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Research Article

One Kind Special Gauss Sums and their Mean Square Values

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In this paper, we introduce one kind special Gauss sums; then, using the elementary and analytic methods to study the mean value properties of these kind sums, we obtain several exact calculating formulae for them.

1. Introduction

Let q and k be two positive integers with $q \geq 3$. For any integer $a \geq 1$ and Dirichlet character $\chi \bmod q$, we write $R_k(a) = [\sqrt[k]{a}]$, where $[x]$ denotes the greatest integer less than or equal to x. Then, we define the summations $G(n, \chi, k; q)$ and $H(n, \chi, k; q)$ as

$$G(n, \chi, k; q) = \sum_{a=1}^{q^k - 1} \chi(R_k(a)) e\left(\frac{na}{q}\right),$$
$$H(n, \chi, k; q) = \sum_{a=1}^{q^k - 1} \chi(R_k(a)) e\left(\frac{n\left(a - R_k^k(a)\right)}{q}\right), \tag{1}$$

where $e(y) = e^{2\pi i y}$ and n is any integer.

It is clear that $G(n, \chi, k; q)$ is a generalization of the classical Gauss sums. In fact, if $k = 1$, then we have $R_1(a) = a$ and

$$G(n, \chi, 1; q) = \sum_{a=1}^{q-1} \chi(a) e\left(\frac{na}{q}\right) = G(n, \chi; q). \tag{2}$$

That is, $G(n, \chi, 1; q)$ becomes the classical Gauss sums $G(n, \chi; q)$. Gauss sums play a very important role in the study of analytic number theory, and many number theory problems are closely related to it, so some scholars have studied the properties of the classical Gauss sums and obtained many meaningful and interesting results; some of them can be found in [1–14]. For this reason, we think that it is necessary and meaningful to study the properties of $G(n, \chi, k; q)$.

On the contrary, $H(n, \chi, 2; q)$ is related to Dirichlet L-function $L(1, \chi)$. In fact, if p is an odd prime, $\chi(-1) = -1$ and $n = 0$, then we have

$$|H(0, \chi, 2; p)| = 2 \cdot \left|\sum_{m=1}^{p-1} m\chi(m)\right| = \frac{2 \cdot p^{3/2}}{\pi} \cdot |L(1, \chi)|. \tag{3}$$

Therefore, the study of the sums $G(n, \chi, n; q)$ and $H(n, \chi, n; q)$ has extensive theoretical significance and application values.

In this study, as an attempt in this direction, we first study the mean value properties of $G(n, \chi, k; q)$ and $H(n, \chi, k; q)$. We shall use the elementary and analytic methods to prove the following several results.

Theorem 1. *Let p be an odd prime. Then, for any integer n with $(n, p) = 1$, we have the identity*

$$\sum_{\chi \bmod p} \left|\sum_{a=1}^{p^2 - 1} \chi(R_2(a)) e\left(\frac{na}{p}\right)\right|^2 = \frac{p(p-1)}{2\sin^2(\pi n/p)} - (p-1). \tag{4}$$

Theorem 2. *Let p be an odd prime. Then, we have the identity*

$$\frac{1}{p-1}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p^3-1}\chi(R_3(a))e\left(\frac{na}{p}\right)\right|^2=\begin{cases}\dfrac{p-\sqrt{p}\cdot\cos(\pi\cdot\overline{2}n/p)}{2\sin^2(\pi n/p)}-1, & \text{if } p\equiv 1 \bmod 12,\\[2ex] \dfrac{p+\sqrt{p}\cdot\cos(\pi\cdot\overline{2}n/p)}{2\sin^2(\pi n/p)}-1, & \text{if } p\equiv 5 \bmod 12,\\[2ex] \dfrac{p-\sqrt{p}\cdot\sin(\pi\cdot\overline{2}n/p)}{2\sin^2(\pi n/p)}-1, & \text{if } p\equiv 7 \bmod 12,\\[2ex] \dfrac{p+\sqrt{p}\cdot\sin(\pi\cdot\overline{2}n/p)}{2\sin^2(\pi n/p)}-1, & \text{if } p\equiv 11 \bmod 12,\end{cases} \tag{5}$$

where $\overline{a}$ denotes the solution of the congruence equation $ax\equiv 1 \bmod p$.

