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Research Article

Entire Solutions for Complex Systems of the Second-Order Partial Differential Difference Equations of Fermat Type

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This article is mainly concerned with the existence and the forms of entire solutions for several systems of the second-order partial differential difference equations of Fermat type $\begin{cases} (\alpha(\partial^2 f_1(z_1,z_2)/\partial z_1^2)+\beta(\partial^2 f_1(z_1,z_2)/\partial z_2^2))^{n_1}+f_2(z_1+c_1,z_2+c_2)^{m_1}=1 \\ (\alpha(\partial^2 f_2(z_1,z_2)/\partial z_1^2)+\beta(\partial^2 f_2(z_1,z_2)/\partial z_2^2))^{n_2}+f_1(z_1+c_1,z_2+c_2)^{m_2}=1 \end{cases}$ and $\begin{cases} (\partial^2 f_1(z_1,z_2)/\partial z_1^2)^2+f_2(z_1+c_1,z_2+c_2)^2=1 \\ (\partial^2 f_2(z_1,z_2)/\partial z_1^2)^2+f_1(z_1+c_1,z_2+c_2)^2=1 \end{cases}$. Our results about the existence and the forms of solutions for these systems generalize the previous theorems given by Xu and Cao, Gao, Liu, and Yang. In addition, we give some examples to explain the existence of solutions of this system in each case.

1. Introduction and Main Results

The issue on the existence and form of solutions for Fermat-type equation $x^m+y^n=1$ has attracted considerable attention from many scholars. Especially, Taylor and Wiles [1, 2] pointed out that this equation does not admit a nontrivial solution in rational numbers for $m=n\geq 3$, and this equation does admit a nontrivial rational solution for $m=n=2$. In fact, the study of this issue should go back to sixty years ago or even earlier; Montel [3] and Gross [4] had pointed out that the entire solutions of the functional equation $f^m+g^n=1$ for $m=n=2$ are $f=\cos a(z)$, $g=\sin a(z)$, where $a(z)$ is an entire function; for $m=n>2$, there are no nonconstant entire solutions.

In 2004, Yang and Li [5] investigated a certain nonlinear differential equational of Malmquist type, by making use of Nevanlinna theory, and obtained the following.

Theorem 1 (See [5]). *Let a_1, a_2, and a_3 be nonzero meromorphic functions. Then, a necessary condition for the differential equation,*

$$a_1 f^2 + a_2 f'^2 = a_3, \tag{1}$$

to have a transcendental meromorphic solution satisfying $T(r,a_k)=S(r,f)$, $k=1,2,3$ is $(a_1/a_3)\equiv$ constant.

In the past ten years, Liu and his collaborators investigated the existence of solutions for a series of complex difference equations and complex differential difference equations of Fermat type, by using the difference Nevanlinna theory for meromorphic functions (see [6–8]), and obtained a lot of interesting original results (see [9–11]). In order to be consistent with the following text, here, we only list one of results are given by Liu.

Theorem 2 (See [10], Theorem 9). *ie transcendental entire solutions with finite order of*

$$f'(z)^2 + f(z+c)^2 = 1, \tag{2}$$

must satisfy $f(z)=\sin(z\pm Bi)$, where B is a constant and $c=2k\pi$ or $c=(2k+1)\pi$, where k is an integer.

In 2019, Liu and Gao [12] further studied the entire solutions of the second-order differential and difference equation with single complex variable and obtained the following.

Theorem 3 (see [12], Theorem 2.1). *Suppose that f is a transcendental entire solution with finite order of the complex differential difference equation*

$$f''(z)^2 + f(z+c)^2 = Q(z). \tag{3}$$

Then, $Q(z) = c_1 c_2$ is a constant, and $f(z)$ satisfies

$$f(z) = \frac{c_1 e^{az+b} + c_2 e^{-az-b}}{2a^2}, \tag{4}$$

where $a, b \in \mathbb{C}$ and $a^4 = 1, c = (\log(-ia^2) + 2k\pi i/a)$, where $k \in \mathbb{Z}$.

In 2016, Gao [13] further investigated the form of solutions for a class of system of differential difference equations corresponding to Theorem 2 and obtained the following.

Theorem 4 (See [13], Theorem 7). *Suppose that (f_1, f_2) is a pair of finite-order transcendental entire solutions for the system of differential difference equations*

$$\begin{cases} [f_1'(z)]^2 + f_2(z+c)^2 = 1, \\ [f_2'(z)]^2 + f_1(z+c)^2 = 1. \end{cases} \tag{5}$$

Then, (f_1, f_2) satisfies

$$\begin{aligned} &(f_1(z), f_2(z)) = (\sin(z - bi), \sin(z - b_1 i)), \\ \text{or } &(f_1(z), f_2(z)) = (\sin(z + bi), \sin(z + b_1 i)), \end{aligned} \tag{6}$$

where b, b_1 are constants and $c = k\pi$, where k is a integer.

For the differential difference equations with several complex variables, Xu and Cao [14, 15] recently investigated the existence of the entire and meromorphic solutions for some Fermat-type partial differential difference equations by using the Nevanlinna theory in several complex variables and obtained the following theorems.

Theorem 5 (See [14], Theorem 7). *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$. Then, the Fermat-type partial differential difference equation,*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^n + f(z_1 + c_1, z_2 + c_2)^m = 1, \tag{7}$$

does not have any transcendental entire solution with finite order, where m and n are two distinct positive integers.

Theorem 6 (See [14], Theorem 8). *Let $c = (c_1, c_2)$ be a constant in $\mathbb{C}^2$. Then, any transcendental entire solution with finite order of the partial differential difference equation,*

$$\left(\frac{\partial f(z_1, z_2)}{\partial z_1}\right)^2 + f(z_1 + c_1, z_2 + c_2)^2 = 1, \tag{8}$$

has the form of $f(z_1, z_2) = \sin(Az_1 + B)$, where A is a constant on $\mathbb{C}$ satisfying $Ae^{iAc_1} = 1$ and B is a constant on $\mathbb{C}$; in the special case, whenever $c_1 = 0$, we have $f(z_1, z_2) = \sin(z_1 + B)$.

Inspired by the form of the abovementioned equations in Theorems 3–6, a question naturally arises: *What will happen about the existence and the form of the solutions when the equations are put into the system and the first-order partial differential is replaced by the second-order partial differentials*? For nearly two decades, although there were a lot of important and meaningful results focusing on the solutions of the complex difference equation of single variable (including [8, 16–21]), as far as we all know, there are few literature about the system of the second-order partial differential difference equations of Fermat type in several complex variables. *It seems that this topic has never been treated before.*

The purpose of this article is concerned with the properties of the solutions for some Fermat-type systems including both the difference operator and the second-order partial differential by making use of the (difference) Nevanlinna theory of several complex variables [22, 23]. We give the existence theorem and the forms of solutions for the Fermat-type systems of the second-order partial differential difference equations, which are generalization of the previous theorems given by Liu, Liu et al., Gao, Xu and Cao, and Xu et al. [9, 10, 13, 14, 24]. Here and below, let $z + w = (z_1 + w_1, z_2 + w_2)$ for any $z = (z_1, z_2), w = (w_1, w_2)$, and $c = (c_1, c_2)$. Now, our main results of this paper are listed as follows.

