Oscillatory Properties for Second-Order Impulsive Neutral Dynamic Equations with Positive and Negative Coefficients on Time Scales

Shuang Zhang and Qiaoluan Li

School of Mathematical Sciences, Hebei Normal University, Shijiazhuang 050024, China

Correspondence should be addressed to Qiaoluan Li; qll71125@163.com

Received 2 June 2020; Revised 6 August 2020; Accepted 20 January 2021; Published 4 February 2021

Academic Editor: Ali Jaballah

Copyright © 2021 Shuang Zhang and Qiaoluan Li. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study oscillatory properties for second-order impulsive neutral dynamic equations with positive and negative coefficients on time scales. By using variable substitution, we obtain sufficient conditions for several dynamic equations to be oscillatory.

1. Introduction

In this paper, we are concerned with the oscillation of the following second-order impulsive neutral dynamic equations with positive and negative coefficients:

\[
\begin{align*}
\left( a(t)z^2(t) \right)^\Delta + p(t)f(x(\tau(t))) - q(t)f(x(\zeta(t))) &= 0, \quad t \in J_T \setminus \{t_k\}, \\
z(t_k^+) &= M_k(z(t_k)), \quad k \in \mathbb{N}, \\
z^\Delta(t_k^+) &= N_k(z^\Delta(t_k)), \quad k \in \mathbb{N},
\end{align*}
\]

where \( z(t) = x(t) + b(t)x(\theta(t)) \) and \( J_T = [t_0, \infty) \cap T, T \) is unbounded above time scale with \( t_k \in T, 0 \leq t_0 < t_1 < t_2 \cdot \cdots < t_k \cdot \cdots \), \( \lim_{k \to \infty} t_k = \infty \), and \( x(t_k^+) = \lim_{h \to 0^+} x(t_k + h) \) and \( x^\Delta(t_k^+) = \lim_{h \to 0^+} x^\Delta(t_k + h) \), which represent right limits of \( x(t) \) and \( x^\Delta(t) \) at \( t = t_k \) in the sense of time scales. Furthermore, if \( t_k \) is right-scattered, then \( x(t_k^+) = x(t_k) \) and \( x^\Delta(t_k^+) = x^\Delta(t_k) \). \( x(t_k) \) and \( x^\Delta(t_k) \) can be defined similarly. Throughout this paper, we assume that \( f \) is continuous on \( T \), all the impulsive points \( t_k \) are right-dense, and \( a(t), p(t), \) and \( q(t) \in \mathcal{C}_{rd}(T, \mathbb{R}^+) \), where \( \mathcal{C}_{rd} \) denotes the set of rd-continuous functions. There exist positive constants \( \alpha_k, \beta_k, c_k, \) and \( d_k \) such that

\[ \alpha_k \leq (M_k(x)/x) \leq \beta_k \quad \text{and} \quad c_k \leq (N_k(x)/x) \leq d_k \quad \text{for} \quad x \neq 0 \quad \text{and} \quad k \in \mathbb{N}. \]

During the past decades, the oscillation of impulsive differential equations and impulsive difference equations has been investigated by many authors [1–3]. In recent years, many researchers focus their attention on the oscillation of dynamic equations on time scales [4–9]. The theory of impulsive dynamic equations has received considerable attention [10]. Moreover, dynamic equations with positive and negative coefficients have been of great interest. Many results on the oscillatory properties for dynamic equations with positive and negative coefficients have been obtained...
However, fewer papers are on the oscillation of impulsive neutral dynamic equations with positive and negative coefficients.

Definition 1. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma: \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \min\{s \in \mathbb{T}: s > t\}$. If $\sigma(t) > t$, then $t$ is called right-scattered. Also, if $t < \inf \mathbb{T}$ and $\sigma(t) = t$, then $t$ is called right-dense.