Theorem 3. *Let p be an odd prime. Then, for any nonprincipal character $\chi \bmod p$, we have the identities*

$$\begin{aligned}\sum_{n=1}^{p-1}\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n(a-R_2^2(a))}{p}\right)\right|^2&=\frac{p(p^2-1)}{12},\\ \sum_{n=1}^{p-1}\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n(a-R_2^2(a))}{p}\right)\right|^4&=\frac{p^2(p^2+11)(p^2-1)}{720},\\ \sum_{n=1}^{p-1}\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n(a-R_2^2(a))}{p}\right)\right|^6&=\frac{p^3(2p^2-11)(p^2+17)(p^2-1)}{60480}.\end{aligned} \tag{6}$$

From Theorems 1 and 2, we can also deduce the following corollaries:

Corollary 1. *Let p be an odd prime. Then, we have the asymptotic formula*

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{a}{p}\right)\right|^2=\frac{p^4}{2\cdot\pi^2}+O(p^3). \tag{7}$$

Corollary 2. *Let p be an odd prime. Then, we have the asymptotic formula*

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p^2-1}\chi(R_3(a))e\left(\frac{a}{p}\right)\right|^2=\frac{p^4}{2\cdot\pi^2}+O(p^{7/2}). \tag{8}$$

Notes. In fact, for any integer $h\geq 4$, using our methods and the results in [15], we can give an exact calculating formula for the general 2hth power mean:

$$\sum_{n=1}^{p-1}\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n(a-R_2^2(a))}{p}\right)\right|^{2h}. \tag{9}$$

However, when h is large, the calculation is more complicated, so we do not consider it.

2. Several Lemmas

This section, we need to give some simple lemmas, which are necessary in the proofs of our theorems. Of course, the proofs of these lemmas also need some knowledge of elementary and analytic number theory, in particular, the properties of the character sums and the classical Gauss sums modulo p. All these can be found in [16, 17], we do not repeat them. First, we have the following.

Lemma 1. *Let p be an odd prime and $k\geq 2$ be a fixed integer. Then, for any nonprincipal Dirichlet character $\chi \bmod p$, we have the identity*

$$\sum_{a=1}^{p^k-1}\chi(R_k(a))e\left(\frac{na}{p}\right)=\sum_{m=1}^{p-1}\chi(m)\cdot\frac{e\left(n(m+1)^k/p\right)-e\left(nm^k/p\right)}{e(n/p)-1},$$
$$\sum_{a=1}^{p^k-1}\chi(R_k(a))e\left(\frac{n\left(a-R_k^k(a)\right)}{p}\right)=\sum_{m=1}^{p-1}\chi(m)\cdot\frac{e\left(n\left((m+1)^k-m^k\right)/p\right)-1}{e(n/p)-1}, \tag{10}$$
$$\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n\left(a-R_2^2(a)\right)}{p}\right)=\frac{\overline{\chi}(2n)e(n/p)\tau(\chi)}{e(n/p)-1},$$

where $\tau(\chi)=\sum_{a=1}^{p-1}\chi(a)e(a/p)$ *denotes the classical Gauss sums.*

Proof. First, we prove the first formula. Similarly, we can deduce the second one. For any positive integer a, it is clear that there are two integers $m\geq 1$ and $0\leq i\leq (m+1)^k-m^k-1$ such that $a=m^k+i$. This number pair (m,i) not only exists, but it is unique. In fact, we have $m=[\sqrt[k]{a}]$ and $0\leq i=a-[\sqrt[k]{a}]^k\leq (m+1)^k-1-m^k$, where $[x]$ denotes the greatest integer $\leq x$. From these results and the properties of geometric series, we have

$$\begin{aligned}\sum_{a=1}^{p^k-1}\chi(R_k(a))e\left(\frac{na}{p}\right)&=\sum_{m=1}^{p-1}\sum_{i=0}^{(m+1)^k-m^k-1}\chi(m)e\left(\frac{n\left(m^k+i\right)}{p}\right)\\&=\sum_{m=1}^{p-1}\chi(m)e\left(\frac{nm^k}{p}\right)\sum_{i=0}^{(m+1)^k-m^k-1}e\left(\frac{ni}{p}\right)\\&=\sum_{m=1}^{p-1}\chi(m)e\left(\frac{nm^k}{p}\right)\frac{e\left(n\left((m+1)^k-m^k\right)/p\right)-1}{e(n/p)-1}\\&=\sum_{m=1}^{p-1}\chi(m)\frac{e\left(n(m+1)^k/p\right)-e\left(nm^k/p\right)}{e(n/p)-1}.\end{aligned} \tag{11}$$