Theorem 7. *Let $c = (c_1, c_2) \in \mathbb{C}^2$, and $m_j, n_j\ (j = 1, 2)$ be positive integers, and α, β be constants in $\mathbb{C}$ that are not zero at the same time. If the following system of Fermat-type partial differential difference equations,*

$$\begin{cases} \left(\alpha \dfrac{\partial^2 f_1(z_1, z_2)}{\partial z_1^2} + \beta \dfrac{\partial^2 f_1(z_1, z_2)}{\partial z_2^2}\right)^{n_1} + f_2(z_1 + c_1, z_2 + c_2)^{m_1} = 1, \\ \left(\alpha \dfrac{\partial^2 f_2(z_1, z_2)}{\partial z_1^2} + \beta \dfrac{\partial^2 f_2(z_1, z_2)}{\partial z_2^2}\right)^{n_2} + f_1(z_1 + c_1, z_2 + c_2)^{m_2} = 1, \end{cases} \tag{9}$$

satisfies one of the conditions

(i) $m_1 m_2 > n_1 n_2$

(ii) $m_j > (n_j / n_j - 1)$ for $n_j \geq 2$, $j = 1, 2$, then system (9) does not have any pair of transcendental entire solution with finite order.

Remark 1. Here, (f, g) is called as a pair of finite-order transcendental entire solutions for the system

$$\begin{cases} f^{n_1} + g^{m_1} = 1, \\ f^{n_2} + g^{m_2} = 1, \end{cases} \tag{10}$$

if f, g are transcendental entire functions and $\rho = \max\{\rho(f), \rho(g)\} < \infty$.

Remark 2. By observing the proof of Theorem 7, it is easy to see that the conclusions of Theorem 7 also hold if $(\partial^2 f_j(z_1, z_2)/\partial z_1^2)$ or $(\partial^2 f_j(z_1, z_2)/\partial z_2^2)$ of system (9) is replaced by $(\partial^2 f_j(z_1, z_2)/\partial z_1 \partial z_2)$, $j = 1, 2$.

Theorem 8. *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then, any pair of transcendental entire solution (f_1, f_2) with finite order for the system of Fermat-type difference equations,*

$$\begin{cases} \left(\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}\right)^2 + f_2(z_1+c_1, z_2+c_2)^2 = 1, \\ \left(\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}\right)^2 + f_1(z_1+c_1, z_2+c_2)^2 = 1, \end{cases} \tag{11}$$

is of the following forms:

$$(f_1(z), f_2(z)) = (\pm\sin(L(z)+B_0), \pm\cos(L(z)+B_0)), \tag{12}$$

or

$$(f_1(z), f_2(z)) = (\pm\cos(L(z)+B_0), \pm\cos(L(z)+B_0)), \tag{13}$$

where $L(z) = a_1z_1 + a_2z_2$, $a_1^4 = 1$, $e^{2iL(c)} = \pm 1$, and B_0 is a constant in $\mathbb{C}$.

The following examples show the existence of solutions for system (11).

Example 1. Let $a = (a_1, a_2) = (1, \pi)$, $c = (c_1, c_2) = (\pi, 1)$, $L(z) = z_1 + \pi z_2$, and

$$(f_1(z), f_2(z)) = (-\sin(L(z)+B_0), -\cos(L(z)+B_0)), \tag{14}$$

where B_0 is a constant in $\mathbb{C}$. Thus, (f_1, f_2) satisfies the following system:

$$\begin{cases} \left(\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}\right)^2 + f_2(z_1+\pi, z_2+1)^2 = 1, \\ \left(\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}\right)^2 + f_1(z_1+\pi, z_2+1)^2 = 1. \end{cases} \tag{15}$$

Example 2. Let $a = (a_1, a_2) = (-i, 1)$, $c = (c_1, c_2) = (-(\pi/2), \pi)$, $L(z) = -iz_1 + z_2$, and

$$(f_1(z), f_2(z)) = (-\cos(L(z)+B_0), \cos(L(z)+B_0)), \tag{16}$$

where B_0 is a constant in $\mathbb{C}$. Thus, (f_1, f_2) satisfies the following system:

$$\begin{cases} \left(\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}\right)^2 + f_2\left(z_1 - \dfrac{\pi}{2}, z_2+\pi\right)^2 = 1, \\ \left(\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}\right)^2 + f_1\left(z_1 - \dfrac{\pi}{2}, z_2+\pi\right)^2 = 1. \end{cases} \tag{17}$$

Theorem 9. *Let $c = (c_1, c_2) \in \mathbb{C}^2$. Then, any pair of transcendental entire solution (f_1, f_2) with finite order for the system of Fermat-type partial differential difference equations,*

$$\begin{cases} \left(\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_2(z_1+c_1, z_2+c_2)^2 = 1, \\ \left(\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_1(z_1+c_1, z_2+c_2)^2 = 1, \end{cases} \tag{18}$$

is of the following forms:

$$(f_1(z), f_2(z)) = (\pm \sin(L(z)+B_0), \pm \cos(L(z)+B_0)), \tag{19}$$

or

$$(f_1(z), f_2(z)) = (\pm \cos(L(z)+B_0), \pm \cos(L(z)+B_0)), \tag{20}$$

where $L(z) = a_1z_1 + a_2z_2$, $[a_1(a_1+a_2)]^2 = 1$, $e^{2iL(c)} = 1$, and B_0 is a constant in $\mathbb{C}$.

We also list two examples to exhibit the existence of solutions for system (18).

Example 3. Let $a = (a_1, a_2) = (\sqrt{2} - 1, 2)$, $c = (c_1, c_2) = (-(\sqrt{2}+1)\pi, \pi)$, $L(z) = (\sqrt{2} - 1)z_1 + 2z_2$, and

$$(f_1(z), f_2(z)) = (\sin(L(z)+B_0), -\cos(L(z)+B_0)), \tag{21}$$

where B_0 is a constant in $\mathbb{C}$. Thus, (f_1, f_2) satisfies the following system:

$$\begin{cases} \left(\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_2(z_1 - (\sqrt{2}+1)\pi, z_2+\pi)^2 = 1, \\ \left(\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_1(z_1 - (\sqrt{2}+1)\pi, z_2+\pi)^2 = 1. \end{cases} \tag{22}$$

Example 4. Let $a = (a_1, a_2) = (-1, 2)$, $c = (c_1, c_2) = (\pi, (\pi/4))$, $L(z) = -z_1 + 2z_2$, and

$$(f_1(z), f_2(z)) = (\cos(L(z) + B_0), \cos(L(z) + B_0)), \tag{23}$$

where B_0 is a constant in $\mathbb{C}$. Thus, (f_1, f_2) satisfies the following system:

$$\begin{cases} \left(\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_2\left(z_1 + \pi, z_2 + \dfrac{\pi}{4}\right)^2 = 1, \\ \left(\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_1\left(z_1 + \pi, z_2 + \dfrac{\pi}{4}\right)^2 = 1. \end{cases} \tag{24}$$

Remark 3. In fact, in view of the proofs of Theorems 8 and 9, it is easy to get that the conclusions of Theorem 9 still holds, if the system (18) is replaced by the following systems:

$$\begin{cases} \left(\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_2(z_1 + c_1, z_2 + c_2)^2 = 1, \\ \left(\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_1(z_1 + c_1, z_2 + c_2)^2 = 1, \end{cases} \tag{25}$$

or

$$\begin{cases} \left(\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_2(z_1 + c_1, z_2 + c_2)^2 = 1, \\ \left(\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2} + \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1 \partial z_2}\right)^2 + f_1(z_1 + c_1, z_2 + c_2)^2 = 1, \end{cases} \tag{26}$$

respectively. The only thing that we need to do is to modify the condition $[a_1(a_1 + a_2)]^2 = 1$ to $(a_1 a_2)^2 = 1$ or $[a_2(a_1 + a_2)]^2 = 1$, respectively.

