Research Article

Exact Values of Zagreb Indices for Generalized T-Sum Networks with Lexicographic Product

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The use of numerical numbers to represent molecular networks plays a crucial role in the study of physicochemical and structural properties of the chemical compounds. For some integer k and a network G , the networks $S_k(G)$ and $R_k(G)$ are its derived networks called as generalized subdivided and generalized semitotal point networks, where S_k and R_k are generalized subdivision and generalized semitotal point operations, respectively. Moreover, for two connected networks, G_1 and G_2 , $G_1[G_2]_{S_k}$ and $G_1[G_2]_{R_k}$ are T -sum networks which are obtained by the lexicographic product of $T(G_1)$ and G_2 , respectively, where $T \in \{S_k, R_k\}$. In this paper, for the integral value $k \geq 1$, we find exact values of the first and second Zagreb indices for generalized T -sum networks. Furthermore, the obtained findings are general extensions of some known results for only $k = 1$. At the end, a comparison among the different generalized T -sum networks with respect to first and second Zagreb indices is also included.

1. Introduction

Topological index (TI) being a molecular descriptor is a mathematical measure that associates a molecular network with a real number and predicts the underlying molecular network's biological, chemical, and structural properties. Molecular descriptors were used by Wiener [1] and Trinajstić and Gutman and Trinajstić [2] to determine the boiling point of paraffin and the total π -electron energy of the molecules, respectively. TIs are also used in the study of cheminformatics to classify molecules in terms of quantitative structure behavior and property relationships. Most notably, all TIs are invariants under the networks' isomorphism parameter [3, 4]. Many TIs exist in the literature for networks. Degree-based TIs, distance-based TIs, and polynomial-based TIs are the three major types of these. The degrees-based TIs are more familiar than the others [5].

In the field of chemical network theory, networks operations are frequently used to discover the new families of networks. Yan et al. [6] explained the subdivision (S_1) and semitotal point (R_1) operations on a molecular network G and attain the Wiener indices of the consequent networks

$S_1(G)$ and $R_1(G)$. After that, Eliasi and Taeri [7] explained the T -sum network $G_1 +_T G_2$ by using Cartesian product of G_1 and $S(G_2)$, where $T \in \{S_1, R_1\}$. Deng et al. [8] defined the T -sum network $(G_1[G_2])$ with the help of lexicographic product and also calculated the first and second Zagreb indices, and Akhter and Imran [9] also calculated the forgotten index of the T -sum networks. For more studies of the TIs under the operations of networks, see [10, 11].

Recently, for some integer $k \geq 1$, Liu et al. [12] defined the generalized subdivision (S_k) and generalized semitotal point (R_k) operations. They also computed the 1st and 2nd Zagreb indices for T -sum networks with the help of Cartesian product, i.e., $(G_1 +_{T_k} G_2)$ for $T_k \in \{S_k, R_k\}$ [13, 14]. In this study, we find the exact values of the Zagreb indices for the generalized T -sum networks which are constructed with the help of generalized subdivision, generalized semitotal point operations, and lexicographic product. In the remaining paper, Section 1 has some previous knowledge related to our work and Section 2 has some basic definition. The key findings are presented in Section 3, and the conclusion and applications of these indices are presented in Section 4.

2. Preliminaries

We consider undirected, connected, and simple networks, where $V(G)$ is a vertex set and $E(G) \subseteq V(G) \times V(G)$ is an edge set. In addition, $|V(G)| = n$ is order and $|E(G)| = m$ is size of G . Each vertices of a molecular network is referred to as an atom, and edges reflect the bonding between atoms.

The number of incident edges is called its degree. The networks $V(P_n) = \{b_j: 1 \leq j \leq n\}$ and $E(P_n) = \{b_j b_{j+1}: 1 \leq j \leq n-1\}$, $V(C_n) = \{b_j: 1 \leq j \leq n\}$ & $E(C_n) = \{b_i b_{i+1}: 1 \leq i \leq n-1\} \cup \{b_n b_1\}$ and $V(K_n) = \{b_i: 1 \leq i \leq n\}$, and $E(K_n) = \{b_i b_j: 1 \leq i, j \leq n\}$ are called path (P_n), cycle (C_n), and complete (K_n), respectively [3].

Definition 1. Let G_1 and G_2 be two networks with their vertex sets as $V(G_1) = \{y_1, y_2, \dots, y_n\}$ and $V(G_2) = \{z_1, z_2, \dots, z_n\}$, respectively. The Cartesian product $G_1 \times G_2$ of these networks is defined as follows:

$$\begin{aligned} V(G_1 \times G_2) &= V(G_1) \times V(G_2), \\ E(G_1 \times G_2) &= \{(y_1, z_1)(y_2, z_2): (y_1, z_1), (y_2, z_2) \in V(G_1) \times V(G_2)\}, \end{aligned} \quad (1)$$

with conditions either $y_1 = y_2$ in $V(G_1)$ and $z_1 z_2 \in E(G_2)$ or $y_1 y_2 \in E(G_1)$ and $z_1 = z_2$ in $V(G_2)$.

