Research Article

Sine Half-Logistic Inverse Rayleigh Distribution: Properties, Estimation, and Applications in Biomedical Data

M. Shrahili ¹, I. Elbatal,² and Mohammed Elgarhy ³

¹Department of Statistics and Operations Research, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

²Department of Mathematics and Statistics, College of Science Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia

³The Higher Institute of Commercial Sciences, Al Mahalla Al Kubra 31951, Algharbia, Egypt

Correspondence should be addressed to Mohammed Elgarhy; m_elgarhy85@sva.edu.eg

Received 10 July 2021; Accepted 7 October 2021; Published 26 October 2021

Academic Editor: Ahmet Ocak Akdemir

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A new lifetime distribution with two parameters, known as the sine half-logistic inverse Rayleigh distribution, is proposed and studied as an extension of the half-logistic inverse Rayleigh model. The sine half-logistic inverse Rayleigh model is a new inverse Rayleigh distribution extension. In the application section, we show that the sine half-logistic inverse Rayleigh distribution is more flexible than the half-logistic inverse Rayleigh and inverse Rayleigh distributions. The statistical properties of the half-logistic inverse Rayleigh model are calculated, including the quantile function, moments, moment generating function, incomplete moment, and Lorenz and Bonferroni curves. Entropy measures such as Rényi entropy, Havrda and Charvat entropy, Arimoto entropy, and Tsallis entropy are proposed for the sine half-logistic inverse Rayleigh distribution. To estimate the sine half-logistic inverse Rayleigh distribution parameters, statistical inference using the maximum likelihood method is used. Applications of the sine half-logistic inverse Rayleigh model to real datasets demonstrate the flexibility of the sine half-logistic inverse Rayleigh distribution by comparing it to well-known models such as half-logistic inverse Rayleigh, type II Topp–Leone inverse Rayleigh, transmuted inverse Rayleigh, and inverse Rayleigh distributions.

1. Introduction

In recent years, inverse and half-inverse problems are studied in general operator theory [1–3], and many statisticians are focusing on generated families of distributions such as Kumaraswamy-G [4], T-X family [5], sine-G [6], type II half logistic-G [7], Weibull-G [8], the Burr type X-G [9], a new power Topp–Leone-G [10], truncated Cauchy power-G [11], beta generalized Marshall–Olkin–Kumaraswamy-G [12], transmuted odd Fréchet-G [13], new Kumaraswamy-G [14], Kumaraswamy Kumaraswamy-G [15], generalized Kumaraswamy-G [16], sine Topp–Leone-G [17], generalized transmuted exponentiated G [18], and Kumaraswamy transmuted-G [19].

A new generated family of distributions which is called the Sine-G (SG) family was introduced in [6]. The distribution function (CDF) of SG is

$$F(x; \varepsilon) = \sin \left[\frac{\pi}{2} G(x; \varepsilon) \right], \quad x \in R, \quad (1)$$

where $G(x; \varepsilon)$ is the CDF of baseline model with parameter vector ε and $F(x; \varepsilon)$ is the CDF derived by the T-X generator proposed in [3]. The probability density function (PDF) of the SG is

$$f(x; \varepsilon) = \frac{\pi}{2} g(x; \varepsilon) \cos \left[\frac{\pi}{2} G(x; \varepsilon) \right], \quad x \in R, \quad (2)$$

respectively.

The inverse Rayleigh (IR) distribution is a useful model for calculating lifetimes. Several authors have developed a number of extensions for the IR distribution in recent years, using various methods of generalization (see, for example, beta IR in [20], transmuted IR (TIR) in [21], modified IR in [22], transmuted modified IR in [23], Kumaraswamy exponentiated IR in [24], weighted IR in [25], odd Fréchet IR in [26], and half-logistic IR (HLIR) in [27]).

The CDF and PDF of HLIR distribution are given by

$$G(x; \lambda, \alpha) = \frac{1 - \left[1 - e^{-(\alpha/x)^2}\right]^\lambda}{1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda}, \quad x, \lambda, \alpha > 0, \quad (3)$$

and

$$g(x; \lambda, \alpha) = \frac{4\lambda\alpha^2 x^{-3} e^{-(\alpha/x)^2} \left[1 - e^{-(\alpha/x)^2}\right]^{\lambda-1}}{\left(1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda\right)^2}, \quad x, \lambda, \alpha > 0, \quad (4)$$

where α is a scale parameter and λ is a shape parameter.