2. Proof of Theorem 7

To prove Theorem 7, we need the following lemmas.

Lemma 1 (See [25, 26]). *Let f be a nonconstant meromorphic function on $\mathbb{C}^n$, and let $I = (i_1, \ldots, i_n)$ be a multi-index with length $|I| = \sum_{j=1}^{n} i_j$. Assume that $T(r_0, f) \geq e$ for some r_0. Then,*

$$m\left(r, \frac{\partial^I f}{f}\right) = S(r, f), \tag{27}$$

holds for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ of finite logarithmic measure $\int_E dt/t < \infty$, where $\partial^I f = (\partial^{|I|} f / (\partial z_1^{i_1}, \ldots, \partial z_n^{i_n}))$.

Lemma 2 (See [22, 23]). *Let f be a nonconstant meromorphic function with finite order on $\mathbb{C}^n$ such that $f(0) \neq 0, \infty$, and let $\varepsilon > 0$. Then, for $c \in \mathbb{C}^n$,*

$$m\left(r, \frac{f(z)}{f(z+c)}\right) + m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f), \tag{28}$$

holds for all $r \geq r_0$ outside a set $E \subset (0, +\infty)$ of finite logarithmic measure $\int_E dt/t < \infty$.

Proof. The proof of Theorem 7: suppose that (f_1, f_2) is a pair of transcendental entire functions with finite order, satisfying system (9); then, it follows that $\alpha(\partial^2 f_1(z_1, z_2)/\partial z_1^2) + \beta(\partial^2 f_1(z_1, z_2)/\partial z_2^2)$ and $\alpha(\partial^2 f_2(z_1, z_2)/\partial z_1^2) + \beta(\partial^2 f_2(z_1, z_2)/\partial z_2^2)$ are transcendental. Here, the following two cases will be considered:

(i) Case 1: $m_1 m_2 > n_1 n_2$: in view of Lemma 2, it yields that

$$m\left(r, \frac{f_j(z_1, z_2)}{f_j(z_1 + c_1, z_2 + c_2)}\right) = S(r, f_j), \quad j = 1, 2, \tag{29}$$

hold for all $r > 0$ outside of a possible exceptional set $E_j \subset [1, +\infty]$ of finite logarithmic measure $\int_{E_j} dt/t < \infty$. Thus, it follows from (29) that

$$\begin{aligned} T(r, f_j(z_1, z_2)) &= m(r, f_j(z_1, z_2)) \\ &\leq m\left(r, \frac{f_j(z_1, z_2)}{f(z_1 + c_1, z_2 + c_2)}\right) + m(r, f_j(z_1 + c_1, z_2 + c_2)) + \log 2, \\ &= m(r, f_j(z_1 + c_1, z_2 + c_2)) + S(r, f_j), \\ &= T(r, f_j(z_1 + c_1, z_2 + c_2)) + S(r, f_j), \quad j = 1, 2, \end{aligned} \tag{30}$$

for all $r \notin E =: E_1 \cup E_2$. In view of (30), Lemma 1, and the Mokhon'ko theorem in several complex variables ([27], Theorem 3.4), it yields that

$$\begin{aligned}
m_1T(r,f_2(z_1,z_2)) &\le m_1T(r,f_2(z_1+c_1,z_2+c_2))+S(r,f_2),\\
&= T(r,f_2(z_1+c_1,z_2+c_2)^{m_1})+S(r,f_2),\\
&= T\left(r,\left(\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)^{n_1}-1\right)+S(r,f_2),\\
&= n_1T\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)+S(r,f_2)+S(r,f_1),\\
&= n_1m\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)+S(r,f_2)+S(r,f_1)\\
&\le n_1\left(m\left(r,\frac{\alpha\partial^2 f_1(z_1,z_2)/\partial z_1^2+\beta\left(\partial^2 f_1(z_1,z_2)/\partial z_2^2\right)}{f_1(z_1,z_2)}\right)+m(r,f_1(z_1,z_2))\right)+S(r,f_1)+S(r,f_2),\\
&= n_1T(r,f_1(z_1,z_2))+S(r,f_1)+S(r,f_2),
\end{aligned} \tag{31}$$

for all $r\in E$. Similarly, we have

$$m_2T(r,f_1(z_1,z_2))\le n_2T(r,f_2(z_1,z_2))+S(r,f_1)+S(r,f_2),\quad r\notin E. \tag{32}$$

In view of (31) and (32), it yields that

$$(m_1m_2-n_1n_2)T\left(r,f_j(z_1,z_2)\right)\le S(r,f_1)+S(r,f_2),\quad r\notin E. \tag{33}$$

Since f_1,f_2 are transcendental and $m_1m_2>n_1n_2$, this is a contradiction.

(ii) Case 2: $m_j>(n_j/n_j-1)$ for $n_j\ge 2$, $j=1,2$: in view of the Nevanlinna second fundamental theorem, Lemma 2, and system (9), it follows that

$$\begin{aligned}
&(n_1-1)T\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)\\
&\le \overline{N}\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)+\sum_{q=1}^{n_1}\overline{N}\left(r,\frac{1}{\alpha\left(\partial^2 f_1(z_1,z_2)/\partial z_1^2\right)+\beta\left(\partial^2 f_1(z_1,z_2)/\partial z_2^2\right)-w_q}\right)\\
&\quad+S\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)\\
&\le \overline{N}\left(r,\frac{1}{\left(\alpha\left(\partial^2 f_1(z_1,z_2)/\partial z_1^2\right)+\beta\left(\partial^2 f_1(z_1,z_2)/\partial z_2^2\right)\right)^{n_1}-1}\right)+S\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)\\
&\le \overline{N}\left(r,\frac{1}{f_2(z_1+c_1,z_2+c_2)}\right)+S(r,f_1)\\
&\le T(r,f_2(z_1+c_1,z_2+c_2))+S(r,f_1)+S(r,f_2),
\end{aligned} \tag{34}$$

where w_q is a root of $w^{n_1}-1=0$. Similarly, we have

$$(n_2-1)T\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)\\ \leq T(r,f_1(z_1+c_1,z_2+c_2))+S(r,f_1)+S(r,f_2). \tag{35}$$

On the other hand, by the Mokhon'ko theorem in several complex variables ([27], Theorem 3.4), it follows from system (9) that

$$\begin{aligned} m_1T(r,f_2(z_1+c_1,z_2+c_2)),\\ &= T(r,f_2(z_1+c_1,z_2+c_2)^{m_1})+S(r,f_2),\\ &= T\left(r,\left(\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)^{n_1}-1\right)+S(r,f_2),\\ &= n_1T\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)+S(r,f_1)+S(r,f_2). \end{aligned} \tag{36}$$

Similarly, we have

$$\begin{aligned} m_2T(r,f_1(z_1+c_1,z_2+c_2)),\\ &= n_2T\left(r,\alpha\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\beta\frac{\partial^2 f_1(z_1,z_2)}{\partial z_2^2}\right)\\ &\quad+S(r,f_1)+S(r,f_2). \end{aligned} \tag{37}$$

In view of (34)–(37) and $m_j>(n_j/n_j-1)$, it follows that

$$\begin{aligned} \left(m_1-\frac{n_1}{n_1-1}\right)T(r,f_2(z_1+c_1,z_2+c_2))\leq S(r,f_1)+S(r,f_2),\\ \left(m_2-\frac{n_2}{n_2-1}\right)T(r,f_1(z_1+c_1,z_2+c_2))\leq S(r,f_1)+S(r,f_2). \end{aligned} \tag{38}$$

It leads to a contradiction with the assumption that f_1, f_2 are transcendental entire functions.