Definition 2. A function $f: \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous if it is continuous at all right-dense points and its left-sided limit exists (and is finite) at a left-dense point. We denote the set of rd-continuous functions by $C_{rd}^{0}(\mathbb{T}, \mathbb{R})$.

Definition 3. Assume $f: \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then, we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s)| - f^{\Delta}(t)[\sigma(t) - s] \leq \varepsilon[\sigma(t) - s],$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of $f$ at $t$.

Definition 4. A function $x$ is a solution of (1), if it satisfies

$$a(t)x^{\Delta}(t) + p(t)f(x(\tau(t))) - q(t)f(x(\varsigma(t))) = 0$$

a.e. on $J_{\mathbb{T}} \setminus \{t_k\}$, $k = 1, 2, \ldots$, and for each $k = 1, 2, \ldots$, $x$ satisfies the impulsive conditions $x(t_k^{+}) = M_k(x(t_k))$ and $x^{\Delta}(t_k^{+}) = N_k(x^{\Delta}(t_k))$, where $x(t) = x(t) + b(t)x(\theta(t))$.

Definition 5. A solution $x$ of (1) is oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all solutions are oscillatory.

For example, Huang and Feng [13] have considered the following equation:

$$x^{\Delta}(t) + p(t)x(t) + q(t)x(t) = 0,$$

with the impulsive conditions

$$x(t_k^{+}) = M_k(x(t_k)), \quad x^{\Delta}(t_k^{+}) = N_k(x^{\Delta}(t_k)),$$

where $p$ and $q \in PC^{1}(\mathbb{T}, \mathbb{R})$ and $d_k > 0$ and $b_k$ are constants; then

$$m(t) \leq m(t_0) \prod_{t_0 < t < t_k} d_k \exp\left(\int_{t_0}^{t} p(s)\Delta s\right)$$

$$+ \sum_{t_0 < t < t_k} \left(\prod_{t_0 < s < t} d_j \exp\left(\int_{t_0}^{s} p(s)\Delta s\right)\right) b_k$$

$$+ \int_{t_0}^{t} \prod_{t_0 < s < t} d_k \exp\left(\int_{s}^{t} p(s)\Delta s\right) q(s)\Delta s, \quad t \geq t_0,$$

where $PC = \{x: J_{\mathbb{T}} \rightarrow \mathbb{R} \text{ which is rd-continuous except at } t_k, k = 1, 2, \ldots, \text{for which } x(t_k^{+}), x(t_k^{-}), x^{\Delta}(t_k^{+}), \text{ and } x^{\Delta}(t_k^{+}) \text{ exist with } x(t_k^{+}) = x(t_k) \text{ and } x^{\Delta}(t_k^{+}) = x^{\Delta}(t_k^{+})\}$.

3. Main Results

In this section, we first consider the dynamic equation

$$(a(t)x^{\Delta}(t))^{\Delta} + p(t)f(x(\tau(t))) - q(t)f(x(\varsigma(t))) = 0,$$

$$z(t_k) = M_k(z(t_k)),$$

$$z^{\Delta}(t_k) = N_k(z^{\Delta}(t_k)),$$

that is, the case $c_k = d_k = \eta_k$ of equation (1).

We assume the following conditions hold:

(H1) $0 < b(t) < 1$

(H2) $\sum_{k=1}^{\infty} \left(\int_{t_k}^{t_{k+1}} (1/a(t))\Delta t\right) \prod_{i=1}^{k} \eta_i = \infty$
(H3) $f$ is nondecreasing, $(f (u)/u > 0$ for $u 
eq 0$, and $f (uv) \geq f (u)f (v)$ for $uv > 0$.