Definition 2. Let G_1 and G_2 be two networks with their vertex sets as $V(G_1) = \{y_1, y_2, \dots, y_n\}$ and $V(G_2) = \{z_1, z_2, \dots, z_n\}$, respectively. Then, the lexicographic product $G_1[G_2]$ of these networks is defined as

$$\begin{aligned} V(G_1[G_2]) &= V(G_1) \times V(G_2), \\ E(G_1[G_2]) &= \{(y_1, z_1)(y_2, z_2): (y_1, z_1), (y_2, z_2) \in V(G_1) \times V(G_2)\}, \end{aligned} \quad (2)$$

with conditions either. $y_1 = y_2$ in $V(G_1)$ and $z_1 z_2 \in E(G_2)$ or $y_1 y_2 \in E(G_1)$ and z_1, z_2 in $V(G_2)$.

For integer $k \geq 1$, Liu et al. [12] defined the following graphs using the generalized subdivision and semitotal point operations:

- (i) $S_k(G)$ network is found out by inserting k -vertices in each edge of G
- (ii) $R_k(G)$ is found out from $S_k(G)$ by connecting the old vertices which are adjacent in G

Eliasi and Taeri [7] proposed the generalized T -sum networks under the operation of two connected networks G_1

and G_2 based on the Cartesian product. Deng et al. [8] defined the generalized T -sum networks under the operation of two connected networks G_1 and G_2 based on the Lexicographic product as follows.

Definition 3. Let G_1 and G_2 be two networks and $T_k \in \{S_k, R_k\}$ and $T_k(G_1)$ be a network with vertex set $V(T_k(G_1))$ and edge set $E(T_k(G_1))$. Then, the generalized T_k -sum network $G_1[G_2]_{T_k}$ is a graph with the vertex set:

$$\begin{aligned} V(G_1[G_2]_{T_k}) &= V(T_k(G_1)) \times V(G_2), \\ E(G_1[G_2]_{T_k}) &= \{(y_1, z_1)(y_2, z_2): (y_1, z_1), (y_2, z_2) \in V(T_k(G_1)) \times V(G_2)\}, \end{aligned} \quad (3)$$

with conditions either $y_1 = y_2$ in $V(T_k(G_1))$ and $z_1 z_2 \in E(G_2)$ or $y_1 y_2 \in E(T_k(G_1))$ and z_1, z_2 in $V(G_2)$.

For more explanation, see Figures 1-4.

3. Main Results

In this part, we will learn about the Zagreb indices of T -sum networks obtained by the operations of generalized subdivision and generalized semitotal point and lexicographic product.

Theorem 1. Let G_1 and G_2 be two networks; then, for some integer k , the first Zagreb index of S_k -sum network is

$$\begin{aligned} M_1(G_1[G_2]_{S_k}) &= 8n_2 e_1 e_2 + n_1 M_1(G_2) + n_2^3 M_1(G_1) \\ &\quad + 4e_1 n_2^3 + 4e_1 n_2^2 (k-1) + 8n_2 e_1 (k-1) \\ &\quad + \sum_{i=1}^{n_2-1} (n_2 - i). \end{aligned} \quad (4)$$

FIGURE 1: (a) $G \cong P_4$, (b) $S_1(G) \cong S_1(P_4)$, and (c) $R_1(G) \cong R_1(P_4)$. FIGURE 2: (a) $S(G_1) \cong S(P_3)$ and (b) $G_2 \cong P_4$. FIGURE 3: $P_3[P_4]_{S_4}$. FIGURE 4: $P_3[P_4]_{R_4}$.

Proof. Let $d(u, v) = d_{G_1[G_2]_{S_k}}(u, v)$ be the degree of a vertex (u, v) in the network $G_1[G_2]_{S_k}$:

$$\begin{aligned}
 M_1(G_1[G_2]_{S_k}) &= \sum_{(u,v) \in V(G_1[G_2]_{S_k})} d^2(u, v) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]_{S_k})} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(S_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1=v_2=v \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(G_1) \\ u_1, u_2 \in V(S_k(G_1)) - V(G_1)}} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(S_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum 1 + \sum 2 + \sum 3 + \sum 4.
 \end{aligned} \tag{5}$$

