Research Article

Novel Construction of Copulas Based on (α, β) Transformation for Fuzzy Random Variables

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The paper introduces a method for the construction of bivariate copulas with the usage of specific values of the parameters α and β ((α, β) transformation) and the parameters κ and λ in their domain. The produced bivariate copulas are defined in four subrectangles of the unit square. The bounds of the produced copulas are investigated, while a novel construction method for fuzzy copulas is introduced, with the usage of the produced copulas via (α, β) transformation in four subrectangles of the unit square. Following this construction procedure, the production of an infinite number of copulas and fuzzy copulas could be possibly achieved. Some applications of the proposed methods are presented.

1. Introduction

Copulas are a significant member of aggregation functions on the unit interval $[0, 1]$. The ability to construct aggregation functions with numerous processes and methods is of great importance, since it is essential for the researchers to not deviate from the real-life data. Sklar [1] presented the concept of copulas, by means of a mathematical tool that describes the stochastic dependence structure within random variables. There are several procedures regarding the construction of copulas, based on given ones, in the literature, such as the construction of asymmetric multivariate copulas [2], which is connected with the product of copulas and the generalization of Archimedean copulas. Another construction of copulas, produced by the gluing of two or more copulas, is presented in [3]. In [4], three types of ordinal sums based on product copula are introduced as construction methods. A representation via the g-ordinal sums of copulas is introduced in [5]. In [6], a method for the construction of bivariate copulas by the modifications of given copulas on some subrectangles of the unit square is contained. Two different representations of 2-increasing aggregation functions, via the lower and the upper margins

and a copula, are provided in [7]. The construction of copulas as a patchwork-like assembly of arbitrary copulas, with nonoverlapping rectangles as patches, is included in [8]. In [9], the set of copulas with the given horizontal section was studied and extended. The family of (α, β) -homogenous copulas was introduced in [10]. A general construction of copulas with given a horizontal and a vertical section is introduced in [11]. One of the most important methods is the flipped and survival copulas [12]. Those are special cases of the (α, β) -transformation, which is a more general construction method [12].

On the other hand, regarding the real-life problems, researchers may handle data possibly imprecise. In order to deal with imprecise or vague information, fuzzy sets [13] are the most adequate tools for someone to establish. In [14], the fuzzy random variables are provided in order to represent the relationship between random experiments results and nonstatistical imprecise data. Thus, the notion of fuzzy copulas is introduced in [15] to describe the stochastic dependence structure between two fuzzy random variables.

This paper shares two main purposes. The first is to provide a novel construction method for copulas, more general than the existing ones, based on the

(α, β) -transformation. The second is to present a new fuzzy copula construction between two fuzzy random variables, via the construction mentioned before. The investigation of fuzzy copulas is of great importance, as in many circumstances, researchers need to fuse or aggregate probabilistic and fuzzy information [16–18]. In the present paper, we aim to provide a novel construction method of copulas, in order to produce a construction procedure of fuzzy copulas that no attempt has been made since the concept of fuzzy copulas was recently developed. The properties of the proposed copula and fuzzy copula construction methods are taken into consideration. Meanwhile, the bounds of the constructed copulas are presented. In conclusion, the proposed methods are illustrated via some numerical examples.

The paper adopts the following structure: in Section 2, some necessary notions of copulas, fuzzy sets, and fuzzy random variables are presented. In Section 3, the novel construction of copulas and the bounds of the produced copulas are presented. In Section 4, the new construction method of fuzzy copulas is introduced. In Section 5, concluding remarks are mentioned.

2. Preliminaries

In this section, some basic definitions are provided, in order for the new construction method of copulas and fuzzy copulas to be introduced.

2.1. Copulas, the Crisp Approach, Notions, and Definitions

Definition 1 (see [1]). A 2-dimensional copula is a function with domain $\text{Dom}C = [0, 1] \times [0, 1]$ and range $\text{Ran}C = [0, 1]$ that is grounded and double-increasing, i.e., it satisfies the following conditions:

$$\begin{aligned} C(x, 0) &= C(0, y) = 0, \\ C(x, 1) &= x, \\ C(1, y) &= y, \\ V_C([x_1, x_2] \times [y_1, y_2]) &\geq 0, \end{aligned} \quad (1)$$

where $x, y, x_1, x_2, y_1, y_2 \in [0, 1]$ and $x_1 \leq x_2, y_1 \leq y_2$.

The definition of survival copula is provided as follows, according to [12].