(H4) $\theta (t), \tau (t),$ and $\zeta (t) \in C_{rd} (T, T)$, $\theta (t) \leq t$, $\lim_{t \to \infty} \theta (t) = \infty$, $\lim_{t \to \infty} \tau (t) = \infty$ and $\lim_{t \to \infty} \zeta (t) = \infty$

(H5) $\alpha_k \geq 1$, $\sum_{k=1}^{\infty} (\beta_k - 1) < \infty$, and $0 < \eta_k \leq 1$

(H6) $\theta (t) \neq t_k$, $\tau (t) \neq t_k$, and $\zeta (t) \neq t_k$ for $t \neq t_k$

**Theorem 1.** Assume (H1)-(H6) are satisfied, if the inequality

\[
\begin{cases}
 y^\Delta (t) + p (t) f (y (\tau (t))) f (S (\tau (t))) \leq 0, \\
y (t_k) \leq \eta_k y (t_k),
\end{cases}
\]

has no eventually positive solution, where

\[S (t) = (1 - b (t)) \int_{T_k}^{t} \frac{1}{a (t)} \Delta t.
\]

Then, every bounded solution of (7) is oscillatory.

**Proof.** Assume that $x (t)$ is an eventually bounded positive solution of (7); then, there exists $T \in \mathbb{T}$ large enough and $T \geq t_0$ such that $x (t) > 0$, $x (\theta (t)) > 0$, $x (\tau (t)) > 0$, and $x (\zeta (t)) > 0$ for all $t \in [T, \infty)$. Let $F (t) = z (t) + \int_{t}^{1} (1/a (s)) f (x (\zeta (s))) \Delta s, t \in (t_k, t_{k+1}]$; we find

\[a (t) F^\Delta (t) = - p (t) f (x (\tau (t))) < 0.
\]

Then, $a (t) F^\Delta (t)$ is decreasing for all $t \in (t_k, t_{k+1}]$. We can claim that $F^\Delta (t)$ is ultimately greater than zero. If not, there must exist a $m \in (t_j, t_{j+1})$, such that $F^\Delta (m) \leq 0$. For $\forall t \in [m, t_{j+1}]$, we have

\[
\int_{m}^{t} F^\Delta (t) \Delta t \leq a (m) F^\Delta (m) \int_{m}^{t} \frac{1}{a (t)} \Delta t.
\]

Making similar analysis on the interval $(t_k, t_{k+1}]$, $k = j + 1, j + 2, \ldots$, we obtain

\[
\int_{t}^{t_{k+1}} F^\Delta (t) \Delta t \leq \eta_k \cdots \eta_{j+1} a (m) F^\Delta (m) \int_{t_k}^{t} \frac{1}{a (t)} \Delta t,
\]

$t \in (t_k, t_{k+1}]$.

Hence,

\[
\int_{m}^{t} F^\Delta (t) \Delta t + \sum_{k=j+1}^{\infty} \int_{t_k}^{t_{k+1}} F^\Delta (t) \Delta t \leq a (m) F^\Delta (m) \left( \int_{m}^{t_{k+1}} \frac{1}{a (t)} \Delta t + \sum_{k=j}^{\infty} \left( \int_{t_k}^{t_{k+1}} \frac{1}{a (t)} \Delta t \left( \prod_{i=j+1}^{k} \eta_i \right) \right) \right).
\]

Furthermore, we obtain

\[
F (\infty) \leq a (m) F^\Delta (m) \left( \int_{m}^{t_{k+1}} \frac{1}{a (t)} \Delta t + \sum_{k=j}^{\infty} \left( \int_{t_k}^{t_{k+1}} \frac{1}{a (t)} \Delta t \left( \prod_{i=j+1}^{k} \eta_i \right) \right) \right) + F (m) + \sum_{k=j}^{\infty} (\beta_k - 1) z (t_k).
\]