Therefore, this completes the proof of Theorem 7.

3. Proofs of Theorems 8 and 9

The following lemma plays the key role in proving Theorems 8 and 9.

Lemma 3 (See [28, 29]). *For an entire function F on $\mathbb{C}^n$, $F(0)\neq 0$ and put $\rho(n_F)=\rho<\infty$. Then, there exist a canonical function f_F and a function $g_F\in\mathbb{C}^n$ such that $F(z)=f_F(z)e^{g_F(z)}$. For the special case $n=1$, f_F is the canonical product of Weierstrass.*

Remark 4. Here, we denote $\rho(n_F)$ to be the order of the counting function of zeros of F.

Lemma 4 (See [30]). *If g and h are entire functions on the complex plane $\mathbb{C}$ and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

(a) the internal function h is a polynomial, and the external function g is of finite order or else

(b) the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order

Lemma 5 (See [31], Lemma 3). *Let $f_j(\equiv 0)$, $j=1,2,3$ be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant and $f_1+f_2+f_3=1$ and such that*

$$\sum_{j=1}^{3}\left\{N_2\left(r,\frac{1}{f_j}\right)+2\overline{N}(r,f_j)\right\}<\lambda T(r,f_1)+O(\log^+T(r,f_1)), \tag{39}$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda<1$ is a positive number. Then, either $f_2=1$ or $f_3=1$.

3.1. The Proof of Theorem 8. Suppose that (f_1,f_2) is a pair of transcendental entire solutions with finite order of system (11). System (11) can be represented as follows:

$$\begin{cases} \left[\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+if_2(z_1+c_1,z_2+c_2)\right]\left[\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}-if_2(z_1+c_1,z_2+c_2)\right]=1,\\ \left[\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}+if_1(z_1+c_1,z_2+c_2)\right]\left[\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}-if_1(z_1+c_1,z_2+c_2)\right]=1. \end{cases} \tag{40}$$

Since f_1, f_2 are transcendental entire functions with finite order, then by (40), we can see that the functions

$$\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+if_2(z_1+c_1,z_2+c_2),$$
$$\frac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}-if_2(z_1+c_1,z_2+c_2),$$
$$\frac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}+if_1(z_1+c_1,z_2+c_2),\tag{41}$$
$$\frac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}-if_1(z_1+c_1,z_2+c_2),$$

have no any zeros and poles. Moreover, by Lemmas 3 and 4, we have that there exist two polynomials $p_1(z), p_2(z)$ such that

$$\begin{cases}\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+if_2(z_1+c_1,z_2+c_2)=e^{ip_1(z_1+c_1,z_2+c_2)},\\[2ex] \dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}-if_2(z_1+c_1,z_2+c_2)=e^{-ip_1(z_1+c_1,z_2+c_2)},\\[2ex] \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}+if_1(z_1+c_1,z_2+c_2)=e^{ip_2(z_1+c_1,z_2+c_2)},\\[2ex] \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}-if_1(z_1+c_1,z_2+c_2)=e^{-ip_2(z_1+c_1,z_2+c_2)}.\end{cases}\tag{42}$$

In view of (42), it yields that

$$\begin{cases}\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}=\dfrac{e^{ip_1(z)}+e^{-ip_1(z)}}{2},\\[2ex] f_2(z+c)=\dfrac{e^{ip_1(z)}-e^{-ip_1(z)}}{2i},\\[2ex] \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}=\dfrac{e^{ip_2(z)}+e^{-ip_2(z)}}{2},\\[2ex] f_1(z+c)=\dfrac{e^{ip_2(z)}-e^{-ip_2(z)}}{2i},\end{cases}\tag{43}$$

which implies

$$\left[\frac{\partial^2 p_2}{\partial z_1^2}+i\left(\frac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(p_1(z+c)+p_2(z))}+\left[\frac{\partial^2 p_2}{\partial z_1^2}-i\left(\frac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(p_1(z+c)-p_2(z))}-e^{2ip_1(z+c)}\equiv 1,\tag{44}$$

$$\left[\frac{\partial^2 p_1}{\partial z_1^2}+i\left(\frac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(p_2(z+c)+p_1(z))}+\left[\frac{\partial^2 p_1}{\partial z_1^2}-i\left(\frac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(p_2(z+c)-p_1(z))}-e^{2ip_2(z+c)}\equiv 1.\tag{45}$$

Now, we claim that $(\partial^2 p_2(z_1,z_2)/\partial z_1^2)-i(\partial p_2(z_1,z_2)/\partial z_1)^2\equiv 0$. If $(\partial p(z_1,z_2)/\partial z_1)\equiv 0$, then equation (44) becomes $e^{2ip_1(z_1+c_1,z_2+c_2)}+1\equiv 0$, and this is impossible since $p_2(z)$ is a nonconstant polynomial. If $(\partial^2 p_2(z_1,z_2)/\partial z_1^2)-i(\partial p_2(z_1,z_2)/\partial z_1)^2\equiv 0$ and $(\partial p_2(z_1,z_2)/\partial z_1)\equiv 0$, then $(\partial u/\partial z_1)=iu^2$, where $u=(\partial p_2(z_1,z_2)/\partial z_1)$. Solving this equation, we have $-(1/u)=iz_1+\varphi_1(z_2)$, that is, $(\partial p_2(z_1,z_2)/\partial z_1)=u=-(1/iz_1+\varphi_1(z_2))$, where $\varphi_1(z_2)$ is a polynomial in z_2. Thus, it follows that $p_2(z_1,z_2)=i\log[iz_1+\varphi_1(z_2)]+\varphi_2(z_2)$, where $\varphi_2(z_2)$ is a polynomial in z_2. This is a contradiction with the assumption of $p_2(z_1,z_2)$ being a nonconstant polynomial. Hence, $(\partial^2 p_2(z_1,z_2)/\partial z_1^2)-i(\partial p_2(z_1,z_2)/\partial z_1)^2\equiv 0$. Similarly, we have $(\partial^2 p_2(z_1,z_2)/\partial z_1^2)+i(\partial p_2(z_1,z_2)/\partial z_1)^2\equiv 0$ and $(\partial^2 p_2(z_1,z_2)/\partial z_1^2)\pm i(\partial p_2(z_1,z_2)/\partial z_1)^2\equiv 0$.