Definition 2 (see [12]). The survival copula of a copula is defined as

$$\hat{C}(x, y) = x + y - 1 + C(1 - x, 1 - y). \quad (2)$$

According to Nelsen [12], let X and Y be two random variables with joint distribution function F_{XY} , with margins F_X and F_Y . Then, there exists a copula C , such that

$$F_{XY}(x, y) = C(F_X(x), F_Y(y)), \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}. \quad (3)$$

Also, if C is a copula and F_X and F_Y are cumulative distribution functions, then $F_{X,Y}$ is a joint distribution

function, with margins F_X and F_Y . If F_X^{-1} and F_Y^{-1} are the inverses of F_X and F_Y , respectively, then

$$C(u, v) = F_{XY}(F_X^{-1}(u), F_Y^{-1}(v)), \quad \forall (u, v) \in [0, 1] \times [0, 1]. \quad (4)$$

As it was mentioned in Section 1, a construction method of copulas is (α, β) -transformation that is defined [12] for parameters α and β , that is, $(\alpha, \beta) \in [0, 1]^2$. The transformation of a copula C into the copula $C_{\alpha, \beta}$ is defined on the unit square by

$$C_{\alpha, \beta}(x, y) = V_C([\alpha(1 - x), x + \alpha(1 - x)] \times [\beta(1 - y), y + \beta(1 - y)]). \quad (5)$$

As a result, for specific values of the parameters α and β (in their domain), different copulas can be constructed. In case that $\alpha = 0$ and $\beta = 0$, we obtain the copula:

$$C_{0,0}(x, y) = C(x, y). \quad (6)$$

In case that $\alpha = 0$ and $\beta = 1$, we obtain the copula:

$$C_{0,1}(x, y) = x - C(x, 1 - y). \quad (7)$$

In case that $\alpha = 1$ and $\beta = 0$, we obtain the copula:

$$C_{1,0}(x, y) = y - C(1 - x, y). \quad (8)$$

In case that $\alpha = 1$ and $\beta = 1$, we obtain the copula:

$$C_{1,1}(x, y) = x + y - 1 + C(1 - x, 1 - y) = \hat{C}(x, y). \quad (9)$$

In the first case, we get the original copula; in the second and third cases, we get the flipped copulas; and in the last case, we get the survival copula.

2.2. Fuzzy Sets, Notions, and Definitions. Let X be a universal set. Each function $\tilde{A}: X \rightarrow [0, 1]$ is called a fuzzy set of X , where $\mu_{\tilde{A}}(x)$'s interpretation is the membership degree of x in the fuzzy set $\tilde{A}$. Crisp (classical) sets are special cases of fuzzy sets, with $\tilde{A}(x) = 0$, or $\tilde{A}(x) = 1$. The α -cuts of a fuzzy set $\tilde{A}$ are defined by $\tilde{A}_{[\alpha]} = \{x: \mu_{\tilde{A}}(x) \geq \alpha\}$, $\forall \alpha \in (0, 1]$, where $\tilde{A}_{[0]}$, which is called support, is the closure in the topology of X of the union of all the α -cuts [19], i.e., $\tilde{A}_{[0]} = \overline{\cup_{\alpha \in (0, 1]} \tilde{A}_{[\alpha]}} = \overline{\{x: \mu_{\tilde{A}}(x) > 0\}}$. Now, a fuzzy set $\tilde{A}$ of $\mathbb{R}$ is called a fuzzy number if

- 1 $\exists x \in \mathbb{R}: \mu_{\tilde{A}}(x) = 1$, which means that $\tilde{A}$ is normal
- 2 $\forall k \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}$; we have
$$\mu_{\tilde{A}}((1 - k)x_1 + kx_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}, \quad (10)$$

which means that $\tilde{A}$ is convex fuzzy set

- 3 $\forall \alpha \in [0, 1]$, $\tilde{A}_{[\alpha]}$ is a nonempty compact interval in $\mathbb{R}$, which means that $\tilde{A}$ has compact support

The interval of the α -cuts is denoted by $\tilde{A}_{[\alpha]} = [\tilde{A}_{[\alpha]}^L, \tilde{A}_{[\alpha]}^R]$, where $\tilde{A}_{[\alpha]}^L = \inf\{x: x \in \tilde{A}_{[\alpha]}\}$ and $\tilde{A}_{[\alpha]}^R = \sup\{x: x \in \tilde{A}_{[\alpha]}\}$. We denote the set of all fuzzy

numbers by $\mathcal{F}(\mathbb{R})$. In [20], based on [21], some of the operations of α -cuts were presented as follows.

Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R})$ and $\tilde{A}_{[\alpha]} = [\tilde{A}_{[\alpha]}^L, \tilde{A}_{[\alpha]}^R]$, $\tilde{B}_{[\alpha]} = [\tilde{B}_{[\alpha]}^L, \tilde{B}_{[\alpha]}^R]$ be their α -cuts, respectively. Then, $\forall \alpha \in [0, 1]$, and the fuzzy addition and the scalar multiplication were defined as follows:

$$\begin{aligned} [\tilde{A} \oplus \tilde{B}]_{[\alpha]} &= [\tilde{A}_{[\alpha]}^L + \tilde{B}_{[\alpha]}^L, \tilde{A}_{[\alpha]}^R + \tilde{B}_{[\alpha]}^R], \\ [\lambda \otimes \tilde{A}]_{[\alpha]} &= [\lambda, \lambda] \left[\tilde{A}_{[\alpha]}^L, \tilde{A}_{[\alpha]}^R \right] = [\lambda \tilde{A}_{[\alpha]}^L, \lambda \tilde{A}_{[\alpha]}^R], \end{aligned} \quad (11)$$

respectively, where the scalar $\lambda \in \mathbb{R}^+$ was identified as the interval $[\lambda, \lambda]$.

Definition 3 (see [22]). Let $\tilde{A}$ be a fuzzy number ($\tilde{A} \in \mathcal{F}(\mathbb{R})$) and $x \in \mathbb{R}$. Then, the index $\text{Cr}: \mathcal{F}(\mathbb{R}) \times \mathbb{R} \rightarrow [0, 1]$ is defined by

$$\text{Cr}\{\tilde{A} \leq x\} = \frac{1}{2} \left[\sup_{y \leq x} \mu_{\tilde{A}}^-(y) + 1 - \sup_{y > x} \mu_{\tilde{A}}^-(y) \right]. \quad (12)$$

This gives the credibility degree, that is, $\tilde{A}$ is less than or equal to x .

Remark 1 (see [22]). Let $\tilde{A} \in \mathcal{F}(\mathbb{R})$ and $x \in \mathbb{R}$, then

$$1 \quad \text{Cr}\{\tilde{A} \leq x\} = 1, \quad (13)$$

if and only if

$$\tilde{A}_0^R \leq x. \quad (14)$$

2 For any fixed $\tilde{A}$, $\text{Cr}\{\tilde{A} \leq x\}$ is a nondecreasing function with respect to x , i.e.,

$$\forall x_1 < x_2, \quad \text{Cr}\{\tilde{A} \leq x_1\} \leq \text{Cr}\{\tilde{A} \leq x_2\}. \quad (15)$$

3 Cr is self-dual, i.e.,

$$\text{Cr}\{\tilde{A} > x\} = 1 - \text{Cr}\{\tilde{A} \leq x\}. \quad (16)$$

Definition 4 (see [23]). Let $\tilde{A} \in \mathcal{F}(\mathbb{R})$ and $\alpha \in (0, 1]$, then the α -pessimistic value of $\tilde{A}$ is given by $\tilde{A}_\alpha = \inf\{x \in \tilde{A}_{[0]}: \text{Cr}\{\tilde{A} \leq x\} \geq \alpha\}$. Furthermore, $\tilde{A}_\alpha$ is a nondecreasing function of $\alpha \in (0, 1]$.

Remark 2 (see [24]). For a given $\tilde{A} \in \mathcal{F}(\mathbb{R})$, let $\tilde{A}_\alpha$ be defined $\forall \alpha \in (0, 1]$, by

$$\tilde{A}_\alpha = \begin{cases} \tilde{A}_{2\alpha}^L, & \alpha \in \left(0, \frac{1}{2}\right], \\ \tilde{A}_{2(1-\alpha)}^R, & \alpha \in \left(\frac{1}{2}, 1\right]. \end{cases} \quad (17)$$

Then, the α -cuts of $\tilde{A}$ are given by

$$\tilde{A}_{[\alpha]} = [\tilde{A}_{\alpha/2}, \tilde{A}_{1-(\alpha/2)}]. \quad (18)$$

Lemma 1 (see [15]). Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R})$ and let $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned} (\tilde{A} \oplus \tilde{B})_\alpha &= \tilde{A}_\alpha + \tilde{B}_\alpha, \\ (\lambda \otimes \tilde{A})_\alpha &= \begin{cases} \lambda \tilde{A}_\alpha, & \lambda \geq 0, \\ \lambda \tilde{A}_{1-\alpha}, & \lambda \leq 0. \end{cases} \end{aligned} \quad (19)$$

Proof. The proof of Lemma 1 can be found in [15]. $\square$

Definition 5 (see [24]). Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R})$, $\forall \alpha \in (0, 1]$. If $\tilde{A}_\alpha \leq \tilde{B}_\alpha$, then $\tilde{A} < \tilde{B}$, and if $\tilde{A}_\alpha = \tilde{B}_\alpha$, then $\tilde{A} = \tilde{B}$.