By (H2) and (H5), we have $F (t) \longrightarrow - \infty$ as $t \longrightarrow \infty$, which contradicts $F (t) > 0$. Therefore, $F^\Delta (t)$ is ultimately greater than zero. There exists $T_1 \geq T$ such that $z^\Delta (t) > F^\Delta (t) > 0$, where $t \geq T_1$. By $z (t_k) \geq a_k z (t_k) \geq z (t_k)$, we have

\[
z (t) \geq \int_{T_1}^{t_k} z^\Delta (s) \Delta s > \int_{T_1}^{t_k} F^\Delta (s) \Delta s
\]

\[
= \int_{T_1}^{t_k} \frac{a (s) F^\Delta (s)}{a (s)} \Delta s \geq a (t) F^\Delta (t) \int_{T_1}^{t_k} \frac{1}{a (s)} \Delta s.
\]

Let $\phi (t) = a (t) F^\Delta (t)$; then, $x (t) \geq z (t) (1 - b (t)) \geq \phi (t) S (t)$. We see $\phi (t)$ is a positive solution of (8), which contradicts the oscillation of (8). We complete the proof.

We assume that the following conditions are satisfied:

(H7) $b (t)$ is bounded

(H8) $\zeta (t) \in C_{rd} (\mathbb{T}, \mathbb{T})$ is bijective, $\tau (t) \in C_{rd} (\mathbb{T}, \mathbb{T})$, $\tau (t) \leq \zeta (t), \zeta^{-1} (\tau (t)) \in C_{rd} (\mathbb{T}, \mathbb{T})$, $\lim_{t \to \infty} \theta (t) = \infty$, and $\lim_{t \to \infty} \tau (t) = \infty$

(H9) $\int_{t_k}^{\infty} \int_{t_k}^{\infty} (1/a (s)) f (s) \Delta s \Delta \eta < \infty$

(H10) There exists $L > 0$ and $(f (u)/u) \geq L$ for $u \neq 0$

(H11) $p (t) - q (\zeta^{-1} (\tau (t))) (\zeta^{-1} (\tau (t)))^\Delta \geq C > 0$

**Theorem 2.** Assume (H2) and (H5)-(H11) are satisfied; then, every bounded solution of (7) is oscillatory.

**Proof.** Assume that $x (t)$ is an eventually bounded positive solution of equation (7); then, there exists $T \in \mathbb{T}$ large enough and $T \geq t_0$ such that $x (t) > 0$, $x (\theta (t)) > 0$, $x (\tau (t)) > 0$, and $x (\zeta (t)) > 0$ for all $t \in [T, \infty)$. Let $u (t) = z (t) - \int_{T}^{t} (1/a (s)) f (s) \Delta s, t \in (t_k, t_{k+1}]$; we obtain that
\[
(a(t)u^\Delta(t))^\Delta = - p(t) - q(\zeta^{-1}(\tau(t)))(\zeta^{-1}(\tau(t)))^\Delta \\
\cdot f(x(\tau(t))) \\
\leq - CLx(\tau(t)) < 0.
\]

Then, \(a(t)u^\Delta(t)\) is decreasing in \((t_k, t_{k+1}]\). We can obtain that \(u^\Delta(t)\) is ultimately greater than zero. There exists \(T_2 \geq T\) such that \(z^\Delta(t) > u^\Delta(t) > 0\) for all \(t \in [T_2, \infty)\).

For \(\forall t \in (t_k, t_{k+1}]\) and \(t \in (T_2, \infty)\), we have
\[
\int_{T_2}^{t} (a(s)u^\Delta(s))^\Delta \Delta s \leq - CL \int_{T_2}^{t} x(\tau(s)) \Delta s. 
\]

Furthermore, we obtain
\[
CL \int_{T_2}^{t} x(\tau(s)) \Delta s \leq a(T_2)u^\Delta(T_2) - a(t)u^\Delta(t) \\
+ \sum_{T_2 \leq t < t_k} (\eta_k - 1)a(t_k)u^\Delta(t_k) \\
\leq a(T_2)u^\Delta(T_2) < \infty.
\]