Now, we prove the third one. Taking $k=2$ in the second formula, note that $(m+1)^2-m^2=2m+1$ and $\sum_{m=1}^{p-1}\chi(m)=0$, from the properties of the classical Gauss sums, and we have

$$\begin{aligned}\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n\left(a-R_2^2(a)\right)}{p}\right)&=\sum_{m=1}^{p-1}\chi(m)\cdot\frac{e(n(2m+1)/p)-1}{e(n/p)-1}\\&=\frac{1}{e(n/p)-1}\left[e\left(\frac{n}{p}\right)\sum_{m=1}^{p-1}\chi(m)e\left(\frac{2nm}{p}\right)-\sum_{m=1}^{p-1}\chi(m)\right]=\frac{\overline{\chi}(2n)e(n/p)\tau(\chi)}{e(n/p)-1}.\end{aligned} \tag{12}$$

This proves Lemma 1. ☐

Lemma 2. *Let p be an odd prime. Then, we have the identity*

$$\sum_{m=1}^{p-1}\left|\frac{e\left((m+1)^3/p\right)-e\left(m^3/p\right)}{e(1/p)-1}\right|^2 = \frac{2p-\chi_2(3)\cdot e(\overline{4}/p)\cdot\tau(\chi_2)-\chi_2(-3)\cdot e(-\overline{4}/p)\cdot\tau(\chi_2)}{4\sin^2(\pi/p)}-1,$$

$$\sum_{m=1}^{p-1}\left|\frac{e\left((m+1)^2/p\right)-e\left(m^2/p\right)}{e(1/p)-1}\right|^2 = \frac{p}{2\sin^2(\pi/p)}-1. \tag{13}$$

Proof. First, for any integer n with $(n,p)=1$, from the properties of the quadratic Gauss sums, we have

$$\sum_{m=0}^{p-1} e\left(\frac{nm^2}{p}\right) = 1+\sum_{m=1}^{p-1}\left(1+\chi_2(m)\right)e\left(\frac{nm}{p}\right) = \left(\frac{n}{p}\right)\cdot\tau(\chi_2), \tag{14}$$

where $\tau^2(\chi_2)=\chi_2(-1)\cdot p$ and $\chi_2=(*/p)$ denotes Legendre's symbol modulo p.

From (14) and the properties of the complete residue system modulo p, we have

$$\begin{aligned}\sum_{m=1}^{p-1} e\left(\frac{3m^2+3m+1}{p}\right) &= -e\left(\frac{1}{p}\right)+\sum_{m=0}^{p-1} e\left(\frac{3m^2+3m+1}{p}\right)\\ &= -e\left(\frac{1}{p}\right)+\sum_{m=0}^{p-1} e\left(\frac{3\cdot\overline{4}(2m+1)^2+\overline{4}}{p}\right) = -e\left(\frac{1}{p}\right)+\sum_{m=0}^{p-1} e\left(\frac{3\cdot\overline{4}\cdot m^2+\overline{4}}{p}\right)\\ &= -e\left(\frac{1}{p}\right)+\chi_2(3\cdot\overline{4})\cdot e\left(\frac{\overline{4}}{p}\right)\cdot\tau(\chi_2) = -e\left(\frac{1}{p}\right)+\chi_2(3)\cdot e\left(\frac{\overline{4}}{p}\right)\cdot\tau(\chi_2).\end{aligned} \tag{15}$$