We now present the sine half-logistic IR (SHLIR) distribution, a new lifetime model with two parameters. Inserting (3) into (1) yields the cdf of the SHLIR distribution as

$$F(x; \lambda, \alpha) = \sin \left[\frac{\pi}{2} \left(\frac{1 - \left[1 - e^{-(\alpha/x)^2}\right]^\lambda}{1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda} \right) \right], \quad x, \lambda, \alpha > 0. \quad (5)$$

The corresponding PDF to (5) is

$$f(x; \lambda, \alpha) = \frac{2\pi\lambda\alpha^2 x^{-3} e^{-(\alpha/x)^2} \left[1 - e^{-(\alpha/x)^2}\right]^{\lambda-1}}{\left(1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda\right)^2} \cos \left[\frac{\pi}{2} \left(\frac{1 - \left[1 - e^{-(\alpha/x)^2}\right]^\lambda}{1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda} \right) \right], \quad x, \lambda, \alpha > 0. \quad (6)$$

The SHLIR distribution's survival function (SF), hazard rate function (HRF), reversed HRF, and cumulative HRF are as follows:

$$R(x; \lambda, \alpha) = 1 - \sin \left[\frac{\pi}{2} \left(\frac{1 - \left[1 - e^{-(\alpha/x)^2}\right]^\lambda}{1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda} \right) \right], \quad (7)$$

$$h(x; \lambda, \alpha) = \frac{2\pi\lambda\alpha^2 x^{-3} e^{-(\alpha/x)^2} \left[1 - e^{-(\alpha/x)^2}\right]^{\lambda-1} \cos \left[\left(\pi/2 \right) \left(\left(1 - \left[1 - e^{-(\alpha/x)^2}\right]^\lambda \right) / \left(1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda \right) \right) \right]}{\left(1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda \right)^2 \left[1 - \sin \left[\left(\pi/2 \right) \left(\left(1 - \left[1 - e^{-(\alpha/x)^2}\right]^\lambda \right) / \left(1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda \right) \right) \right] \right]}, \quad (8)$$

$$\tau(x; \lambda, \alpha) = \frac{2\pi\lambda\alpha^2 x^{-3} e^{-(\alpha/x)^2} \left[1 - e^{-(\alpha/x)^2}\right]^{\lambda-1}}{\left(1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda \right)^2} \cot \left[\frac{\pi}{2} \left(\frac{1 - \left[1 - e^{-(\alpha/x)^2}\right]^\lambda}{1 + \left[1 - e^{-(\alpha/x)^2}\right]^\lambda} \right) \right], \quad (9)$$

$$H(x; \lambda, \alpha) = -\ln \left[1 - \sin \left[\frac{\pi}{2} \left(\frac{1 - [1 - e^{-(\alpha/x)^2}]^\lambda}{1 + [1 - e^{-(\alpha/x)^2}]^\lambda} \right) \right] \right] \quad (10)$$

Figures 1 and 2 show plots of the SHLIR PDF and HRF for various parameter values.

We can conclude from Figures 1 and 2 that the PDF of the SHLIR distribution can be unimodal and right skewed. SHLIR distribution HRF can be J-shaped and increasing.

The remainder of this paper is structured as follows. Section 2 discusses some of the structural characteristics of the SHLIR distribution, such as the quantile function, moments, incomplete moments, Lorenz and Bonferroni curves, and various measures of entropy. Section 3 discusses maximum likelihood (ML) parameter estimators for the SHLIR distribution. Section 4 implements simulation schemes. In Section 5, two sets of real-world data

applications are used to demonstrate the potential of the SHLIR distribution in comparison to other distributions. The article concludes with some closing remarks.

2. Statistical Characteristics

Some statistical properties of the SHLIR distribution are obtained in this section.

2.1. Linear Representation. In this section, we will go over the most important linear PDF combinations for SHLIR distribution.