Therefore, \(x(t)\) and \(z(t)\) are integrable. However, we know \(z(t)\) is increasing in \((T_2, \infty)\) and nonintegrable from \(z^\Delta(t) > 0\) and \(z(t_k) \geq z(t')\), which contradicts the integrability of \(z(t)\). We complete the proof. \(\square\)

Next, we study the equation
\[
\begin{aligned}
(a(t)z^\Delta(t))^\Delta + p(t)f(x(\sigma(t))) &= 0, \\
z(t_k) &= M_k(z(t_k)), \\
z^\Delta(t_k) &= \eta_k z^\Delta(t_k),
\end{aligned}
\]

i.e., the case \(q(t) \equiv 0, \; \tau(t) = \sigma(t)\), and \(c_k = d_k = \eta_k\) of (1).

We assume the following conditions hold:

(H12) \(0 < b(t) < (H(t)B_k/H(\theta(t)))\), where

\[
H(t) = \int_{t}^{\infty} (1/a(s)) \Delta s, B_k = \int_{\theta(t)}^{\infty} \frac{a_k}{\theta(\theta(t))} \Delta s.
\]

If \(H(t) \neq H(\theta(t))\),

\[
H(t) = \int_{t}^{\infty} (1/a(s)) \Delta s, B_k = \int_{\theta(t)}^{\infty} \frac{a_k}{\theta(\theta(t))} \Delta s.
\]

We can make similar analysis on the intervals \((t_j, t_{j+1}]\) and \((t_{f(u)}, u]\), such that \(j = k + 1, \ldots, j = \frac{u - 1}{I(u)}\). Thus,

\[
\begin{aligned}
H(t) &= \int_{t}^{\infty} (1/a(s)) \Delta s, B_k = \int_{\theta(t)}^{\infty} \frac{a_k}{\theta(\theta(t))} \Delta s.
\end{aligned}
\]

Therefore, we can claim that \(z(t)/H(t)\) is increasing in \((t_k, t_{k+1}]\).

If \(f_k \notin (\theta(t), t)\), we obtain

\[
\begin{aligned}
\frac{z(t)}{H(t)} &\geq \frac{z(\theta(t))}{H(\theta(t))} \\
\end{aligned}
\]

If \(f_k \in (\theta(t), t)\), we obtain

\[
\begin{aligned}
\frac{z(t)}{H(t)} &\geq \prod_{\theta(t) < t < t_k} \frac{a_k z(\theta(t))}{H(\theta(t))}.
\end{aligned}
\]
So
\[ \frac{z(t)}{H(t)} \geq B_k \frac{z(\theta(t))}{H(\theta(t))} \] \hspace{1cm} (28)

Therefore,
\[ x(t) = z(t) - b(t)x(\theta(t)) \geq z(t) - b(t)z(\theta(t)) \]
\[ \geq z(t) \left(1 - b(t) \frac{H(\theta(t))}{H(t)} \frac{1}{B_k}\right). \] \hspace{1cm} (29)

Let \( \omega(t) = \left((a(t)z^\Delta(t))/z(t)\right), \ t \in (t_k, t_{k+1}]; \) we have
\[ \omega(t) \leq \omega(t_0) \prod_{t_0 < t < t_k} \frac{\eta_k}{\alpha_k} - L \int_{t_0}^t \prod_{t_0 < s < t_k} \frac{\eta_k}{\alpha_k} p(s)f \left(1 - b(\sigma(s)) \frac{H(\theta(s))}{H(\sigma(s))} \frac{1}{B_k}\right) \Delta s \]
\[ = \prod_{t_0 < t < t_k} \frac{\eta_k}{\alpha_k} \left(\omega(t_0) - L \int_{t_0}^t \prod_{t_0 < s < t} \frac{\alpha_k}{\eta_k} p(s)f \left(1 - b(\sigma(s)) \frac{H(\theta(s))}{H(\sigma(s))} \frac{1}{B_k}\right) \Delta s\right). \] \hspace{1cm} (30)

We get a contradiction as \( t \to \infty. \) We complete the proof.