Now, first, we calculate

$$\begin{aligned}
 \sum 1 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [n_2 d_{G_1}(u) + d_{G_2}(v_1) + n_2 d_{G_1}(u) + d_{G_2}(v_2)] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [2n_2 d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2)] \\
 &= \sum_{u \in V(G_1)} e_2 \cdot 2n_2 d(u) + \sum_{u \in V(G_1)} M_1(G_2) = 2e_1 e_2 (2n_2) + n_1 M_1(G_2) = 4n_2 e_1 e_2 + n_1 M_1(G_2), \\
 \sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(S_k(G_1))} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u \in (v(G_1)), a \in (E(G_1)) \\ u \text{ and } a \text{ are incident}}} [d(u, v_1) + d(a, v_2)] \\
 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u \in (v(G_1)), a \in (E(G_1))} [n_2 d_{G_1} u + d_{G_2} v_1 + 2n_2] \\
 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u \in (v(G_1))} [d_{G_1} u] [n_2 d_{G_1} u + d_{G_2} v_1 + 2n_2] \\
 &= n_2 n_2 n_2 \sum_{u \in (v(G_1))} [d_{G_1}^2 u] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} 2e_1 [d_{G_2} v_1 + 2n_2] \\
 &= n_2^3 M_1(G_1) + 2e_1 [2e_2 n_2 + 2n_2 n_2 n_2] = n_2^3 M_1(G_1) + 4n_2 e_1 e_2 + 4e_1 n_2^3,
 \end{aligned}$$

$$\begin{aligned}
\sum 3 &= \sum_{v_1=v_2=v \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} [d(u_1, v_1) + d(u_2, v_2)] \\
&\quad u_1 u_2 \in V(S_k(G_1)) - V(G_1) \\
&= n_2(k-1)e_1[2n_2 + 2n_2] = 4e_1 n_2^2(k-1), \\
\sum 4 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_1))} [d(u_1, v_1) + d(u_2, v_2)] \\
&= [2n_2 + 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + [2n_2 + 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1 \\
&= 8n_2 \sum_{i=1}^{n_2-1} (n_2 - i)(k-2)e_1 + 8n_2 \sum_{i=1}^{n_2-1} (n_2 - i)e_1 = 8n_2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i).
\end{aligned} \tag{6}$$

So, the result is

$$\begin{aligned}
&= 8n_2 e_1 e_2 + n_1 M_1(G_2) + n_2^3 M_1(G_1) + 4e_1 n_2^3 \\
&\quad + 4e_1 n_2^2(k-1) + 8n_2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i).
\end{aligned} \tag{7}$$

□

Theorem 2. Let G_1 and G_2 be two networks; then, for some integer k , the second Zagreb index of S_k -sum network is

$$\begin{aligned}
M_2[G_1[G_2]_{S_k}] &= n_2^2 e_2 M_1(G_1) + 2n_2 e_1 M_1(G_2) + n_1 M_2(G_2) + 2n_2^4 M_1(G_1) + 8n_2^2 e_1 e_2 + 4e_1 n_2^2(k-1) \\
&\quad + 8n_2^2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i).
\end{aligned} \tag{8}$$

Proof. Let $d(u, v) = d_{G_1[G_2]_{S_k}}(u, v)$ be the degree of a vertex (u, v) in the graph $G_1[G_2]_{S_k}$:

$$\begin{aligned}
M_1(G_1[G_2]_{S_k}) &= \sum_{(u,v) \in V(G_1[G_2]_{S_k})} d^2(u, v) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]_{S_k})} [d(u_1, v_1)d(u_2, v_2)] \\
&= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d(u, v_1)d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_1))} [d(u_1, v_1)d(u_2, v_2)] \\
&= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\
&= \sum 1 + \sum 2 + \sum 3 + \sum 4.
\end{aligned} \tag{9}$$

Now, first, we calculate

$$\begin{aligned}
 \sum 1 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d(u, v_1) d(u, v_2)] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [n_2^2 d_{G_1}^2(u) + n_2 d_{G_1}(u) [d_{G_2}(v_1) + d_{G_2}(v_2)] + d_{G_2}(v_1) d_{G_2}(v_2)] \\
 &= \sum_{u \in V(G_1)} [n_2^2 e_2 d_{G_1}^2(u) + n_2 d_{G_1}(u) M_1(G_2) + M_2(G_2)] = n_2^2 e_2 M_1(G_1) + 2n_2 e_1 M_1(G_2) + n_1 M_2(G_2), \\
 \sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_1))} [d(u_1, v_1) d(u_2, v_2)] \\
 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u \in (v(G_1)), a \in (E(G_1)) \\ u \text{ and } a \text{ are incident}}} [d(u, v_1) d(a, v_2)] \\
 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u \in (v(G_1)), a \in (E(G_1)) \\ u \text{ and } a \text{ are incident}}} [n_2 d_{G_1}(u) + d_{G_2}(v_1)] [2n_2] \\
 &= n_2 n_2 2n_2 \sum_{u \in (v(G_1))} [d_{G_1}^2(u)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} 2e_1 [2n_2 d_{G_2}(v_1)] \\
 &= 2n_2^4 M_1(G_1) + 2n_2 n_2 2e_1 2e_2 = 2n_2^4 M_1(G_1) + 8n_2^2 e_1 e_2, \\
 \sum 3 &= \sum_{v_1=v_2=v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 u_2 \in V(S_k(G_1)) - V(G_1)}} [d(u_1, v_1) d(u_2, v_2)] \\
 &= n_2 (k-1) e_1 [2n_2 2n_2] = 4e_1 n_2^2 (k-1), \\
 \sum 4 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_1))} [d(u_1, v_1) d(u_2, v_2)] \\
 &= [2n_2 \times 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i) (k-2) e_1 + [2n_2 \times 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i) e_1 \\
 &= 8n_2^2 \sum_{i=1}^{n_2-1} (n_2 - i) (k-2) e_1 + 8n_2^2 \sum_{i=1}^{n_2-1} (n_2 - i) e_1 = 8n_2^2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i).
 \end{aligned} \tag{10}$$

