a(t)F^{\Delta}(t)$ is decreasing for all $t \in (t_k, t_{k+1}]$. We can claim that $F^{\Delta}(t)$ is ultimately greater than zero. If not, there must exist a $m \in (t_j, t_{j+1}]$, such that $F^{\Delta}(m) \leq 0$.

For $\forall t \in [m, t_{j+1}]$, we have

$$\begin{aligned} &a\left(t_{j+1}\right)F^{\Delta}\left(t_{j+1}\right) \leq a(t)F^{\Delta}(t) \leq a(m)F^{\Delta}(m) \leq 0, \\ &\int_m^{t_{j+1}} F^{\Delta}(t)\Delta t \leq a(m)F^{\Delta}(m)\int_m^{t_{j+1}} \frac{1}{a(t)}\Delta t. \end{aligned} \tag{11}$$

Making similar analysis on the interval $(t_k, t_{k+1}]$, $k = j+1, j+2, \ldots$, we obtain

$$\int_{t_k}^{t_{k+1}} F^{\Delta}(t)\Delta t \leq \eta_k \cdots \eta_{j+1} a(m)F^{\Delta}(m)\int_{t_k}^{t_{k+1}} \frac{1}{a(t)}\Delta t, \quad t \in (t_k, t_{k+1}]. \tag{12}$$

Hence,

$$\int_m^{t_{j+1}} F^{\Delta}(t)\Delta t + \sum_{k=j+1}^{\infty}\int_{t_k}^{t_{k+1}} F^{\Delta}(t)\Delta t \leq a(m)F^{\Delta}(m)\left(\int_m^{t_{j+1}} \frac{1}{a(t)}\Delta t + \sum_{k=j+1}^{\infty}\left(\int_{t_k}^{t_{k+1}} \frac{1}{a(t)}\Delta t \prod_{i=j+1}^{k}\eta_i\right)\right). \tag{13}$$

Furthermore, we obtain

$$F(\infty) \leq a(m)F^{\Delta}(m)\left(\int_m^{t_{j+1}} \frac{1}{a(t)}\Delta t + \sum_{k=j+1}^{\infty}\left(\int_{t_k}^{t_{k+1}} \frac{1}{a(t)}\Delta t \prod_{i=j+1}^{k}\eta_i\right)\right) + F(m) + \sum_{k=j+1}^{\infty}(\beta_k - 1)z(t_k). \tag{14}$$

By (*H2*) and (*H5*), we have $F(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$, which contradicts $F(t) > 0$. Therefore, $F^{\Delta}(t)$ is ultimately greater than zero. There exists $T_1 \geq T$ such that $z^{\Delta}(t) > F^{\Delta}(t) > 0$, where $t \geq T_1$. By $z(t_k^+) \geq \alpha_k z(t_k) \geq z(t_k)$, we have

$$\begin{aligned} z(t) &\geq \int_{T_1}^{t} z^{\Delta}(s)\Delta s > \int_{T_1}^{t} F^{\Delta}(s)\Delta s \\ &= \int_{T_1}^{t} \frac{a(s)F^{\Delta}(s)}{a(s)}\Delta s \geq a(t)F^{\Delta}(t)\int_{T_1}^{t} \frac{1}{a(s)}\Delta s. \end{aligned} \tag{15}$$

Let $\varphi(t) = a(t)F^{\Delta}(t)$; then, $x(t) \geq z(t)(1 - b(t)) \geq \varphi(t)S(t)$. We see $\varphi(t)$ is a positive solution of (8), which contradicts the oscillation of (8). We complete the proof. □

We assume that the following conditions are satisfied:

(H7) $b(t)$ is bounded

(H8) $\zeta(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$ is bijective, $\tau(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq \zeta(t)$, $\zeta^{-1}(\tau(t)) \in C_{\mathrm{rd}}^1(\mathbb{J}_{\mathbb{T}}, \mathbb{T})$, $\lim_{t\longrightarrow\infty}\theta(t) = \infty$, and $\lim_{t\longrightarrow\infty}\tau(t) = \infty$