The sine function's series:

$$\sin(Z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} Z^{2i}. \quad (11)$$

By inserting (11) in (6), we get

$$f(x) = \sum_{i=0}^{\infty} \frac{(-1)^i \pi^{2i+1} \lambda \alpha^2 x^{-3} e^{-(\alpha/x)^2} [1 - e^{-(\alpha/x)^2}]^{\lambda-1} [1 - [1 - e^{-(\alpha/x)^2}]^\lambda]^{2i}}{(2i)! 2^{2i-1} \left(1 + [1 - e^{-(\alpha/x)^2}]^\lambda\right)^{2(i+1)}}, \quad (12)$$

where $f(x) = f(x; \lambda, \alpha)$.

Consider the following well-known binomial expansions (for $0 < a < 1$):

$$(1+a)^{-n} = \sum_{j=0}^{\infty} (-1)^j \binom{n+j-1}{j} a^j. \quad (13)$$

Thus, inserting (13) in (12), we get

$$f(x) = \sum_{i,j=0}^{\infty} \frac{\lambda \alpha^2 (-1)^{i+j} \pi^{2i+1}}{x^3 (2i)! 2^{2i-1}} \binom{2i+j+1}{j} e^{-(\alpha/x)^2} [1 - e^{-(\alpha/x)^2}]^{\lambda(j+1)-1} [1 - [1 - e^{-(\alpha/x)^2}]^\lambda]^{2i}. \quad (14)$$

Again, we can using the binomial expansion in the following term:

$$\left[1 - [1 - e^{-(\alpha/x)^2}]^\lambda \right]^{2i} = \sum_{k=0}^{\infty} (-1)^k \binom{2i}{k} [1 - e^{-(\alpha/x)^2}]^{k\lambda}. \quad (15)$$

Therefore, by inserting (15) in (14),

$$f(x) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^{i+j+k} \pi^{2i+1}}{(2i)! 2^{2i-1}} \binom{2i+j+1}{j} \binom{2i}{k} \lambda \alpha^2 x^{-3} e^{-(\alpha/x)^2} [1 - e^{-(\alpha/x)^2}]^{\lambda(k+j+1)-1}. \quad (16)$$

Again, we can using the binomial expansion in the following term:

FIGURE 1: PDF plots for SHLIR distribution.

FIGURE 2: HRF plots for the SHLIR distribution.

$$\left[1 - e^{-(\alpha/x)^2}\right]^{\lambda(k+j+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{\lambda(k+j+1)-1}{m} e^{-m(\alpha/x)^2}. \quad (17)$$

Therefore, by inserting (17) in (16), we can write the PDF of SHLIR as

$$f(x) = 2\alpha^2 \sum_{i=0}^{\infty} \Delta_m x^{-3} e^{-(m+1)(\alpha/x)^2}, \quad (18)$$

where $\Delta_m = \sum_{i,j,k=0}^{\infty} ((-1)^i + j + k + m\pi^{2i+1}) / ((2i)! 2^{2i})$
 $\binom{2i+j+1}{j} \binom{2i}{k} \binom{\lambda(k+j+1)-1}{m} \lambda.$

2.2. Quantile Function. By inverting (5), we can obtain the quantile function of the SHLIR distribution, say $Q(u) = F^{-1}(u)$ of X , as follows:

$$Q(u) = \frac{\alpha}{\sqrt{-\ln\left(1 - \left((1 - (2/\pi)\sin^{-1} u)/(1 + (2/\pi)\sin^{-1} u)\right)^{1/\lambda}\right)}}, \quad (19)$$

where u is thought of as a uniform random variable on $(0, 1)$.

2.3. Moments. If X has PDF (6), then its moment can be calculated using the following relation.

$$\mu_r^\lambda = E(X^r) = \int_0^\infty x^r f(x) dx. \quad (20)$$

Substituting (18) into (20) yields

$$\mu_r^\lambda = 2\alpha^2 \sum_{m=0}^\infty \Delta_m \int_0^\infty x^{r-3} e^{-(m+1)(\alpha/x)^2} dx. \quad (21)$$

Let $y = (\alpha/x)^2$; then,

$$\mu_r^\lambda = \alpha^r \sum_{m=0}^\infty \Delta_m \int_0^\infty y^{-r/2} e^{-(m+1)y} dy. \quad (22)$$