In view of Lemma 5 and (44) and (45), it follows that

$$\begin{aligned}&\left[\frac{\partial^2 p_2}{\partial z_1^2}+i\left(\frac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(p_1(z+c)+p_2(z))}\equiv 1\\ \text{or}\ &\left[\frac{\partial^2 p_2}{\partial z_1^2}-i\left(\frac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(p_1(z+c)-p_2(z))}\equiv 1,\\ &\left[\frac{\partial^2 p_1}{\partial z_1^2}+i\left(\frac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(p_2(z+c)+p_1(z))}\equiv 1\\ \text{or}\ &\left[\frac{\partial^2 p_1}{\partial z_1^2}-i\left(\frac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(p_2(z+c)-p_1(z))}\equiv 1.\end{aligned}\tag{46}$$

Now, the four cases will be taken into account below.

(i) Case 1

$$\begin{cases}\left[\dfrac{\partial^2 p_2}{\partial z_1^2}+i\left(\dfrac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(p_1(z+c)+p_2(z))}\equiv 1,\\[2ex] \left[\dfrac{\partial^2 p_1}{\partial z_1^2}+i\left(\dfrac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(p_2(z+c)+p_1(z))}\equiv 1.\end{cases}\tag{47}$$

Since $p_1(z), p_2(z)$ are polynomials, from (47), it follows that $p_1(z+c)+p_2(z)\equiv C_1$ and $p_2(z+c)+p_1(z)\equiv C_2$, and here and below, C_1, C_2 are constants. Thus, it yields that $p_1(z+2c)-p_1(z)\equiv C_1-C_2$ and $p_2(z+2c)-p_2(z)\equiv C_2-C_1$. Hence, we have that $p_1(z)=L(z)+H(z)+B_1$, $p_2(z)=-L(z)-H(z)+B_2$, where L is a linear function of the form $L(z)=a_1z_1+a_2z_2$, $a_1(\neq 0), a_2, B_1, B_2$ are constants, and $H(z):=H(s)$, $H(s)$ is a polynomial in s in $\mathbb{C}$, $s=c_2z_1-c_1z_2$.

Here, we will prove that $H(z) \equiv 0$. If $\deg_s H = n$, equation (47) implies

$$\frac{d^2 H}{ds^2} + i\left(\frac{dH}{ds}\right)^2 \equiv \zeta_0, \tag{48}$$

that is,

$$\frac{d^2 H}{ds^2} \equiv \zeta_0 - i\left(\frac{dH}{ds}\right)^2, \tag{49}$$

where $\zeta_0 \in \mathbb{C}$. By comparing the degree of s in both sides of the abovementioned equation, we have $n-2 = 2(n-1)$, that is, $n=0$. Thus, the form of $L(z)+H(z)+B$ is still the linear form of $A_1 z_1 + A_2 z_2 + B$, which means that $H(z) \equiv 0$. Thus, this means that $(\partial^2 p_1/\partial z_1^2) \equiv (\partial^2 p_2/\partial z_1^2) \equiv 0$. Substituting these into (47), we have

$$\begin{cases} i a_1^2 e^{iL(c)+i(B_1+B_2)} \equiv 1, \\ i a_1^2 e^{-iL(c)+i(B_1+B_2)} \equiv 1. \end{cases} \tag{50}$$

In addition, in view of (44)–(47), it follows that

$$\begin{cases} \left[\dfrac{\partial^2 p_2}{\partial z_1^2} - i\left(\dfrac{\partial p_2}{\partial z_1}\right)^2\right] e^{i(-p_1(z+c)-p_2(z))} \equiv 1, \\ \left[\dfrac{\partial^2 p_1}{\partial z_1^2} - i\left(\dfrac{\partial p_1}{\partial z_1}\right)^2\right] e^{i(-p_2(z+c)-p_1(z))} \equiv 1, \end{cases} \tag{51}$$

which means that

$$\begin{cases} -i a_1^2 e^{-iL(c)-i(B_1+B_2)} \equiv 1, \\ -i a_1^2 e^{iL(c)-i(B_1+B_2)} \equiv 1. \end{cases} \tag{52}$$

Thus, we can deduce from (50) and (52) that

$$\begin{aligned} a_1^4 &= 1, \\ e^{2iL(c)} &= 1, \\ a_1^2 e^{iL(c)+i(B_1+B_2)} &\equiv -i. \end{aligned} \tag{53}$$

In view of (43), f_1, f_2 are of the forms

$$f_1(z) = \frac{e^{i(-L(z)+B_2)+iL(c)} - e^{i(L(z)-B_2)-iL(c)}}{2i}, \tag{54}$$

$$f_2(z) = \frac{e^{i(L(z)+B_1)-iL(c)} - e^{i(-L(z)-B_1)+iL(c)}}{2i}. \tag{55}$$

If $a_1^2 = 1$ and $e^{iL(c)} = 1$, then $L(c) = 2k\pi$ and $e^{i(B_1+B_2)} = -i$. Thus, it follows from (54) and (55) that

$$f_1(z_1, z_2) = \frac{-e^{i(L(z)-B_2)} + e^{i(-L(z)+B_2)}}{2i} = -\sin(L(z)+B_0), \tag{56}$$

where $B_0 = -B_2$, and

$$\begin{aligned} f_2(z_1, z_2) &= \frac{e^{i(L(z)+B_1)} - e^{i(-L(z)-B_1)}}{2i} = \frac{e^{i(L(z)-B_2)} e^{i(B_1+B_2)} - e^{i(-L(z)+B_2)} e^{-i(B_1+B_2)}}{2i}, \\ &= -\frac{e^{i(L(z)-B_2)} + e^{i(-L(z)+B_2)}}{2} = -\cos(L(z)+B_0). \end{aligned} \tag{57}$$

If $a_1^2 = 1$ and $e^{iL(c)} = -1$, then $L(c) = (2k+1)\pi$ and $e^{i(B_1+B_2)} = i$. Thus, it follows from (54) and (55) that

$$\begin{aligned} f_1(z_1, z_2) &= \frac{e^{i(L(z)-B_2)} - e^{i(-L(z)+B_2)}}{2i} = \sin(L(z)+B_0), \\ f_2(z_1, z_2) &= \frac{-e^{i(L(z)+B_1)} + e^{i(-L(z)-B_1)}}{2i} = \frac{-e^{i(L(z)-B_2)} e^{i(B_1+B_2)} + e^{i(-L(z)+B_2)} e^{-i(B_1+B_2)}}{2i}, \\ &= -\frac{e^{i(L(z)-B_2)} + e^{i(-L(z)+B_2)}}{2} = -\cos(L(z)+B_0). \end{aligned} \tag{58}$$

If $a_1^2 = -1$ and $e^{iL(c)} = 1$, then $L(c) = 2k\pi$ and $e^{i(B_1+B_2)} = i$. Thus, it follows from (54) and (55) that

$$\begin{aligned} f_1(z_1, z_2) &= -\sin(L(z)+B_0), \\ f_2(z_1, z_2) &= \cos(L(z)+B_0). \end{aligned} \tag{59}$$

If $a_1^2=-1$ and $e^{iL(c)}=-1$, then $L(c)=(2k+1)\pi$ and $e^{i(B_1+B_2)}=-i$. Thus, it follows from (54) and (55) that

$$\begin{aligned} f_1(z_1,z_2)&=\sin(L(z)+B_0),\\ f_2(z_1,z_2)&=\cos(L(z)+B_0). \end{aligned} \tag{60}$$

(ii) Case 2

$$\begin{cases} \left[\dfrac{\partial^2 p_2}{\partial z_1^2}+i\left(\dfrac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(p_1(z+c)+p_2(z))}\equiv 1,\\ \left[\dfrac{\partial^2 p_1}{\partial z_1^2}-i\left(\dfrac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(p_2(z+c)-p_1(z))}\equiv 1. \end{cases} \tag{61}$$

Since $p_1(z),p_2(z)$ are polynomials, from (61), it follows that $p_1(z+c)+p_2(z)\equiv C_1$ and $p_2(z+c)-p_1(z)\equiv C_2$, which imply that $p_2(z+2c)+p_2(z)\equiv C_1+C_2$, and this is a contradiction with the condition of $p_2(z)$ being a nonconstant polynomial.