(H9) $\int_{t_0}^{\infty}(1/a(\eta))\int_{\zeta^{-1}(\tau(\eta))}^{\eta} q(s)\Delta s \Delta\eta < \infty$

(H10) There exists $L > 0$ and $(f(u)/u) \geq L$ for $u \neq 0$

(H11) $p(t) - q(\zeta^{-1}(\tau(t)))(\zeta^{-1}(\tau(t)))^{\Delta} \geq C > 0$

**Theorem 2.** *Assume (H2) and (H5)–(H11) are satisfied; then, every bounded solution of (7) is oscillatory.*

*Proof.* Assume that $x(t)$ is an eventually bounded positive solution of equation (7); then, there exists $T \in \mathbb{T}$ large enough and $T \geq t_0$ such that $x(t) > 0$, $x(\theta(t)) > 0$, $x(\tau(t)) > 0$, and $x(\zeta(t)) > 0$ for all $t \in [T, \infty)$. Let $u(t) = z(t) - \int_T^t (1/a(\eta))\int_{\zeta^{-1}(\tau(\eta))}^{\eta} q(s)f(x(\zeta(s)))\Delta s \Delta\eta, t \in (t_k, t_{k+1}]$; we obtain that

$$\left(a(t)u^{\Delta}(t)\right)^{\Delta} = -\left(p(t) - q\left(\zeta^{-1}(\tau(t))\left(\zeta^{-1}(\tau(t))\right)^{\Delta}\right)\right.$$
$$\cdot f(x(\tau(t)))$$
$$\leq -CLx(\tau(t)) < 0. \tag{16}$$

Then, $a(t)u^{\Delta}(t)$ is decreasing in $(t_k, t_{k+1}]$. We can obtain that $u^{\Delta}(t)$ is ultimately greater than zero. There exists $T_2 \geq T$ such that $z^{\Delta}(t) > u^{\Delta}(t) > 0$ for all $t \in [T_2, \infty)$.

For $\forall t \in (t_k, t_{k+1}]$ and $t \in (T_2, \infty)$, we have

$$\int_{T_2}^{t}\left(a(s)u^{\Delta}(s)\right)^{\Delta}\Delta s \leq -CL\int_{T_2}^{t} x(\tau(s))\Delta s. \tag{17}$$

Furthermore, we obtain

$$\begin{aligned} CL\int_{T_2}^{t} x(\tau(s))\Delta s &\leq a(T_2)u^{\Delta}(T_2) - a(t)u^{\Delta}(t) \\ &\quad + \sum_{T_2 < t_k < t}(\eta_k - 1)a(t_k)u^{\Delta}(t_k) \\ &\leq a(T_2)u^{\Delta}(T_2) < \infty. \end{aligned} \tag{18}$$

Therefore, $x(t)$ and $z(t)$ are integrable. However, we know $z(t)$ is increasing in $(T_2, \infty)$ and nonintegrable from $z^{\Delta}(t) > 0$ and $z(t_k^{+}) \geq z(t_k)$, which contradicts the integrability of $z(t)$. We complete the proof. □

Next, we study the equation

$$\begin{cases} \left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)f(x(\sigma(t))) = 0, \\ z(t_k^{+}) = M_k(z(t_k)), \\ z^{\Delta}(t_k^{+}) = \eta_k z^{\Delta}(t_k), \end{cases} \tag{19}$$

i.e., the case $q(t) \equiv 0$, $\tau(t) = \sigma(t)$, and $c_k = d_k = \eta_k$ of (1).