Then, μ_r^λ becomes

$$\mu_r^\lambda = \alpha^r \sum_{m=0}^\infty \Delta_m \frac{\Gamma(1 - (r/2))}{(m+1)^{1-(r/2)}}, \quad \frac{r}{2} < 1. \quad (23)$$

The SHLIR distribution's moment generating function is given by

$$M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r) = \alpha^r \sum_{r,m=0}^\infty \Delta_m \frac{t^r}{r!} \frac{\Gamma(1 - (r/2))}{(m+1)^{1-(r/2)}}, \quad \frac{r}{2} < 1. \quad (24)$$

The incomplete moments of SHLIR are defined by

$$\phi_s(t) = \int_0^t x^s f(x) dx. \quad (25)$$

Using (18), $\phi_s(t)$ will be

$$\phi_s(t) = 2\alpha^2 \sum_{m=0}^\infty \Delta_m \int_0^t x^{s-3} e^{-(m+1)(\alpha/x)^2} dx = \alpha^s \sum_{m=0}^\infty \Delta_m \frac{\Gamma(1 - (s/2), (m+1)(\alpha/t)^2)}{(m+1)^{1-(s/2)}}, \quad (26)$$

where $\Gamma(s, t) = \int_0^t x^{s-1} e^{-x} dx$ is the lower incomplete gamma function.

The Lorenz and Bonferroni curves are obtained as follows:

$$L_F(x) = \frac{\phi_1(x)}{E(X)} = \frac{\alpha \sum_{m=0}^\infty \Delta_m \left(\Gamma\left(\frac{1}{2}, (m+1)(\alpha/t)^2\right) / (m+1)^{1/2} \right)}{\sum_{m=0}^\infty \Delta_m \left((\alpha\sqrt{\pi}) / (m+1)^{1/2} \right)} = \frac{\sum_{m=0}^\infty \left(\left(\Delta_m \Gamma\left(\frac{1}{2}, (m+1)(\alpha/t)^2\right) \right) / (m+1)^{1/2} \right)}{\sum_{m=0}^\infty \left((\Delta_m \sqrt{\pi}) / (m+1)^{1/2} \right)}, \quad (27)$$

and

$$B_F(x) = \frac{L_F(x)}{F(x)} = \frac{\sum_{m=0}^\infty \left(\Delta_m \Gamma\left(\frac{1}{2}, (m+1)(\alpha/t)^2\right) / (i+1)^{1/2} \right)}{\sin\left[\left(\frac{\pi}{2}\right)\left(\left(1 - \left[1 - e^{-(\alpha/x)^2}\right]^\lambda\right)\right)\right] \sum_{m=0}^\infty \left(\Delta_m \sqrt{\pi} / (m+1)^{1/2} \right)}. \quad (28)$$

2.4. Entropies. The entropy of the SHLIR model can be measured by various measures such as Rényi entropy (RE) [28], Havrda and Charvat entropy (HCE) [29], Arimoto entropy (AE) [30], and Tsallis entropy (TE) [31]. These measures of entropy are mentioned in Table 1.

$\int_0^\infty f^\gamma(x)dx$ is very complicated to calculate, so it will be solved numerically.

3. Maximum Likelihood Estimation

To obtain the ML estimators (MLEs) of the SHLIR model with parameters α and λ , let $X_1, \dots, X_n$ be observed values from this distribution. As a result, the log-likelihood function can be written as

$$\ell = n \log 2 + n \log \lambda + 2n \log \alpha - 3 \sum_{i=1}^n \log(x_i) + (\lambda - 1) \sum_{i=1}^n \log(T_i) - 2 \sum_{i=1}^n \log(1 + [T_i]^\lambda) + \sum_{i=1}^n \log \cos \left[\frac{\pi}{2} \left(\frac{1 - [T_i]^\lambda}{1 + [T_i]^\lambda} \right) \right]. \quad (29)$$