(iii) Case 3

$$\begin{cases} \left[\dfrac{\partial^2 p_2}{\partial z_1^2}-i\left(\dfrac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(p_1(z+c)-p_2(z))}\equiv 1,\\ \left[\dfrac{\partial^2 p_1}{\partial z_1^2}+i\left(\dfrac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(p_2(z+c)+p_1(z))}\equiv 1. \end{cases} \tag{62}$$

Since $p_1(z),p_2(z)$ are polynomials, from (62), it follows that $p_1(z+c)-p_2(z)\equiv C_1$ and $p_2(z+c)+p_1(z)\equiv C_2$, which imply that $p_1(z+2c)+p_1(z)\equiv C_1+C_2$, and this is also a contradiction.

(iv) Case 4

$$\begin{cases} \left[\dfrac{\partial^2 p_2}{\partial z_1^2}-i\left(\dfrac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(p_1(z+c)-p_2(z))}\equiv 1,\\ \left[\dfrac{\partial^2 p_1}{\partial z_1^2}-i\left(\dfrac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(p_2(z+c)-p_1(z))}\equiv 1. \end{cases} \tag{63}$$

Since $p_1(z),p_2(z)$ are polynomials, then from (63), it follows that $p_1(z+c)-p_2(z)\equiv C_1$ and $p_2(z+c)-p_1(z)\equiv C_2$. This means that $p_1(z+2c)-p_1(z)\equiv C_1+C_2$ and $p_2(z+2c)-p_2(z)\equiv C_1+C_2$. Similar to the argument as in case 1 in Theorem 8, we can deduce that $p_1(z)=L(z)+B_1, p_2(z)=L(z)+B_2$, where L is a linear function of the form $L(z)=a_1z_1+a_2z_2$, $a_1(\neq 0), a_2, B_1, B_2$ are constants. Hence, it follows that $(\partial^2 p_1/\partial z_1^2)\equiv(\partial^2 p_2/\partial z_1^2)\equiv 0$. Substituting these into (63), we have

$$\begin{cases} -ia_1^2e^{iL(c)+i(B_1-B_2)}\equiv 1,\\ -ia_1^2e^{iL(c)+i(B_2-B_1)}\equiv 1. \end{cases} \tag{64}$$

In addition, in view of (44)–(47), it follows that

$$\begin{cases} \left[\dfrac{\partial^2 p_2}{\partial z_1^2}+i\left(\dfrac{\partial p_2}{\partial z_1}\right)^2\right]e^{i(-p_1(z+c)+p_2(z))}\equiv 1,\\ \left[\dfrac{\partial^2 p_1}{\partial z_1^2}+i\left(\dfrac{\partial p_1}{\partial z_1}\right)^2\right]e^{i(-p_2(z+c)+p_1(z))}\equiv 1, \end{cases} \tag{65}$$

which means that

$$\begin{cases} ia_1^2e^{-iL(c)+i(B_2-B_1)}\equiv 1,\\ ia_1^2e^{-iL(c)+i(B_1-B_2)}\equiv 1. \end{cases} \tag{66}$$

Thus, we can deduce from (63) and (64) that

$$\begin{aligned} a_1^4&=1,\\ e^{2iL(c)}&=-1,\\ a_1^2e^{iL(c)+i(B_1-B_2)}&\equiv i. \end{aligned} \tag{67}$$

In view of (43), f_1,f_2 are of the forms

$$f_1(z)=\frac{e^{i(L(z)+B_2)-iL(c)}-e^{i(-L(z)-B_2)+iL(c)}}{2i}, \tag{68}$$

$$f_2(z)=\frac{e^{i(L(z)+B_1)-iL(c)}-e^{i(-L(z)-B_1)+iL(c)}}{2i}. \tag{69}$$

If $a_1^2=1$ and $e^{iL(c)}=i$, then $L(c)=(2k+(1/2))\pi$ and $e^{i(B_1-B_2)}=1$. Thus, it follows from (68) and (69) that

$$f_1(z_1,z_2)=\frac{-ie^{i(L(z)+B_2)}-ie^{i(-L(z)-B_2)}}{2i}=-\cos(L(z)+B_0), \tag{70}$$

where $B_0=B_2$, and

$$\begin{aligned} f_2(z_1,z_2)&=\frac{-ie^{i(L(z)+B_1)}-ie^{i(-L(z)-B_1)}}{2i}\\ &=\frac{-ie^{i(L(z)+B_2)}e^{i(B_1-B_2)}-ie^{i(-L(z)-B_2)}e^{i(B_2-B_1)}}{2i}\\ &=-\frac{e^{i(L(z)+B_2)}+e^{i(-L(z)-B_2)}}{2}=-\cos(L(z)+B_0). \end{aligned} \tag{71}$$

If $a_1^2=1$ and $e^{iL(c)}=-i$, then $L(c)=(2k-(1/2))\pi$ and $e^{i(B_1-B_2)}=-1$. Thus, it follows from (68) and (69) that

$$\begin{aligned} f_1(z_1,z_2)&=\cos(L(z)+B_0),\\ f_2(z_1,z_2)&=-\cos(L(z)+B_0). \end{aligned} \tag{72}$$

If $a_1^2=-1$ and $e^{iL(c)}=i$, then $L(c)=(2k+(1/2))\pi$ and $e^{i(B_1-B_2)}=-1$. Thus, it follows from (68) and (69) that

$$\begin{aligned} f_1(z_1,z_2)&=-\cos(L(z)+B_0),\\ f_2(z_1,z_2)&=\cos(L(z)+B_0). \end{aligned} \tag{73}$$

If $a_1^2=-1$ and $e^{iL(c)}=-i$, then $L(c)=(2k-(1/2))\pi$ and $e^{i(B_1-B_2)}=1$. Thus, it follows from (68) and (69) that

$$\begin{aligned} f_1(z_1,z_2)&=\cos(L(z)+B_0),\\ f_2(z_1,z_2)&=\cos(L(z)+B_0). \end{aligned} \tag{74}$$

Thus, in view of Cases 1–4, this completes the proof of Theorem 8.