So, the final result is

$$= n_2^2 e_2 M_1(G_1) + 2n_2 e_1 M_1(G_2) + n_1 M_2(G_2) + 2n_2^4 M_1(G_1) + 8n_2^2 e_1 e_2 + 4e_1 n_2^2 (k-1) + 8n_2^2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \tag{11}$$

Theorem 3. Let G_1 and G_2 be two networks; then, for some integer k , the first Zagreb index of R_k -sum network is

$$M_1(G_1[G_2]_{R_k}) = 16n_2 e_1 e_2 + n_1 M_1(G_2) + 4n_2^3 M_1(G_1) + 4e_1 n_2^3 + 4e_1 n_2^2 (k-1) + 8n_2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \tag{12}$$

Proof. Let $d(u, v) = d_{G_1[G_2]_{R_k}}(u, v)$ be the degree of a vertex (u, v) in the graph $G_1[G_2]_{R_k}$.

$$\begin{aligned}
 M_1(G_1[G_2]_{R_k}) &= \sum_{(u,v) \in V(G_1[G_2]_{R_k})} d^2(u, v) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]_{R_k})} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_2))} \\
 &\quad \cdot [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1=v_2=v \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(R_k(G_1)) \\ u_1, u_2 \in V[R_k(G_1)-V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum 1 + \sum 2 + \sum 3 + \sum 4.
 \end{aligned} \tag{13}$$

Now, first, we calculate

$$\begin{aligned}
 \sum 1 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[n_2 d_{R_{kG_1}} u + d_{G_2} v_1 + n_2 d_{R_{kG_1}} u + d_{G_2} v_2 \right] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[2n_2 d_{R_{kG_1}} u + d_{G_2} v_1 + d_{G_2} v_2 \right] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[4n_2 d_{G_1} u + d_{G_2} v_1 + d_{G_2} v_2 \right] \\
 &= 4n_2 \sum_{v_1, v_2 \in E(G_2)} \left[\sum_{u \in V(G_1)} [d_{G_1} u] + \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d_{G_2} v_1 + d_{G_2} v_2] \right] \\
 &= 4n_2 \sum_{v_1, v_2 \in E(G_2)} (e_1) + \sum_{u \in V(G_1)} M_1(G_2) \\
 &= 4n_2 (2e_2)(e_1) + n_1 M_1(G_2) = 8n_2 e_1 e_2 + n_1 M_1(G_2), \\
 \sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(G_1) \\ u_1 \in V(G_1), u_2 \in V[R_k(G_1)-V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(R_k(G_1)) \\ u_1 \in V(G_1), u_2 \in V[R_k(G_1)-V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1=v_2=v \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(R_k(G_1)) \\ u_1, u_2 \in V[R_k(G_1)-V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)],
 \end{aligned}$$

$$\begin{aligned}
\sum 2 &= \sum 2' + \sum 2'', \\
\sum 2' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left(n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 \right) + \left(n_2 d_{R_{kG_1}} u_2 + d_{G_2} v_2 \right) \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} n_2 \left(d_{R_{kG_1}} u_1 + d_{R_{kG_1}} u_2 \right) + (d_{G_2} v_1 + d_{G_2} v_2) \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} 2n_2 (d_{G_1} u_1 + d_{G_1} u_2) + (d_{G_2} v_1 + d_{G_2} v_2) \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} 2n_2 M_1(G_1) + 2e_2 e_1 n_2 + 2e_1 e_2 n_2 \\
&= n_2 n_2 2n_2 M_1(G_1) + 4e_1 e_2 n_2 = 2n_2^3 M_1(G_1) + 4e_1 e_2 n_2, \\
\sum 2'' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} \left[n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 + n_2 d_{R_{kG_1}} u_2 \right] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} \left[2n_2 d_{G_1} u_1 + d_{G_2} v_1 + 2n_2 \right] \tag{14} \\
&= n_1 n_2 2n_2 \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} d_{G_1}(u_1) + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(R_{k2})} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d_{G_2} v_1 + 2n_2] \\
&= 2n_2^3 \sum_{u_1 \in V(G_1)} d_{G_1}^2 u_1 \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 \in V(G_1)} d_{G_1} u_1 [d_{G_2} v_1 + 2n_2] \\
&= 2n_2^3 M_1(G_1) + 2e_1 \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} [d_{G_2} v_1 + 2n_2] \\
&= 2n_2^3 M_1(G_1) + 2e_1 [2e_2 n_2 + 2n_2 n_2 n_2] = 2n_2^3 M_1(G_1) + 4e_1 e_2 n_2 + 4e_1 n_2^3, \\
\sum 3 &= \sum_{v_1 = v_2 = v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R_k(G_1)) \\ u_1 u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
&= n_2 (k-1) e_1 [2n_2 + 2n_2] = 4e_1 n_2^2 (k-1), \\
\sum 4 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} [d(u_1, v_1) + d(u_2, v_2)] \\
&= [2n_2 + 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + [2n_2 + 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1 \\
&= 8n_2 \sum_{i=1}^{n_2-1} (n_2 - i)(k-2)e_1 + 8n_2 \sum_{i=1}^{n_2-1} (n_2 - i)e_1
\end{aligned}$$