We assume the following conditions hold:

(H12) $0 \leq b(t) < (H(t)B_k/H(\theta(t)))$, where $H(t) = \int_t^{\infty}(1/a(s))\Delta s$, $B_k = \begin{cases} 1, & t_k \notin (\theta(t), t), \\ \prod_{\theta(t)<t_k<t}\alpha_k, & \text{other} \end{cases}$

(H13) $0 < \alpha_k \leq \beta_k \leq 1$, $\eta_k \leq 1$, and $\theta(t) \neq t_k$ for $t \neq t_k$

(H14) $\int_{t_0}^{t_1}(1/a(s))\Delta s + \eta_1\int_{t_1}^{t_2}(1/a(s))\Delta s + \eta_2\eta_1\int_{t_2}^{t_3}(1/a(s))\Delta s + \cdots = \infty$

(H15) $\theta(t) \leq t$ and $\theta(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$ is increasing, $\lim_{t\longrightarrow\infty}\theta(t) = \infty$, and $\sigma(\theta(t)) = \theta(\sigma(t))$

(H16) There exists $L > 0$ and $(f(u)/u) \geq L$ for $u \neq 0$; $f$ is nondecreasing and $f(uv) \geq f(u)f(v)$ for $uv > 0$

**Theorem 3.** *Assume (H12)–(H16) are satisfied. If*

$$\int_{t_0}^{\infty}\prod_{t_0<t_k<s}\frac{\alpha_k}{\eta_k}p(s)f(1 - b(\sigma(s)))\frac{H(\theta(\sigma(s)))}{H(\sigma(s))B_k}\Delta s = \infty, \tag{20}$$

*then (19) is oscillatory.*

*Proof.* Assume that (19) has a nonoscillatory solution $x(t)$, which is eventually positive; then, there exists $T \in \mathbb{T}$ large enough and $T \geq t_0$ such that $x(t) > 0$ and $x(\theta(t)) > 0$ for all $t \in [T, \infty)$. We obtain that $(a(t)z^{\Delta}(t))^{\Delta} = -p(t) f(x(\sigma(t))) < 0$, where $t \in (t_k, t_{k+1}]$. Therefore, $a(t)z^{\Delta}(t)$ is decreasing in $(t_k, t_{k+1}]$. We denote $I(u) = \max\{i\colon t_0 < t_i < u\}$.

For $t_k < t \leq s \leq t_{k+1}$, we have

$$a(t_{k+1})z^{\Delta}(t_{k+1}) \leq a(s)z^{\Delta}(s) \leq a(t)z^{\Delta}(t). \tag{21}$$

Hence,

$$\int_t^{t_{k+1}} z^{\Delta}(s)\Delta s \leq a(t)z^{\Delta}(t)\int_t^{t_{k+1}}\frac{1}{a(s)}\Delta s, \quad s \in [t, t_{k+1}]. \tag{22}$$

We can make similar analysis on the intervals $(t_j, t_{j+1}]$ and $(t_{I(u)}, u]$, such that $j = k+1, \ldots, I(u) - 1$. Thus,

$$\begin{aligned} z(u) - z(t) &\leq \int_t^{t_{k+1}} z^{\Delta}(s)\Delta s + \sum_{j=k+1}^{I(u)-1}\int_{t_j}^{t_{j+1}} z^{\Delta}(s)\Delta s + \int_{t_{I(u)}}^{u} z^{\Delta}(s)\Delta s \\ &\leq a(t)z^{\Delta}(t)\left(\int_t^{t_{k+1}}\frac{1}{a(s)}\Delta s + \eta_{k+1}\int_{t_{k+1}}^{t_{k+2}}\frac{1}{a(s)}\Delta s + \cdots \eta_{I(u)}\cdots\eta_{k+1}\int_{t_{I(u)}}^{u}\frac{1}{a(s)}\Delta s\right). \end{aligned} \tag{23}$$

By $(H14)$, we obtain $z^{\Delta}(t)$ is ultimately greater than zero. Then, there exists $T_3 \geq T$ such that $z^{\Delta}(t) > 0$ for all $t \in [T_3, \infty)$, and

$$z(t) \geq -a(t)z^{\Delta}(t)\int_t^{u}\frac{1}{a(s)}\Delta s. \tag{24}$$

Letting $u \longrightarrow \infty$, we have

$$z(t) \geq -a(t)z^{\Delta}(t)\int_t^{\infty}\frac{1}{a(s)}\Delta s = -a(t)z^{\Delta}(t)H(t). \tag{25}$$

Therefore, we can claim that $(z(t)/H(t))$ is increasing in $(t_k, t_{k+1}]$.