For \( q(t) \equiv 0 \) and \( \alpha_k = \beta_k = 1 \) in (1), it can be written as
\[ \left(a(t)z^\Delta(t)\right)\Delta + p(t)f(x(\tau(t))) = 0, \]
\[ z(t_k) = z(t_k), \]
\[ z^\Delta(t_k) = N_k(z^\Delta(t_k)). \] \hspace{1cm} (31)

We assume that the following conditions hold for (32):

(H17) \( \theta(t), \tau(t) \in C_{\infty}(\mathbb{T}, \mathbb{T}), \theta(t) \leq r, \tau(t) \leq t, \lim_{t \to \infty} \theta(t) = \infty, \) and \( \lim_{t \to \infty} \tau(t) = \infty. \)

(H18) \( \theta(t) \neq t_k \) and \( \tau(t) \neq t_k \) for \( t \neq t_k \)

(H19) \( \int_0^\infty \prod_{t_0 \leq t < T} \frac{1}{p(t)f(1 - b(\tau(t)))K(t)} \Delta t = \infty. \)

Lemma 2. Assume (H1) is satisfied and \( x(t) \) is an eventually positive solution of equation (32); then, \( z^\Delta(t) \geq 0 \) and \( z^\Delta(t_k) \geq 0, \) where \( t \in (t_k, t_{k+1}]. \)

Proof. Because \( x(t) \) is an eventually positive solution of equation (32), we obtain
\[ (a(t)z^\Delta(t))\Delta = -p(t)f(x(\tau(t))) < 0, \ t \in (t_k, t_{k+1}]. \] \hspace{1cm} (32)

Then, \( a(t)z^\Delta(t) \) is decreasing in \( (t_k, t_{k+1}]. \) If there exists \( t_j \) such that \( z^\Delta(t_j) = -l < 0 \) for \( l > 0 \) and we obtain for \( \forall t \in (t_j, t_{j+1}), \) \( a(t)z^\Delta(t) \leq -a(t_j)c_jc_{j+1}, \ldots, c_{j+l}, \) then
\[ z^\Delta(t) \leq -\frac{a(t_j)}{a(t)} l \prod_{t_j \leq s < t} c_k, \] \hspace{1cm} (33)
\[ z(t_j) = z(t_k). \]

By Lemma 1, we obtain for \( t > t_j, \)

\[ z(t) \leq z(t_j) - l \int_{t_j}^t \frac{a(t)}{a(s)} \prod_{t_j \leq s < t} c_k \Delta s. \] \hspace{1cm} (34)

\[ \omega(t) \leq \frac{-p(t)f(x(\sigma(t)))}{z(\sigma(t))} \leq -p(t)Lf\left(1 - b(\sigma(t)) \frac{H(\theta(\sigma(t)))}{H(\sigma(t))} \frac{1}{B_k}\right), \] \hspace{1cm} (35)

We get a contradiction as \( t \to \infty. \) Then, \( z^\Delta(t_k) \geq 0. \)

Theorem 4. Assume (H1) and (H16)–(H19) are satisfied. If \( d_k \leq 1 \) and
\[ \int_{t_0}^\infty \prod_{t \leq t < T} \frac{1}{p(t)f(1 - b(\tau(t)))K(t)} \Delta t = \infty, \] \hspace{1cm} (36)

then (32) is oscillatory.