So, the final result is

$$= 16n_2e_1e_2 + n_1M_1(G_2) + 4n_2^3M_1(G_1) + 4e_1n_2^3 + 4e_1n_2^2(k-1) + 8n_2e_1(k-1) \sum_{i=1}^{n_2-1}. \quad (15)$$

□

Theorem 4. Let G_1 and G_2 be two networks; then, for some integer k , the second Zagreb index of R_k -sum network is

$$M_2(G_1[G_2]_{R_k}) = 8n_2^2e_2M_1(G_1) + 4e_1n_2M_1(G_2) + n_1M_2(G_2) + 4n_2^4M_2(G_1) + 4n_2^4M_1(G_2) + 8n_2^2e_1e_2 + 4e_1e_2^2 + 4e_1n_2^2(k-1) + 8n_2^2e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \quad (16)$$

Proof. Let $d(u, v) = d_{G_1[G_2]_{R_k}}(u, v)$ be the degree of a vertex (u, v) in the network $G_1[G_2]_{R_k}$:

$$\begin{aligned} M_1(G_1[G_2]_{R_k}) &= \sum_{(u,v) \in V(G_1[G_2]_{R_k})} d^2(u, v) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]_{R_k})} [d(u_1, v_1)d(u_2, v_2)] \\ &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1)d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_2))} [d(u_1, v_1)d(u_2, v_2)] \\ &\quad + \sum_{v_1=v_2=v \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_1))} [d(u_1, v_1) + d(u_2, v_2)] \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\ &= \sum 1 + \sum 2 + \sum 3 + \sum 4. \end{aligned} \quad (17)$$

Now, first, we calculate

$$\begin{aligned} \sum 1 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1)d(u, v_2)] \\ &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[n_2 d_{R_{kG_1}} u + d_{G_2} v_1 \right] \left[n_2 d_{R_{kG_1}} u + d_{G_2} v_2 \right] \\ &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[2n_2 d_{G_1} u + d_{G_2} v_1 \right] \left[2n_2 d_{G_1} u + d_{G_2} v_2 \right] \\ &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[4n_2^2 d_{G_1}^2 u + 2n_2 d_{G_1} u [d_{G_2} v_1 + d_{G_2} v_2] \right] \\ &= 4e_2 n_2^2 M_1(G_1) + 4e_1 n_2 M_1(G_2) + n_1 M_2(G_2), \end{aligned}$$

$$\begin{aligned}
\sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) d(u_2, v_2)] \\
&\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) d(u_2, v_2)], \\
\sum 2 &= \sum 2' + \sum 2'', \\
\sum 2' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} [d(u_1, v_1) \cdot d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} [d(u_1, v_1) d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left[n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 \right] \left[n_2 d_{R_{kG_1}} u_2 + d_{G_2} v_2 \right] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left[2n_2 d_{G_1} u_1 + d_{G_2} v_1 \right] \left[n_2 d_{G_1} u_2 + d_{G_2} v_2 \right] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left[4n_2^2 d_{G_1} u_1 d_{G_1} u_2 + 2n_2 d_{G_1} u_1 d_{G_2} v_2 + 2n_2 d_{G_1} u_2 d_{G_2} v_1 + d_{G_2} v_1 d_{G_2} v_2 \right] \\
&= 4n_2^4 M_2(G_1) + 4e_2 n_2^2 \sum_{u_1 u_2 \in E(G_1)} [d_{G_1} u_1 + d_{G_1} u_2] + 4e_1 e_2^2 = 4n_2^4 M_2(G_1) + 4e_2 n_2^2 M_1(G_1) + 4e_1 e_2^2,
\end{aligned}$$

$$\begin{aligned}
\sum 2'' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} \left[n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 \right] \left[n_2 d_{R_{kG_1}} u_2 \right] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} \left[n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 \right] [2n_2] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} \left[4n_2^2 [d_{G_1} u_1 + 2n_2 d_{G_2} v_1] \right] \\
&= 4n_2^4 \sum_{u_1 u_2 \in E(N_2(G_1))} d_{G_1} u_1 + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} [2n_2 d_{G_2} v_1] \\
&= 4n_2^4 \sum_{u_1 \in V(G_1)} d_{G_1}^2 u_1 + 4e_2 n_2^2 \sum_{u_1 \in V(G_1)} d_{G_1} u_1 = 4n_2^4 M_1(G_2) + 8e_1 e_2 n_2^2,
\end{aligned}$$

$$\sum 3 = \sum_{v_1 = v_2 = v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R_k(G_1)) \\ u_1 u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) d(u_2, v_2)],$$