Example 1. Let $x \in \mathbb{R}$ and $\tilde{A}$ be a nonsymmetric triangular fuzzy number with membership function given by

$$\mu_{\tilde{A}}^-(x) = \begin{cases} 0, & -\infty < x \leq 5, \\ \frac{x-5}{3}, & 5 \leq x \leq 8, \\ 1, & x = 8, \\ 9-x, & 8 \leq x \leq 9, \\ 0, & 9 \leq x < +\infty. \end{cases} \quad (20)$$

Then, the credibility degree that $\tilde{A}$ is less than or equal to x is given by

$$\text{Cr}\{\tilde{A} \leq x\} = \begin{cases} 0, & -\infty < x \leq 5, \\ \frac{x-5}{6}, & 5 \leq x \leq 8, \\ \frac{1}{2}, & x = 8, \\ \frac{x-7}{2}, & 8 \leq x \leq 9, \\ 1, & 9 \leq x < +\infty. \end{cases} \quad (21)$$

As a result, we obtain the α -pessimistic values of $\tilde{A}$ by

$$\tilde{A}_\alpha = \begin{cases} 6\alpha + 5, & 0 < \alpha \leq \frac{1}{2}, \\ 2\alpha + 7, & \frac{1}{2} \leq \alpha \leq 1. \end{cases} \quad (22)$$

2.3. Fuzzy Random Variables. The concept of fuzzy random variables is one of the most adequate tools to handling the results of random experiments, expressed in nonexact terms. In order to integrate randomness and vagueness, random fuzzy sets and random fuzzy numbers [25], are introduced. In most real-life problems, the nature of the data of the experiments is affected by fuzziness, and the procedure of

the extraction of the data of the experiments is affected by randomness. Thus, the definition of fuzzy random variables to be considered in this paper was given in [24], as in the following definition.

Definition 6. (see [24]). Let Ω be the set of all possible outcomes of a random experiment, let $\mathcal{A}$ be the σ -algebra of the subsets of Ω , and let $\mathbb{P}$ be the probability measure on the measurable space $(\Omega, \mathcal{A})$. Now, suppose that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ describes the random experiment. If $\forall \alpha \in [0, 1]$, $\tilde{X}_\alpha: \Omega \rightarrow \mathbb{R}$ is a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$, then $\tilde{X}: \Omega \rightarrow \mathcal{F}(\mathbb{R})$ is called a fuzzy random variable.

The notion of fuzzy random variables was introduced in [14, 26]. In [27], the notion of fuzzy random variables was formalized with the following approach: let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. If $\forall \alpha \in [0, 1]$, the two mappings $\tilde{X}_\alpha^L: \Omega \rightarrow \mathbb{R}$ and $\tilde{X}_\alpha^R: \Omega \rightarrow \mathbb{R}$ are random variables, then $\tilde{X}: \Omega \rightarrow \mathcal{F}(\mathbb{R})$ is a fuzzy random variable.

Remark 3. (see [28]). In the next relationships, summarize the data of the two dimensional variable $(\tilde{X}_\alpha^L, \tilde{X}_\alpha^R)$, in the one-dimensional variable $\tilde{X}_\alpha$.

$$\tilde{X}_\alpha = \begin{cases} \tilde{X}_{2\alpha}^L, & 0 < \alpha \leq \frac{1}{2}, \\ \tilde{X}_{2(1-\alpha)}^R, & \frac{1}{2} \leq \alpha \leq 1, \end{cases} \quad (23)$$

$$\tilde{X}_{[\alpha]} = [\tilde{X}_{\alpha/2}, \tilde{X}_{1-(\alpha/2)}], \quad \alpha \in (0, 1].$$

Definition 7 (see [15]). If $\tilde{X}_\alpha$ and $\tilde{Y}_\alpha$ are independent $\forall \alpha \in [0, 1]$, then $\tilde{X}$ and $\tilde{Y}$ are independent fuzzy random variables. If $\tilde{X}_\alpha$ and $\tilde{Y}_\alpha$ are identically distributed $\forall \alpha \in [0, 1]$, then $\tilde{X}$ and $\tilde{Y}$ are identically distributed fuzzy random variables.