Proof. Suppose to the contrary that \( x(t) \) is an eventually positive solution of (32); then, there exists \( T_0 \) large enough and \( T_0 \geq t_0 \) such that \( x(t) > 0, x(\theta(t)) > 0, \) and \( x(\tau(t)) > 0 \) for all \( t \in [T, \infty). \) Then, \( a(t)z^\Delta(t) \) is decreasing in \( (t_k, t_{k+1}]. \) According to \( a(t_k)z^\Delta(t_k) \leq a(t_k)z^\Delta(t_k), \) we obtain \( a(t)z^\Delta(t) \) is decreasing in \( (T, \infty). \)

For \( \forall t \in (t_k, t_{k+1} \) and \( t \geq \tau(t) \geq T, \) we have
\[ \frac{z(\sigma(t))}{z(\tau(t))} \leq K(t), \] \hspace{1cm} (37)

where
\[ K(t) = \frac{Q(\tau(t))}{Q(\sigma(t))}, \] \hspace{1cm} (38)

and
\[ Q(t) = \int_{T}^{\infty} \frac{1}{a(s)} \Delta s, \] \hspace{1cm} (39)
It can be proved similar to Lemma 1 of [11] and so its proof is omitted here.

From Lemma 2, we have $z^\Delta (t)\geq 0$ and $z^\Delta (t_k)\geq 0$, where $t\geq T$ and $t\in T$. Let $\omega (t) = (a(t)z^\Delta (z(t))$, $t \in (t_k, t_{k+1}]$. We have

$$\omega^\Delta (t) \leq \frac{-p(t)f(x(\sigma(t)))}{z(\sigma(t))} \leq -p(t)Lf(1-b(\tau(t))K(t)), $$

$$\omega(t_k^+) \leq d_k \omega(t_k).$$

Applying Lemma 1, we obtain for $t > t_0$,

$$\omega(t) \leq \omega(t_0) \prod_{t_0 < t_k < t} d_k \int_{t_k}^{t} \int_{t_k}^{t} d_k p(s) \cdot f(1-b(\tau(s)))K(s) \Delta s,$$

$$= \prod_{t_0 < t_k < t} d_k \left( \omega(t_0) - L \int_{t_k}^{t} \int_{t_k}^{t} \frac{1}{d_k} p(s) \right.$$

$$\left. \cdot f(1-b(\tau(s)))K(s) \Delta s \right).$$

We get a contradiction as $t \longrightarrow \infty$. This completes the proof of Theorem 4. \hfill \Box

4. Examples

Example 1. Consider the equation

$$\left\{ \begin{array}{l}
\left( \frac{1}{e^\Delta (t)} \right)^\Delta + p(t)x(t-2) - q(t)x(t+2) = 0, \quad t \in J_{\tau} \setminus \{2k\}, \\
z(2k^+) = M_k(z(2k)), \quad k \in N, \\
z^\Delta (2k^+) = \frac{k}{k+1} z^\Delta (2k), \quad k \in N,
\end{array} \right. $$

(42)

where $z(t) = x(t) + (1/t)x(t-2)$ and $J_{\tau} := [2, \infty)$. Here, $\alpha_k = 1 + (1/k^2 + 1)$, $\beta_k = 1 + (1/k^2)$, and $\eta_k = (k/(k+1))$. Obviously, ($H1$) and ($H3$)–($H6$) are satisfied:

$$\sum_{k=1}^{\infty} \left( \int_{t_k}^{t_{k+1}} \frac{1}{a(t)} \Delta t \int_{t_k}^{k} \eta_i \right) \left( \int_{k}^{k+1} \frac{1}{i} \right) = \frac{1}{2} \sum_{i=1}^{\infty} \int_{t_k}^{t_{k+1}} e^{\Delta t} + \frac{1}{3} \sum_{j=1}^{\infty} \left( \int_{t_k}^{t_{k+1}} e^{\Delta t} \right) \cdot \Delta t < \infty,$$

$$p(t) - q(\zeta^{-1} (\tau(t))) \left( \zeta^{-1} (\tau(t)) \right)^\Delta = e^{\Delta} - e^{\Delta - \tau(t)} \geq e^{\Delta} - 1 > 0.$$ 

(46)

So ($H9$) and ($H11$) are satisfied. By Theorem 2, we see that every bounded solution of (45) is oscillatory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was partially supported by NNSF of China (11971145).

References