TABLE 1: Numerical comparison for generalized network $G(P_n[P_m]_{T_k})$ for $k = 1$.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(5, 5)	4460	22920	10350	80636
(5, 6)	5634	29312	13184	104952
(5, 7)	6808	35704	16018	129268
(5, 8)	7982	42096	18852	153584
(5, 9)	9156	48488	21686	177900
(5, 10)	10330	54880	24520	202216
(6, 5)	7530	45752	17562	162672
(6, 6)	9516	58560	22380	211844
(6, 7)	11502	71368	27198	261016
(6, 8)	13488	841767	32016	310188
(6, 9)	15474	96984	36834	359360
(6, 10)	17460	109792	41652	408532
(7, 5)	11744	823647	27494	294900
(7, 6)	14846	105484	35048	384216
(7, 7)	17948	128604	42602	473532
(7, 8)	21050	151724	50156	562848
(7, 9)	24152	174844	57710	652164
(7, 10)	27254	197964	65264	741480
(8, 5)	17282	137400	40578	494408
(8, 6)	21852	176048	51740	644388
(8, 7)	26422	214696	62902	794368
(8, 8)	30992	253344	74064	944348
(8, 9)	35562	291992	85226	1094328
(8, 10)	40132	330640	96388	1244308
(9, 5)	24324	216176	57246	780780
(9, 6)	30762	277080	73008	1017944
(9, 7)	37200	337984	88770	1255108
(9, 8)	43638	398888	104532	1492272
(9, 9)	50076	459792	120294	1729436
(9, 10)	56514	520696	136056	1966600
(10, 5)	33050	324680	77930	1176096
(10, 6)	41804	416272	99404	1533732
(10, 7)	50558	507864	120878	1891368
(10, 8)	59312	599456	142352	2249004
(10, 9)	68066	691048	163826	2606640
(10, 10)	76820	782640	185300	2964276

$$\sum_{v_1=v_2=v \in V(G_2)} \sum_{u_1 u_2 \in E(G_2)} [d(u_1, v_1)d(u_2, v_2)] = n_2(k-1)e_1[2n_2 \cdot 2n_2] = 4e_1n_2^3(k-1),$$

$$\begin{aligned} \sum 4 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} [d(u_1, v_1)d(u_2, v_2)] \\ &= [2n_2 \times 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + [2n_2 \times 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1 \\ &= 8n_2^2 \sum_{i=1}^{n_2-1} (n_2 - i)(k-2)e_1 + 8n_2^2 \sum_{i=1}^{n_2-1} (n_2 - i)e_1 = 8n_2^2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \quad (18) \end{aligned}$$

So, the final result is

$$\begin{aligned} &= 8n_2^2 e_2 M_1(G_1) + 4e_1 n_2 M_1(G_2) + n_1 M_2(G_2) \\ &\quad + 4n_2^4 M_2(G_1) + 4n_2^4 M_1(G_2) + 8n_2^2 e_1 e_2 + 4e_1 e_2^2 \\ &\quad + 4e_1 n_2^3(k-1) \\ &\quad + 8n_2^2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \end{aligned} \quad (19)$$

□

4. Conclusion

Let $G_1 = P_n$ and $G_2 = P_m$ be two networks; then, we have $n_1 = |V(G_1)| = n$, $n_2 = |V(G_2)| = m$, $e_1 = |E(G_1)| = n-1$, $e_2 = |E(G_2)| = m-1$, $M_1(G_1) = 4n-6$, $M_1(G_2) = 4m-6$, $M_2(G_1) = 4(n-2)$, and $M_2(G_2) = 4(m-2)$.

Applying these values on the above derived results in Theorems 1–4, we obtain Tables 1–4. The graphical