If $t_k \notin (\theta(t), t)$, we obtain

$$\frac{z(t)}{H(t)} \geq \frac{z(\theta(t))}{H(\theta(t))}. \tag{26}$$

If $t_k \in (\theta(t), t)$, we obtain

$$\frac{z(t)}{H(t)} \geq \prod_{\theta(t)<t_k<t}\alpha_k\frac{z(\theta(t))}{H(\theta(t))}. \tag{27}$$

So

$$\frac{z(t)}{H(t)} \geq B_k \frac{z(\theta(t))}{H(\theta(t))}. \tag{28}$$

Therefore,

$$\begin{aligned} x(t) &= z(t) - b(t)x(\theta(t)) \geq z(t) - b(t)z(\theta(t)) \\ &\geq z(t)\left(1 - b(t)\frac{H(\theta(t))}{H(t)}\frac{1}{B_k}\right). \end{aligned} \tag{29}$$

Let $\omega(t) = ((a(t)z^{\Delta}(t))/z(t))$, $t \in (t_k, t_{k+1}]$; we have

$$\begin{aligned} \omega^{\Delta}(t) &\leq \frac{-p(t)f(x(\sigma(t)))}{z(\sigma(t))} \\ &\leq -p(t)Lf\left(1 - b(\sigma(t))\frac{H(\theta(\sigma(t)))}{H(\sigma(t))}\frac{1}{B_k}\right), \\ \omega(t_k^+) &\leq \frac{\eta_k}{\alpha_k}\omega(t_k). \end{aligned} \tag{30}$$

By using Lemma 1, we obtain for $t > t_0$,

$$\begin{aligned} \omega(t) &\leq \omega(t_0)\prod_{t_0 < t_k < t}\frac{\eta_k}{\alpha_k} - L\int_{t_0}^{t}\prod_{s < t_k < t}\frac{\eta_k}{\alpha_k}p(s)f\left(1 - b(\sigma(s))\frac{H(\theta(\sigma(s)))}{H(\sigma(s))}\frac{1}{B_k}\right)\Delta s \\ &= \prod_{t_0 < t_k < t}\frac{\eta_k}{\alpha_k}\left(\omega(t_0) - L\int_{t_0}^{t}\prod_{t_0 < t_k < s}\frac{\alpha_k}{\eta_k}p(s)f\left(1 - b(\sigma(s))\frac{H(\theta(\sigma(s)))}{H(\sigma(s))}\frac{1}{B_k}\right)\Delta s\right). \end{aligned} \tag{31}$$

We get a contradiction as $t \longrightarrow \infty$. We complete the proof. □

For $q(t) \equiv 0$ and $\alpha_k = \beta_k = 1$ in (1), it can be written as

$$\begin{cases} \left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)f(x(\tau(t))) = 0, \\ z(t_k^+) = z(t_k), \\ z^{\Delta}(t_k^+) = N_k\left(z^{\Delta}(t_k)\right). \end{cases} \tag{32}$$

We assume that the following conditions hold for (32):

(H17) $\theta(t), \tau(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\theta(t) \leq t$, $\tau(t) \leq t$, $\lim_{t\longrightarrow\infty}\theta(t) = \infty$, and $\lim_{t\longrightarrow\infty}\tau(t) = \infty$