The ML equations of the SHLIR distribution are given by

$$\frac{\partial \ell}{\partial \alpha} = \frac{2n}{\alpha} + 2\alpha(\lambda - 1) \sum_{i=1}^n \frac{e^{-(\alpha/x_i)^2}}{x_i^2(T_i)} - 4\alpha\lambda \sum_{i=1}^n \frac{e^{-(\alpha/x_i)^2} [T_i]^{\lambda-1}}{x_i^2(1 + [T_i]^\lambda)} + 2\pi\alpha\lambda \sum_{i=1}^n \frac{e^{-(\alpha/x_i)^2} [T_i]^{\lambda-1}}{x_i^2(1 + [T_i]^\lambda)^2} \tan \left[\frac{\pi}{2} \left(\frac{1 - [T_i]^\lambda}{1 + [T_i]^\lambda} \right) \right], \quad (30)$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \log(T_i) - 2 \sum_{i=1}^n \frac{[T_i]^\lambda \ln[T_i]}{1 + [T_i]^\lambda} + 2 \sum_{i=1}^n \frac{[T_i]^\lambda \ln[1 - e^{-(\alpha/x_i)^2}]}{(1 + [T_i]^\lambda)^2} \tan \left[\frac{\pi}{2} \left(\frac{1 - [T_i]^\lambda}{1 + [T_i]^\lambda} \right) \right], \quad (31)$$

where $T_i = 1 - e^{-(\alpha/x_i)^2}$. Equating $\partial \ell / \partial \alpha$ and $\partial \ell / \partial \lambda$ with zeros and solving simultaneously, we obtain the ML estimators of α and λ .

4. Numerical Results

A numerical result is evaluated and compared to evaluate and compare the behaviour of the estimates in terms of their mean square errors (MSEs). From the SHLIR model, we generate 5000 random samples $X_1, \dots, X_n$ of sizes $n = 10, 20, 30, 50, 100$, and 200 . Four distinct sets of parameters are taken into account, and their ML estimates (MLEs) are computed. The MSEs of the estimated unknown parameters are then computed. In Table 2, the simulated outcomes are listed, and the following observations are found.

For all estimates, the MSEs decrease as sample sizes increase.

5. Applications

Two data analyses are provided in this section to assess the goodness of fit of the SHLIR model in comparison to some known distributions such as type II Topp–Leone IR (TIITLIR) in [32], TIR, and IR distributions.

Maximized likelihood (A1), Akaike information criterion (A2), consistent Akaike information criterion (A3), Bayesian information criterion (A4), and Hannan–Quinn information

criterion (HQIC) were used to compare the models. The model with the lowest values of A1, A2, A3, A4, and A5 is thought to be the best fit for the proposed data.

Data I: Bjerkedal [33] observed and reported the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli.

Data II: they represent the waiting times (in minutes) before service of 100 bank customers, observed and reported by Ghitany et al. [34].

Figures 3 and 4 show the fitted cumulative function (ECDF) of the SHLIR distribution, as well as the ECDFs of the compared models (HLIR, TIITLIR, TIR, and IR) for the first and second datasets.

According to Figures 3 and 4, the SHLIR distribution is the best fit when compared to the other models mentioned above for the two datasets.

Tables 3 and 4 show the ML estimates and standard errors (SEs) for the SHLIR model when compared to some known distributions such as HLIR, TIITLIR, TIR, and IR. Tables 5 and 6 also show the corresponding measures of fit statistic using A1, A2, A3, A4, and A5.

Also, Tables 5 and 6 confirm that the SHLIR distribution is the best fit among the other models for the two datasets, as the SHLIR distribution has the lowest values of A1, A2, A3, A4, and A5.

TABLE 1: Various measures of entropy for a distribution with PDF $f(x)$ at γ .

The measures	Formula
RE	$I_R(\gamma) = (1/1 - \gamma)\text{Log}[\int_0^\infty f^\gamma(x)dx], \quad \gamma \neq 1, \gamma > 0.$
HCE	$HC_R(\gamma) = (1/(2^{1-\gamma} - 1))[\int_0^\infty f^\gamma(x)dx - 1], \quad \gamma \neq 1, \gamma > 0.$
AE	$A_R(\gamma) = (\gamma/1 - \gamma)[(\int_0^\infty f^\gamma(x)dx)^{1/\gamma} - 1], \quad \gamma \neq 1, \gamma > 0.$
TE	$T_R(\gamma) = (1/\gamma - 1)[1 - \int_0^\infty f^\gamma(x)dx], \quad \gamma \neq 1, \gamma > 0.$