TABLE 2: Numerical comparison for generalized networks $G(P_n[P_m]_{N_k})$ for $k = 2$.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(5, 5)	6660	31120	12350	103136
(5, 6)	8374	39572	15684	129952
(5, 7)	10088	48024	19018	156768
(5, 8)	11802	56476	22352	183584
(5, 9)	13516	64928	25686	210400
(5, 10)	15230	73380	29020	237216
(6, 5)	11226	63368	21018	209328
(6, 6)	14124	80592	26700	263684
(6, 7)	17022	97816	32382	318040
(6, 8)	19920	115040	38064	372396
(6, 9)	22818	132264	43746	426752
(6, 10)	25716	149488	49428	481108
(7, 5)	17512	115796	32982	381336
(7, 6)	22042	147288	41908	480256
(7, 7)	26572	178780	50834	579176
(7, 8)	31102	210272	59760	678096
(7, 9)	35632	241764	68686	777016
(7, 10)	40162	273256	77612	875936
(8, 5)	25794	195448	48770	641864
(8, 6)	32476	248624	61980	808228
(8, 7)	39158	301800	75190	974592
(8, 8)	45840	354976	88400	1140956
(8, 9)	52522	408152	101610	1307320
(8, 10)	59204	461328	114820	1473684
(9, 5)	36348	310424	68910	1016976
(9, 6)	45774	394908	87588	1280384
(9, 7)	55200	479392	106266	1543792
(9, 8)	64626	563876	124944	1807200
(9, 9)	74052	648360	143622	2070608
(9, 10)	83478	732844	162300	2334016
(10, 5)	49450	469880	93930	1536096
(10, 6)	62284	597792	119404	1933732
(10, 7)	75118	725704	144878	2331368
(10, 8)	87952	853616	170352	2729004
(10, 9)	100786	981528	195826	3126640
(10, 10)	113620	1109440	221300	3524276

TABLE 3: Numerical comparison for generalized networks $G(P_n[P_m]_{N_k})$ for $k = 3$.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(5, 5)	8460	39520	14750	100636
(5, 6)	10634	50072	18684	129952
(5, 7)	12808	60624	22618	159268
(5, 8)	14982	71176	26552	188584
(5, 9)	17156	81728	30486	217900
(5, 10)	19330	92280	34420	247216
(6, 5)	14442	81224	25050	204144
(6, 6)	18156	102912	31740	263684
(6, 7)	21870	124600	38430	323224
(6, 8)	25584	146288	45120	382764
(6, 9)	29298	167976	51810	442304
(6, 10)	33012	189664	58500	501844
(7, 5)	22720	149508	39254	371732
(7, 6)	28566	189428	49748	480256
(7, 7)	34412	229348	60242	588780
(7, 8)	40258	269268	70736	697304
(7, 9)	46104	309188	81230	805828
(7, 10)	51950	349108	91724	914352

TABLE 3: Continued.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(8, 5)	33666	253816	57986	625480
(8, 6)	42332	321584	73500	808228
(8, 7)	50998	389352	89014	990976
(8, 8)	59664	457120	104528	1173724
(8, 9)	68330	524888	120042	1356472
(8, 10)	76996	592656	135556	1539220
(9, 5)	47652	405032	81870	990732
(9, 6)	59922	513168	103788	1280384
(9, 7)	72192	621304	125706	1570036
(9, 8)	84462	729440	147624	1859688
(9, 9)	96732	837576	169542	2149340
(9, 10)	109002	945712	191460	2438992
(10, 5)	65050	615480	111530	1496096
(10, 6)	81804	779792	141404	1933732
(10, 7)	98558	944104	171278	2371368
(10, 8)	115312	1108416	201152	2809004
(10, 9)	132066	1272728	231026	3246640
(10, 10)	148820	1437040	260900	3684276

TABLE 4: Numerical comparison for generalized networks $G(P_n[P_m]_{N_k})$ for $k = 4$.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(5, 5)	10460	47920	16750	110636
(5, 6)	13134	60572	21184	142452
(5, 7)	15808	73224	25618	174268
(5, 8)	18482	85876	30052	206084
(5, 9)	21156	98528	34486	237900
(5, 10)	23830	111180	38920	269716
(6, 5)	17898	99080	28506	224880
(6, 6)	22476	125232	36060	289604
(6, 7)	27054	151384	43614	354328
(6, 8)	31632	177536	51168	419052
(6, 9)	36210	203688	58722	483776
(6, 10)	40788	229840	66276	548500
(7, 5)	28208	183220	44742	410148
(7, 6)	35426	231568	56608	528276
(7, 7)	42644	279916	68474	646404
(7, 8)	49862	328264	80340	764532
(7, 9)	57080	376612	92206	882660
(7, 10)	64298	424960	104072	1000788
(8, 5)	41858	312184	66178	691016
(8, 6)	52572	394544	83740	890148
(8, 7)	63286	476904	101302	1089280
(8, 8)	74000	559264	118864	1288412
(8, 9)	84714	641624	136426	1487544
(8, 10)	95428	723984	153988	1686676
(9, 5)	59316	499640	93534	1095708
(9, 6)	74502	631428	118368	1411604
(9, 7)	89688	763216	143202	1727500
(9, 8)	104874	895004	168036	2043396
(9, 9)	120060	1026792	192870	2359292
(9, 10)	135246	1158580	217704	2675188
(10, 5)	81050	761080	127530	1656096
(10, 6)	101804	961792	161404	2133732
(10, 7)	122558	1162504	195278	2611368
(10, 8)	143312	1363216	229152	3089004
(10, 9)	164066	1563928	263026	3566640
(10, 10)	184820	1764640	296900	4044276

FIGURE 5: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $k = 1$.