(H18) $\theta(t) \neq t_k$ and $\tau(t) \neq t_k$ for $t \neq t_k$

(H19) $\int_{t_0}^{\infty}\prod_{t_0 \leq t_k < s}c_k(1/(a(s)))\Delta s = \infty$

**Lemma 2.** *Assume (H19) is satisfied and $x(t)$ is an eventually positive solution of equation (32); then, $z^{\Delta}(t) \geq 0$ and $z^{\Delta}(t_k) \geq 0$, where $t \in (t_k, t_{k+1}]$.*

*Proof.* Because $x(t)$ is an eventually positive solution of equation (32), we obtain

$$\left(a(t)z^{\Delta}(t)\right)^{\Delta} = -p(t)f(x(\tau(t))) < 0, \quad t \in (t_k, t_{k+1}]. \tag{33}$$

Then, $a(t)z^{\Delta}(t)$ is decreasing in $(t_k, t_{k+1}]$. If there exists $t_j$ such that $z^{\Delta}(t_j) = -l < 0$ for $l > 0$ and we obtain for $\forall t \in (t_{j+n}, t_{j+n+1}]$, $a(t)z^{\Delta}(t) \leq -a(t_j)c_jc_{j+1}, \ldots, c_{j+n}l$, then

$$\begin{aligned} z^{\Delta}(t) &\leq \frac{-a(t_j)}{a(t)}l\prod_{t_j \leq t_k < t}c_k, \\ z(t_k^+) &= z(t_k). \end{aligned} \tag{34}$$

By Lemma 1, we obtain for $t > t_j$,

$$z(t) \leq z(t_j) - l\int_{t_j}^{t}\frac{a(t_j)}{a(s)}\prod_{t_j \leq t_k < s}c_k\Delta s. \tag{35}$$

We get a contradiction as $t \longrightarrow \infty$. Then, $z^{\Delta}(t_k) \geq 0$. For $\forall t \in (t_{k-1}, t_k]$,

$$z^{\Delta}(t) \geq \frac{a(t_k)}{a(t)}z^{\Delta}(t_k) \geq 0. \tag{36}$$

We complete the proof. □

**Theorem 4.** *Assume (H1) and (H16)–(H19) are satisfied. If $d_k \leq 1$ and*

$$\int_{t_0}^{\infty}\prod_{t_0 < t_k < t}\frac{1}{d_k}p(t)f(1 - b(\tau(t)))K(t)\Delta t = \infty, \tag{37}$$

*where*

$$\begin{aligned} K(t) &= \frac{Q(\tau(t))}{Q(\sigma(t))}, \\ Q(t) &= \int_{T}^{t}\frac{1}{a(s)}\Delta s, \end{aligned} \tag{38}$$

*then (32) is oscillatory.*

*Proof.* Suppose to the contrary that $x(t)$ is an eventually positive solution of (32); then, there exists $T \in \mathbb{T}$ large enough and $T \geq t_0$ such that $x(t) > 0$, $x(\theta(t)) > 0$, and $x(\tau(t)) > 0$ for all $t \in [T, \infty)$. Then, $a(t)z^{\Delta}(t)$ is decreasing in $(t_k, t_{k+1}]$. According to $a(t_k^+)z^{\Delta}(t_k^+) \leq a(t_k)z^{\Delta}(t_k)$, we obtain $a(t)z^{\Delta}(t)$ is decreasing in $(T, \infty)$.

For $\forall t \in (t_k, t_{k+1}]$ and $t \geq \tau(t) \geq T$, we have

$$\frac{z(\sigma(t))}{z(\tau(t))} \leq \frac{1}{K(t)}. \tag{39}$$

It can be proved similar to Lemma 1 of [11] and so its proof is omitted here.