3.2. The Proof of Theorem 9. Suppose that (f_1,f_2) is a pair of transcendental entire solutions with finite order of system (18). System (18) can be represented as follows:

$$\begin{cases} \left[\dfrac{\partial^2 f_1(z)}{\partial z_1^2}+\dfrac{\partial^2 f_1(z)}{\partial z_1\partial z_2}+if_2(z+c)\right]\left[\dfrac{\partial^2 f_1(z)}{\partial z_1^2}+\dfrac{\partial^2 f_1(z)}{\partial z_1\partial z_2}-if_2(z+c)\right]=1,\\ \left[\dfrac{\partial^2 f_2(z)}{\partial z_1^2}+\dfrac{\partial^2 f_2(z)}{\partial z_1\partial z_2}+if_1(z+c)\right]\left[\dfrac{\partial^2 f_2(z)}{\partial z_1^2}+\dfrac{\partial^2 f_2(z)}{\partial z_1\partial z_2}-if_1(z+c)\right]=1. \end{cases} \tag{75}$$

Since f_1,f_2 are transcendental entire functions with finite order, then by (75), we can see that the functions

$$\begin{aligned} &\frac{\partial^2 f_1(z)}{\partial z_1^2}+\frac{\partial^2 f_1(z)}{\partial z_1\partial z_2}+if_2(z+c),\\ &\frac{\partial^2 f_1(z)}{\partial z_1^2}+\frac{\partial^2 f_1(z)}{\partial z_1\partial z_2}-if_2(z+c),\\ &\frac{\partial^2 f_2(z)}{\partial z_1^2}+\frac{\partial^2 f_2(z)}{\partial z_1\partial z_2}+if_1(z+c),\\ &\frac{\partial^2 f_2(z)}{\partial z_1^2}+\frac{\partial^2 f_2(z)}{\partial z_1\partial z_2}-if_1(z+c), \end{aligned} \tag{76}$$

have no any zeros and poles. Moreover, by Lemmas 3 and 4, we have that there exist two polynomials $p_1(z)$, $p_2(z)$ such that

$$\begin{cases} \dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1\partial z_2}+if_2(z_1+c_1,z_2+c_2)=e^{ip_1(z_1+c_1,z_2+c_2)},\\ \dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial^2 f_1(z_1,z_2)}{\partial z_1\partial z_2}-if_2(z_1+c_1,z_2+c_2)=e^{-ip_1(z_1+c_1,z_2+c_2)},\\ \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1\partial z_2}+if_1(z_1+c_1,z_2+c_2)=e^{ip_2(z_1+c_1,z_2+c_2)},\\ \dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1^2}+\dfrac{\partial^2 f_2(z_1,z_2)}{\partial z_1\partial z_2}-if_1(z_1+c_1,z_2+c_2)=e^{-ip_2(z_1+c_1,z_2+c_2)}. \end{cases} \tag{77}$$

In view of (77), it yields that

$$\begin{cases} \dfrac{\partial^2 f_1(z)}{\partial z_1^2} + \dfrac{\partial^2 f_1(z)}{\partial z_1 \partial z_2} = \dfrac{e^{ip_1(z)} + e^{-ip_1(z)}}{2}, \\ f_2(z+c) = \dfrac{e^{ip_1(z)} - e^{-ip_1(z)}}{2i}, \\ \dfrac{\partial^2 f_2(z)}{\partial z_1^2} + \dfrac{\partial^2 f_2(z)}{\partial z_1 \partial z_2} = \dfrac{e^{ip_2(z)} + e^{-ip_2(z)}}{2}, \\ f_1(z+c) = \dfrac{e^{ip_2(z)} - e^{-ip_2(z)}}{2i}, \end{cases} \tag{78}$$

which implies

$$(P_{21} + iP_{22})e^{i(p_1(z+c)+p_2(z))} + (P_{21} - iP_{22})e^{i(p_1(z+c)-p_2(z))} - e^{2ip_1(z+c)} \equiv 1, \tag{79}$$

$$(P_{11} + iP_{12})e^{i(p_2(z+c)+p_1(z))} + (P_{11} - iP_{12})e^{i(p_2(z+c)-p_1(z))} - e^{2ip_2(z+c)} \equiv 1, \tag{80}$$

where

$$\begin{aligned} P_{11} &= \frac{\partial^2 p_1}{\partial z_1^2} + \frac{\partial^2 p_1}{\partial z_1 \partial z_2}, \\ P_{12} &= \left(\frac{\partial p_1}{\partial z_1}\right)^2 + \frac{\partial p_1}{\partial z_1}\frac{\partial p_1}{\partial z_2}, \\ P_{21} &= \frac{\partial^2 p_2}{\partial z_1^2} + \frac{\partial^2 p_2}{\partial z_1 \partial z_2}, \\ P_{22} &= \left(\frac{\partial p_2}{\partial z_1}\right)^2 + \frac{\partial p_2}{\partial z_1}\frac{\partial p_2}{\partial z_2}. \end{aligned} \tag{81}$$

Now, we claim that $P_{21} - iP_{22} \equiv 0$. If $P_{21} - iP_{22} \equiv 0$; then, equation (79) becomes $(P_{21} + iP_{22})e^{i(p_1(z+c)+p_2(z))} - e^{2ip_1(z_1+c_1,z_2+c_2)} \equiv 1$. If $P_{21} + iP_{22} \equiv 0$, then it yields $e^{2ip_1(z_1+c_1,z_2+c_2)} \equiv -1$, and this is a contradiction with the condition of p_1 being a nonconstant polynomial. If $P_{21} + iP_{22} \equiv 0$, we have

$$(P_{21} + iP_{22})e^{i(p_1(z+c)+p_2(z))} \equiv e^{2ip_1(z_1+c_1,z_2+c_2)} + 1. \tag{82}$$

By making use of the Mokhon'ko theorem in several complex variables ([27], Theorem 3.4), in view of (82), it follows that

$$T\left(r, e^{2ip_1(z_1+c_1,z_2+c_2)}\right) = T\left(r, (P_{21} + iP_{22})e^{i(p_1(z+c)+p_2(z))}\right) + O(1). \tag{83}$$

In view of the Nevanlinna second fundamental theorem, (82) and (83), it follows that

$$\begin{aligned} &T\left(r, e^{2ip_2(z_1+c_1,z_2+c_2)}\right) \\ &\le N\left(r, \frac{1}{e^{2ip_2(z_1+c_1,z_2+c_2)}}\right) + N\left(r, \frac{1}{e^{2ip_2(z_1+c_1,z_2+c_2)} + 1}\right) \\ &\quad + S\left(r, e^{2ip_2(z_1+c_1,z_2+c_2)}\right) \\ &\le N\left(r, \frac{1}{(P_{21} + iP_{22})e^{i(p_1(z+c)+p_2(z))}}\right) \\ &\quad + S\left(r, e^{2ip_2(z_1+c_1,z_2+c_2)}\right) \\ &\le N\left(r, \frac{1}{P_{21} + iP_{22}}\right) + S\left(r, e^{2ip_2(z_1+c_1,z_2+c_2)}\right) \\ &\le O(T(r, p_2)) + S\left(r, e^{2ip_2(z_1+c_1,z_2+c_2)}\right), \end{aligned} \tag{84}$$

outside possibly a set of finite Lebesgue measure. This is a contradiction with the fact that

$$\lim_{r \longrightarrow +\infty} \frac{T\left(r, e^{2ip_2}\right)}{T(r, p_2)} = +\infty, \tag{85}$$

for $p_2(z)$ being a nonconstant polynomial. Hence, $P_{21} - iP_{22} \equiv 0$. Similarly, we have $P_{21} + iP_{22} \equiv 0$ and $P_{11} + iP_{12} \equiv 0$.