Applying formula (15), we may immediately deduce

$$\begin{aligned}\sum_{m=1}^{p-1}\left|\frac{e\left((m+1)^3/p\right)-e\left(m^3/p\right)}{e(1/p)-1}\right|^2 &= \sum_{m=1}^{p-1}\left|\frac{e\left((3m^2+3m+1)/p\right)-1}{e(1/p)-1}\right|^2\\ &= \frac{1}{4\sin^2(\pi/p)}\cdot\sum_{m=1}^{p-1}\left(2-e\left(\frac{3m^2+3m+1}{p}\right)-e\left(\frac{-\left(3m^2+3m+1\right)}{p}\right)\right)\\ &= \frac{1}{4\sin^2(\pi/p)}\cdot\left[2p-2-\sum_{m=1}^{p-1} e\left(\frac{3m^2+3m+1}{p}\right)-\sum_{m=1}^{p-1} e\left(\frac{-\left(3m^2+3m+1\right)}{p}\right)\right]\\ &= \frac{2p-2+e(1/p)+e(-1/p)-\chi_2(3)\cdot e(\overline{4}/p)\cdot\tau(\chi_2)-\chi_2(-3)\cdot e(-\overline{4}/p)\cdot\tau(\chi_2)}{4\sin^2(\pi/p)}\\ &= \frac{2p-\chi_2(3)\cdot e(\overline{4}/p)\cdot\tau(\chi_2)-\chi_2(-3)\cdot e(-\overline{4}/p)\cdot\tau(\chi_2)}{4\sin^2(\pi/p)}-1.\end{aligned} \tag{16}$$

This proves the first formula in Lemma 2.

Similarly, note that the identity

$$\sum_{m=0}^{p-1} e\left(\frac{nm}{p}\right) = \begin{cases} p, & \text{if } (n, p) = p, \\ 0, & \text{if } (n, p) = 1, \end{cases} \tag{17}$$

and we also have

$$\begin{aligned} \sum_{m=1}^{p-1} \left| \frac{e\left((m+1)^2/p\right) - e\left(m^2/p\right)}{e(1/p) - 1} \right|^2 &= \sum_{m=1}^{p-1} \left| \frac{e(2m+1/p) - 1}{e(1/p) - 1} \right|^2 \\ &= \frac{1}{4\sin^2(\pi/p)} \cdot \sum_{m=1}^{p-1} \left(2 - e\left(\frac{2m+1}{p}\right) - e\left(\frac{-2m-1)}{p}\right)\right) \\ &= \frac{1}{4\sin^2(\pi/p)} \cdot \left[2p - 2 - \sum_{m=1}^{p-1} e\left(\frac{2m+1}{p}\right) - \sum_{m=1}^{p-1} e\left(\frac{-2m-1}{p}\right)\right] \\ &= \frac{1}{4\sin^2(\pi/p)} \cdot \left[2p - 2 + e\left(\frac{1}{p}\right) + e\left(\frac{-1}{p}\right)\right] = \frac{p}{2\sin^2(\pi/p)} - 1. \end{aligned} \tag{18}$$

This proves Lemma 2. □

Lemma 3. *Let $q \geq 3$ be an integer. Then, we have the identities*

$$\begin{aligned} \sum_{a=1}^{q-1} \frac{1}{\sin^2(\pi a/q)} &= \frac{q^2 - 1}{3}, \\ \sum_{a=1}^{q-1} \frac{1}{\sin^4(\pi a/q)} &= \frac{\left(q^2 + 11\right)\left(q^2 - 1\right)}{45}, \\ \sum_{a=1}^{q-1} \frac{1}{\sin^6(\pi a/q)} &= \frac{\left(2q^2 - 11\right)\left(q^2 + 17\right)\left(q^2 - 1\right)}{945}. \end{aligned} \tag{19}$$

Proof. See [18] or Corollary 1 in [15]. □

3. Proofs of the Theorems

Applying three simple lemmas in Section 2, we can easily complete the proofs of our theorems. First, we prove Theorem 1. From the orthogonality of the characters modulo p and Lemma 1 with $k = 2$, we have

$$\begin{aligned} &\frac{1}{p-1} \sum_{\chi \bmod p} \left| \sum_{a=1}^{p^2-1} \chi\left(R_2(a)\right) e\left(\frac{na}{p}\right) \right|^2 \\ &= \frac{1}{4\sin^2(\pi n/p)} \cdot \sum_{m=1}^{p-1} \left| e\left(\frac{n(m+1)^2}{p}\right) - e\left(\frac{nm^2}{p}\right) \right|^2 = \frac{p}{2\sin^2(\pi n/p)} - 1. \end{aligned} \tag{20}$$

This proves Theorem 1.