From Lemma 2, we have $z^{\Delta}(t) \geq 0$ and $z^{\Delta}(t_k) \geq 0$, where $t \geq T$ and $t \in \mathbb{T}$. Let $\omega(t) = (a(t)z^{\Delta}(t)/z(t))$, $t \in (t_k, t_{k+1}]$. We have

$$\begin{aligned}\omega^{\Delta}(t) &\leq \frac{-p(t)f(x(\tau(t)))}{z(\sigma(t))}\\ &\leq -p(t)Lf(1-b(\tau(t))K(t)),\\ \omega(t_k^+) &\leq d_k\omega(t_k).\end{aligned} \tag{40}$$

Applying Lemma 1, we obtain for $t > t_0$,

$$\begin{aligned}\omega(t) &\leq \omega(t_0)\prod_{t_0<t_k<t} d_k - L\int_{t_0}^{t}\prod_{s<t_k<t} d_k p(s)\\ &\quad\cdot f(1-b(\tau(s)))K(s)\Delta s\\ &= \prod_{t_0<t_k<t} d_k\Bigg(\omega(t_0) - L\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{1}{d_k}p(s)\\ &\quad\cdot f(1-b(\tau(s)))K(s)\Delta s\Bigg).\end{aligned} \tag{41}$$

We get a contradiction as $t \longrightarrow \infty$. This completes the proof of Theorem 4. □

## 4. Examples

*Example 1.* Consider the equation

$$\begin{cases}\left(\dfrac{1}{e^t}z^{\Delta}(t)\right)^{\Delta} + p(t)x(t-2) - q(t)x(t+2) = 0, & t \in \mathbb{J}_{\mathbb{T}}\backslash\{2k\},\\ z(2k^+) = M_k(z(2k)), & k \in N,\\ z^{\Delta}(2k^+) = \dfrac{k}{k+1}z^{\Delta}(2k), & k \in N,\end{cases} \tag{42}$$

where $z(t) = x(t) + (1/t)x(t-2)$ and $\mathbb{J}_{\mathbb{T}}: = [2,\infty)$. Here, $\alpha_k = 1 + (1/(k^2+1))$, $\beta_k = 1 + (1/k^2)$, $\eta_k = (k/(k+1))$, and $p(t)S(t-2) = 1$. Obviously, $(H1)$ and $(H3)$–$(H6)$ are satisfied:

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\int_{t_k}^{t_{k+1}}\frac{1}{a(t)}\Delta t\prod_{i=1}^{k}\eta_i\right) &= \sum_{k=1}^{\infty}\left(\int_{2k}^{2k+2}e^t\Delta t\prod_{i=1}^{k}\frac{i}{i+1}\right)\\ &= \frac{1}{2}\int_2^4 e^t\Delta t + \frac{1}{3}\int_4^6 e^t\Delta t + \cdots\\ &> \int_2^{\infty}\Delta t > \infty.\end{aligned} \tag{43}$$

So $(H2)$ is satisfied. Taking account of Theorem 1 of [1], we know that

$$\begin{cases} y'(t) + y(t-2) \leq 0,\\ y(t_k^+) \leq \dfrac{k}{k+1}y(t_k),\end{cases} \tag{44}$$

has no eventually positive solution. By Theorem 1, it is clear that every bounded solution of (42) is oscillatory.

*Example 2.* Consider the following equation:

$$\begin{cases}\left(\dfrac{t^2}{e^t}z^{\Delta}(t)\right)^{\Delta} + e^t x(t+2) - e^{-t}x(t+4) = 0, & t \in \mathbb{J}_{\mathbb{T}}\backslash\{2k\},\\ z(2k^+) = M_k(z(2k)), & k \in N,\\ z^{\Delta}(2k^+) = \dfrac{k}{k+1}z^{\Delta}(2k), & k \in N,\end{cases} \tag{45}$$

where $z(t) = x(t) + (1/t)x(t-2)$ and $\mathbb{J}_{\mathbb{T}}: = \cup_{k=2}^{\infty}[k, k+(1/2)]$. Here, $\alpha_k = 1 + (1/(k^2+1))$, $\beta_k = 1 + (1/k^2)$, and $\eta_k = (k/(k+1))$. Obviously, $(H2)$, $(H5)$–$(H8)$, and $(H10)$ are satisfied:

$$\begin{aligned}&\int_{t_0}^{\infty}\frac{1}{a(\eta)}\int_{\zeta^{-1}(\tau(\eta))}^{\eta}q(s)\Delta s\Delta\eta = \int_2^{\infty}\frac{e^{\eta}}{\eta^2}\int_{\eta-2}^{\eta}e^{-s}\Delta s\Delta\eta < \infty,\\ &p(t) - q\left(\zeta^{-1}(\tau(t))\right)\left(\zeta^{-1}(\tau(t))\right)^{\Delta} = e^t - e^{2-t} \geq e^2 - 1 > 0.\end{aligned} \tag{46}$$

So $(H9)$ and $(H11)$ are satisfied. By Theorem 2, we see that every bounded solution of (45) is oscillatory.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Acknowledgments

This research was partially supported by NNSF of China (11971145).

## References

[1] D. D. Bainov, M. B. Dimitrova, and V. A. Petrov, "Oscillatory properties of solutions of impulsive differential equations with several retarded arguments," *Georgian Mathematical Journal*, vol. 5, no. 3, pp. 201–212, 1998.

[2] F. Jian, J. Shen, and Z. Ji, "Asymptotic behavior of impulsive neutral delay differential equations with positive and negative coefficients of euler form," *Advances in Difference Equations*, vol. 2018, p. 83, 2018.

[3] M. Peng, "Oscillation criteria for second-order impulsive delay difference equations," *Applied Mathematics and Computation*, vol. 146, no. 1, pp. 227–235, 2003.

[4] L. Erbe, A. Peterson, and S. H. Saker, "Oscillation criteria for second-order nonlinear dynamic equations on time scales," *Journal of the London Mathematical Society*, vol. 67, no. 3, pp. 701–714, 2003.

[5] S. Saker, "Oscillation of second-order forced nonlinear dynamic equations on time scales," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 23, no. 23, pp. 1–17, 2005.

[6] S. H. Saker, "Oscillation criteria of second-order half-linear dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 177, no. 2, pp. 375–387, 2005.

[7] Y. Sui and Z. Han, "Oscillation of second order neutral dynamic equations with deviating arguments on time scales," *Advances in Difference Equations*, vol. 2018, p. 337, 2018.

[8] B. G. Zhang and Z. Shanliang, "Oscillation of second-order nonlinear delay dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 49, no. 4, pp. 599–609, 2005.

[9] Y. Zhou, A. Alsaedi, and B. Ahmad, "Oscillation and nonoscillation theorems of neutral dynamic equations on time scales," *Advances in Difference Equations*, vol. 2019, p. 404, 2019.

[10] R. P. Agarwal, M. Benchohra, D. O'Regan, and A. Ouahab, "Second order impulsive dynamic equations on time scales," *Journal of Dynamics and Differential Equations*, vol. 11, pp. 223–234, 2004.

[11] X.-H. Deng, Q.-R. Wang, and R. P. Agarwal, "Oscillation and nonoscillation for second order neutral dynamic equations with positive and negative coefficients on time scales," *Advances in Difference Equations*, vol. 2014, no. 1, p. 115, 2014.

[12] X. Huang and X. Deng, "Properties of third-order nonlinear delay dynamic equations with positive and negative coefficients," *Advances in Difference Equations*, vol. 2019, p. 292, 2019.

[13] M. G. Huang and W. Z. Feng, "Oscillation of second-order nonliear impulsive dynamic equations on time scales," *Electronic Journal of Differential Equations*, vol. 72, pp. 1–13, 2007.

[14] M. G. Huang and W. Z. Feng, "Oscillation criteria for impulsive dynamic equations on time scales," *Electronic Journal of Differential Equations*, vol. 169, pp. 1–9, 2007.