In view of Lemma 5 and (79) and (80), it follows that

$$\begin{aligned} &(P_{21} + iP_{22})e^{i(p_1(z+c)+p_2(z))} \equiv 1 \\ \text{or } &(P_{21} - iP_{22})e^{i(p_1(z+c)-p_2(z))} \equiv 1, \\ &(P_{11} + iP_{12})e^{i(p_2(z+c)+p_1(z))} \equiv 1 \\ \text{or } &(P_{11} - iP_{12})e^{i(p_2(z+c)-p_1(z))} \equiv 1. \end{aligned} \tag{86}$$

Now, we will consider the four cases below.

Case 1

$$\begin{cases} (P_{21} + iP_{22})e^{i(p_1(z+c)+p_2(z))} \equiv 1, \\ (P_{11} + iP_{12})e^{i(p_2(z+c)+p_1(z))} \equiv 1. \end{cases} \tag{87}$$

Since $p_1(z), p_2(z)$ are polynomials, from (87), it follows that $p_1(z+c) + p_2(z) \equiv C_1$ and $p_2(z+c) + p_1(z) \equiv C_2$. Thus, it yields that $p_1(z+2c) - p_1(z) \equiv C_1 - C_2$ and $p_2(z+2c) - p_2(z) \equiv C_2 - C_1$. Hence, similar to the argument as in case 1 of Theorem 8, we have that $p_1(z) = L(z) + B_1, p_2(z) = -L(z) + B_2$, where L is a linear function of the form $L(z) = a_1 z_1 + a_2 z_2$, $a_1(\neq 0), a_1 \neq -a_2, B_1, B_2$ are constants, which means that $(\partial^2 p_1/\partial z_1^2) \equiv (\partial^2 p_2/\partial z_1^2) \equiv (\partial^2 p_2/\partial z_1 \partial z_2) \equiv 0$. Substituting these into (87), we have

$$\begin{cases} ia_1(a_1 + a_2)e^{iL(c)+i(B_1+B_2)} \equiv 1, \\ ia_1(a_1 + a_2)e^{-iL(c)+i(B_1+B_2)} \equiv 1. \end{cases} \tag{88}$$

In addition, in view of (79)–(87), it follows that

$$\begin{cases} [P_{21} - iP_{22}]e^{i(-p_1(z+c)-p_2(z))} \equiv 1, \\ [P_{11} - iP_{12}]e^{i(-p_2(z+c)-p_1(z))} \equiv 1, \end{cases} \tag{89}$$

which means that

$$\begin{cases} -ia_1(a_1 + a_2)e^{-iL(c)-i(B_1+B_2)} \equiv 1, \\ -ia_1(a_1 + a_2)e^{iL(c)-i(B_1+B_2)} \equiv 1. \end{cases} \tag{90}$$

Thus, we can deduce from (88) and (90) that

$$\begin{aligned} a_1^2(a_1 + a_2)^2 &= 1, \\ e^{2iL(c)} &= 1, \\ a_1(a_1 + a_2)e^{iL(c)+i(B_1+B_2)} &\equiv -i. \end{aligned} \tag{91}$$

Similar to the argument as in the proof of Theorem 8 and by combining with (91), we have that (f_1, f_2) is of the form

$$(f_1, f_2) = (\pm \sin(L(z) + B_0), \pm \cos(L(z) + B_0)). \tag{92}$$

Case 2

$$\begin{cases} (P_{21} + iP_{22})e^{i(p_1(z+c)+p_2(z))} \equiv 1, \\ (P_{11} - iP_{12})e^{i(p_2(z+c)-p_1(z))} \equiv 1. \end{cases} \tag{93}$$

Since $p_1(z), p_2(z)$ are polynomials, from (93), it follows that $p_1(z+c) + p_2(z) \equiv C_1$ and $p_2(z+c) - p_1(z) \equiv C_2$, which imply that $p_2(z+2c) + p_2(z) \equiv C_1 + C_2$, and this is a contradiction with the condition of $p_2(z)$ being a nonconstant polynomial.

Case 3

$$\begin{cases} (P_{21} - iP_{22})e^{i(p_1(z+c)-p_2(z))} \equiv 1, \\ (P_{11} + iP_{12})e^{i(p_2(z+c)+p_1(z))} \equiv 1. \end{cases} \tag{94}$$

Since $p_1(z), p_2(z)$ are polynomials, then from (94), it follows that $p_1(z+c) - p_2(z) \equiv C_1$ and $p_2(z+c) + p_1(z) \equiv C_2$, which imply that $p_1(z+2c) + p_1(z) \equiv C_1 + C_2$, and this is also a contradiction.

Case 4

$$\begin{cases} (P_{21} - iP_{22})e^{i(p_1(z+c)-p_2(z))} \equiv 1, \\ (P_{11} - iP_{12})e^{i(p_2(z+c)-p_1(z))} \equiv 1. \end{cases} \tag{95}$$

Since $p_1(z), p_2(z)$ are polynomials, from (95), it follows that $p_1(z+c) - p_2(z) \equiv C_1$ and $p_2(z+c) - p_1(z) \equiv C_2$. This means that $p_1(z+2c) - p_1(z) \equiv C_1 + C_2$ and $p_2(z+2c) - p_2(z) \equiv C_1 + C_2$. Thus, similar to the argument as in case 1 of Theorem 2, we can deduce that $p_1(z) = L(z) + B_1, p_2(z) = L(z) + B_2$, where L is a linear function of the form $L(z) = a_1 z_1 + a_2 z_2$, $a_1(\neq 0), a_1 \neq -a_2, B_1, B_2$ are constants. Hence, it follows that $(\partial^2 p_1/\partial z_1^2) \equiv (\partial^2 p_2/\partial z_1^2) \equiv (\partial^2 p_2/\partial z_1 \partial z_2) \equiv 0$. Substituting these into (95), we have

$$\begin{cases} -ia_1(a_1 + a_2)e^{iL(c)+i(B_1-B_2)} \equiv 1, \\ -ia_1(a_1 + a_2)e^{iL(c)+i(B_2-B_1)} \equiv 1. \end{cases} \tag{96}$$

In addition, in view of (80)–(87), it follows that

$$\begin{cases} [P_{21} + iP_{22}]e^{i(-p_1(z+c)+p_2(z))} \equiv 1, \\ [P_{11} + iP_{12}]e^{i(-p_2(z+c)+p_1(z))} \equiv 1, \end{cases} \tag{97}$$

which means that

$$\begin{cases} ia_1(a_1+a_2)e^{-iL(c)+i(B_2-B_1)} \equiv 1, \\ ia_1(a_1+a_2)e^{-iL(c)+i(B_1-B_2)} \equiv 1. \end{cases} \quad (98)$$

Thus, we can deduce from (96) and (98) that

$$a_1^2(a_1+a_2)^2 = 1,$$
$$e^{2iL(c)} = -1, \quad (99)$$
$$a_1(a_1+a_2)e^{iL(c)+i(B_1-B_2)} \equiv i.$$

Similar to the argument as in the proof of Theorem 8, and by combining with (99), we can deduce that (f_1, f_2) is of the form

$$(f_1, f_2) = (\pm \cos(L(z)+B_0), \pm \cos(L(z)+B_0)). \quad (100)$$

Thus, in view of cases 1–4, this completes the proof of Theorem 8.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest in the manuscript.

Authors' Contributions

H. Y. Xu. conceptualized the study and wrote the original draft; S. M. Liu and H. Y. Xu. reviewed and edited the manuscript and acquired funding.

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