TABLE 2: MLEs and MSEs of SHLIR distribution.

n	$\alpha = 1.5, \lambda = 0.8$		$\alpha = 1.5, \lambda = 0.5$		$\alpha = 1.5, \lambda = 1$		$\alpha = 0.5, \lambda = 0.5$	
	MLEs	MSEs	MLEs	MSEs	MLEs	MSEs	MLEs	MSEs
10	1.716	0.305	1.825	0.617	1.699	0.266	0.609	0.072
	0.994	0.290	0.618	0.091	1.317	0.708	0.618	0.091
20	1.630	0.133	1.685	0.214	1.619	0.105	0.564	0.026
	0.907	0.082	0.557	0.024	1.139	0.127	0.557	0.024
30	1.570	0.068	1.593	0.107	1.558	0.058	0.533	0.015
	0.859	0.037	0.539	0.017	1.089	0.077	0.539	0.017
50	1.548	0.040	1.560	0.059	1.540	0.035	0.522	6.335*
	0.841	0.023	0.520	6.079*	1.046	0.036	0.520	6.076*
100	1.526	0.021	1.525	0.028	1.516	0.017	0.510	2.775*
	0.820	8.132*	0.510	2.736*	1.020	0.016	0.510	2.736*
200	1.510	8.196*	1.519	0.014	1.514	6.478*	0.508	1.225*
	0.810	4.216*	0.507	1.178*	1.012	5.964*	0.507	1.178*

* indicates that the value has multiplied by 10^{-3} .

FIGURE 3: Fitted CDFs of models for data I.

FIGURE 4: Fitted CDFs of models for data II.

TABLE 3: The MLE and SE of the model parameters for data I.

Model	MLEs and SEs	
SHLIR (α, λ)	0.439 (0.05)	0.487 (0.061)
HLIR (α, λ)	0.436 (0.05)	0.579 (0.07)
TIITLIR (α, λ)	0.325 (0.036)	0.404 (0.058)
TIR (α, λ)	0.352 (0.426)	-0.942 (0.351)
IR (α)	0.68 (0.04)	

TABLE 4: The MLE and SE of the model parameters for data I.

Model	MLEs and SEs	
SHLIR (α, λ)	2.419 (0.226)	0.499 (0.052)
HLIR (α, λ)	2.404 (0.226)	0.589 (0.06)
TIITLIR (α, λ)	1.824 (0.162)	0.43 (0.051)
TIR (α, λ)	9.978 (1.136)	-0.812 (0.085)
IR (α)	3.619 (0.181)	

TABLE 5: The values of A1, A2, A3, A4, and A5 for data I.

Model	Goodness-of-fit criteria				
	A1	A2	A4	A5	A3
SHLIR	223.087	227.087	226.802	228.9	227.261
HLIR	260.586	264.586	264.301	266.399	264.76
TIITLIR	280.492	284.492	284.207	286.305	284.666
TIR	280.538	284.538	284.253	286.351	284.712
IR	327.518	329.518	329.375	330.424	329.575

TABLE 6: The values of A1, A2, A3, A4, and A5 for data II.

Model	Goodness-of-fit criteria				
	A1	A2	A4	A5	A3
SHLIR	628.185	632.185	632.185	634.293	632.308
HLIR	680.806	684.806	684.806	686.915	684.93
TIITLIR	700.214	704.214	704.214	706.323	704.338
TIR	720.665	724.665	724.665	726.774	724.706
IR	759.629	761.629	761.629	762.683	761.67

6. Conclusion

This article investigates a new two-model distribution known as the sine half-logistic inverse Rayleigh (SHLIR). Some fundamental statistical properties of the SHLIR model are calculated and discussed, including the quantile function, moments, moment generating function, incomplete moment, and Lorenz and Bonferroni curves. Entropy measures such as Rényi entropy, Havrda and Charvat entropy, Arimoto entropy, and Tsallis entropy are investigated. The model parameter estimation is discussed using the ML method. Applications to two real datasets show that the SHLIR model outperforms other well-known competitive models such as the HLIR, TIITLIR, TIR, and IR models in terms of fit.

Data Availability

The numerical dataset used to carry out the analysis reported in this article is available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by King Saud University, Deanship of Scientific Research, College of Science Research Center.

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