FIGURE 6: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $k = 2$.

representations of the abovementioned results in tables are shown in Figures 5–8. In Figure 9, we summarize the results of generalized T -sum networks from $k = 1$ to $k = 4$.

For $k = 1$,

- (a) $G_1(P_n[P_m]S) = 8m^3n - 10m^3 + 8m^2n - 8m^2 - 4mn + 8m - 6n$
- (b) $G_2(P_n[P_m]S) = 8m^4n - 12m^4 + 12m^3n - 14m^3 - 4m^2n + 6m^2 + 12m - 8n$
- (c) $G_1(P_n[P_m]R) = 20m^3n - 28m^3 + 16m^2n - 16m^2 - 12mn + 16m - 6n$

$$(d) G_2(P_n[P_m]R) = 32m^4n - 56m^4 + 40m^3n - 56m^3 - 20m^2n + 36m^2 - 36mn + 32m - 4n - 4$$

For $k = 2$,

- (a) $G_1(P_n[P_m]S) = 8m^3n - 10m^3 + 12m^2n - 12m^2 + 4mn + 8m - 6n + (8mn - 8m)\sum_{i=1}^{m-1}(m-i)$
- (b) $G_2(P_n[P_m]S) = 8m^4n - 12m^4 + 12m^3n - 14m^3 + 2m^2 - 8mn + 12m - 8n + (8m^2n - 8m^2)\sum_{i=1}^{m-1}(m-i)$
- (c) $G_1(P_n[P_m]R) = 20m^3n - 28m^3 + 20m^2n - 20m^2 - 12mn + 16m - 6n + (8mn - 8m)\sum_{i=1}^{m-1}(m-i)$

FIGURE 7: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $k = 3$.

FIGURE 8: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $k = 4$.

$$(d) \quad G_2(P_n[P_m]R) = 32m^4n - 56m^4 + 44m^3n - 60m^3 - 20m^2n + 36m^2 - 36mn + 32m - 4n - 4 + (8m^2n - 8m^2)\sum_{i=1}^{m-1}(m-i)$$

For $k = 3$,

$$(a) \quad G_1(P_n[P_m]S) = 8m^3n - 10m^3 + 16m^2n - 16m^2 - 4mn + 8m - 6n + (16mn - 16m)\sum_{i=1}^{m-1}(m-i)$$

$$(b) \quad G_2(P_n[P_m]S) = 8m^4n - 12m^4 + 12m^3n - 14m^3 + 4m^2n - 2m^2 - 8mn + 12m - 8n + (16m^2n - 16m)\sum_{i=1}^{m-1}(m-i)$$

$$(c) \quad G_1(P_n[P_m]R) = 20m^3n - 28m^3 + 28m^2n - 28m^2 - 12mn + 16m - 6n + (16mn - 16m)\sum_{i=1}^{m-1}(m-i)$$

$$(d) \quad G_2(P_n[P_m]R) = 32m^4n - 56m^4 + 48m^3n - 64m^3 - 20m^2n + 36m^2 - 36mn + 32m - 4n - 4 + (16m^2n - 16m)\sum_{i=1}^{m-1}(m-i)$$

For $k = 4$,

$$(a) \quad G_1(P_n[P_m]S) = 8m^3n - 10m^3 + 20m^2n - 20m^2 - 4mn + 8m - 6n + (24mn - 24m)\sum_{i=1}^{m-1}(m-i)$$

FIGURE 9: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $M_2(P_n[P_m]_{R_k})$, where $k \in \{1, 2, 3, 4\}$.

- (b) $G_2(P_n[P_m]S) = 8m^4n - 12m^4 + 12m^3n - 14m^3 + 8m^2n - 6m^2 - 8mn + 12m - 8n + (24m^2n - 24m^2)\text{sum}(m-i)$
- (c) $G_1(P_n[P_m]R) = 20m^3n - 28m^3 + 32m^2n - 32m^2 - 12mn + 16m - 6n + (24mn - 24m)\text{sum}(m-i)$
- (d) $G_2(P_n[P_m]R) = 32m^4n - 56m^4 + 52m^3n - 68m^3 - 20m^2n + 36m^2 - 36mn + 32m - 4n - 4 + (24m^2n - 24m^2)\text{sum}(m-i)$

In this paper, we proved the result of 1st and 2nd Zagreb indices of networks $G_1[G_2]_{T_k}$, by using generalized subdivision and generalized semitotal point operation and lexicographic product. The first and second Zagreb indices are in the form $M_i(G_1[G_2]_{T_k})$, where $(i = 1, 2)$. We plot the network of these derived exact values individually and after that plot a combine network for $k \in \{1, 2, 3, 4\}$ to compare these results. In this result, it is clearly shown that the 2nd Zagreb indices $M_2(G_1[G_2]_{T_k})$ shows better result as compared to others' exact values.

Data Availability

All the data are included within this paper. However, more details of the data can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors have no conflicts of interest.

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