Proof of Theorem 2. From the orthogonality of the characters modulo p and Lemma 1 with $k=3$, we have

$$\begin{aligned}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p^3-1}\chi(R_3(a))e\left(\frac{na}{p}\right)\right|^2 &= \frac{p-1}{4\sin^2(\pi n/p)}\cdot\sum_{m=1}^{p-1}\left|e\left(\frac{n(m+1)^3}{p}\right)-e\left(\frac{nm^3}{p}\right)\right|^2\\ &= (p-1)\cdot\left[\frac{2p-\chi_2(3)\cdot e(\overline{4}n/p)\cdot\tau(\chi_2)-\chi_2(-3)\cdot e(-\overline{4}n/p)\cdot\tau(\chi_2)}{4\sin^2(\pi n/p)}-1\right].\end{aligned} \tag{21}$$

If $p\equiv 1 \bmod 12$, then $\chi_2(3)=\chi_2(-1)=1$ and $\tau(\chi_2)=\sqrt{p}$. From (21), we have

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p^3-1}\chi(R_3(a))e\left(\frac{na}{p}\right)\right|^2 = \frac{(p-1)(p-\sqrt{p}\cdot\cos(\pi\cdot\overline{2}n/p))}{2\sin^2(\pi n/p)}-(p-1). \tag{22}$$

If $p\equiv 5 \bmod 12$, then $\chi_2(3)=-1$, $\chi_2(-1)=1$, and $\tau(\chi_2)=\sqrt{p}$. From (21), we have

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p^3-1}\chi(R_3(a))e\left(\frac{na}{p}\right)\right|^2 = \frac{(p-1)(p+\sqrt{p}\cdot\cos(\pi\cdot\overline{2}n/p))}{2\sin^2(\pi n/p)}-(p-1). \tag{23}$$

If $p\equiv 7 \bmod 12$, then $\chi_2(3)=-1$, $\chi_2(-1)=-1$, and $\tau(\chi_2)=i\cdot\sqrt{p}$, $i^2=-1$. From (21), we have

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p^3-1}\chi(R_3(a))e\left(\frac{na}{p}\right)\right|^2 = \frac{(p-1)(p-\sqrt{p}\cdot\sin(\pi\cdot\overline{2}n/p))}{2\sin^2(\pi n/p)}-(p-1). \tag{24}$$

If $p\equiv 11 \bmod 12$, then $\chi_2(3)=1$, $\chi_2(-1)=-1$, and $\tau(\chi_2)=i\cdot\sqrt{p}$. From (21), we have

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p^3-1}\chi(R_3(a))e\left(\frac{na}{p}\right)\right|^2 = \frac{(p-1)(p+\sqrt{p}\cdot\sin(\pi\cdot\overline{2}n/p))}{2\sin^2(\pi n/p)}-(p-1). \tag{25}$$

Now, Theorem 2 follows from (22), (23), (24), and (25).

Now, we prove Theorem 3. For any nonprincipal character modulo p, note that $|\tau(\chi)|=\sqrt{p}$; from the third formula in Lemma 1, we have

$$\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n(a-R_2^2(a))}{p}\right)\right|^2 = \frac{p}{4\sin^2(\pi n/p)}. \tag{26}$$

From (26) and Lemma 3, we have

$$\sum_{n=1}^{p-1}\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n(a-R_2^2(a))}{p}\right)\right|^2 = \sum_{n=1}^{p-1}\frac{p}{4\sin^2(\pi n/p)} = \frac{p(p^2-1)}{12}. \tag{27}$$

Similarly, from (26) and Lemma 3, we also have

$$\sum_{n=1}^{p-1}\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n(a-R_2^2(a))}{p}\right)\right|^4 = \sum_{n=1}^{p-1}\frac{p^2}{16\sin^4(\pi n/p)} = \frac{p^2(p^2+11)(p^2-1)}{720}, \tag{28}$$

$$\sum_{n=1}^{p-1}\left|\sum_{a=1}^{p^2-1}\chi(R_2(a))e\left(\frac{n(a-R_2^2(a))}{p}\right)\right|^6 = \sum_{n=1}^{p-1}\frac{p^3}{64\cdot\sin^6(\pi n/p)} = \frac{p^3(2p^2-11)(p^2+17)(p^2-1)}{60480}. \tag{29}$$

Now, Theorem 3 follows from (27)– (29).

This completes the proofs of all our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

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