3. The Novel Construction Method of Copulas

In this section, a novel construction method of copulas is provided, via the (α, β) -transformation, in four different subrectangles of the unit square. This becomes feasible by the jointing process of the four cases produced in Section 2.1 for specific values of the parameters α and β , with the adequate adjustments. The construction is achieved through the following theorem.

Theorem 1. Let κ and λ be fixed in $[0, 1]$ and $C: [0, 1] \times [0, 1] \rightarrow [0, 1]$ be any copula. Then, the function $\mathfrak{C}$, defined by

$$\mathfrak{C}(x, y) = \begin{cases} C\left(\frac{x}{\kappa}, \frac{y}{\lambda}\right)\kappa\lambda, & (x, y) \in [0, \kappa] \times [0, \lambda], \\ x - C\left(\frac{x}{\kappa}, \frac{1-y}{1-\lambda}\right)\kappa(1-\lambda), & (x, y) \in [0, \kappa] \times [\lambda, 1], \\ y - C\left(\frac{1-x}{1-\kappa}, \frac{y}{\lambda}\right)(1-\kappa)\lambda, & (x, y) \in [\kappa, 1] \times [0, \lambda], \\ x + y - 1 + C\left(\frac{1-x}{1-\kappa}, \frac{1-y}{1-\lambda}\right)(1-\kappa)(1-\lambda), & (x, y) \in [\kappa, 1] \times [\lambda, 1], \end{cases} \quad (24)$$

is also a copula.

Proof. The proof that the function $\mathfrak{C}$ is well defined and the boundary conditions of Definition 1 are satisfied is straightforward. In order to prove that $\mathfrak{C}$ is double-

increasing, the proof that $\mathfrak{C}$ is double-increasing in each one of the four rectangles of the domain is needed. For the first rectangle, we have $0 \leq x_1 \leq x_2 \leq \kappa$ and $0 \leq y_1 \leq y_2 \leq \lambda$, and with the usage of the first branch of $\mathfrak{C}$, we obtain

$$V_{\mathfrak{C}}([x_1, x_2] \times [y_1, y_2]) = \kappa\lambda \left(V_C\left(\left[\frac{x_1}{\kappa}, \frac{x_2}{\kappa}\right] \times \left[\frac{y_1}{\lambda}, \frac{y_2}{\lambda}\right]\right) \right) \geq 0 \quad (25)$$

since $\kappa, \lambda \in [0, 1]$ and C is double-increasing as a copula. Hence, $\mathfrak{C}$ is double-increasing in the rectangle $[0, \kappa] \times [0, \lambda]$. For the rectangle $[0, \kappa] \times [\lambda, 1]$, we have $0 \leq x_1 \leq x_2 \leq \kappa$ and

$$V_{\mathfrak{C}}([x_1, x_2] \times [y_1, y_2]) = \kappa(1 - \lambda) \left(V_C \left(\left[\frac{x_1}{\kappa}, \frac{x_2}{\kappa} \right] \times \left[\frac{1 - y_2}{1 - \lambda}, \frac{1 - y_1}{1 - \lambda} \right] \right) \right) \geq 0 \quad (26)$$

since $\kappa, 1 - \lambda \in [0, 1]$ and C is double-increasing as a copula. Hence, $\mathfrak{C}$ is double-increasing in the rectangle $[0, \kappa] \times [\lambda, 1]$. For the rectangle $[\kappa, 1] \times [0, \lambda]$, we have $\kappa \leq x_1 \leq x_2 \leq 1$ and

$$V_{\mathfrak{C}}([x_1, x_2] \times [y_1, y_2]) = (1 - \kappa)\lambda \left(V_C \left(\left[\frac{1 - x_2}{1 - \kappa}, \frac{1 - x_1}{1 - \kappa} \right] \times \left[\frac{y_1}{\lambda}, \frac{y_2}{\lambda} \right] \right) \right) \geq 0 \quad (27)$$

since $1 - \kappa, \lambda \in [0, 1]$ and C is double-increasing as a copula. Hence, $\mathfrak{C}$ is double-increasing in the rectangle $[\kappa, 1] \times [0, \lambda]$. For the rectangle $[\kappa, 1] \times [\lambda, 1]$, we have $\kappa \leq x_1 \leq x_2 \leq 1$ and $\lambda \leq y_1 \leq y_2 \leq 1$, and with the usage of the fourth branch of $\mathfrak{C}$, we obtain

$$V_{\mathfrak{C}}([x_1, x_2] \times [y_1, y_2]) = (1 - \kappa)(1 - \lambda) \left(V_C \left(\left[\frac{1 - x_2}{1 - \kappa}, \frac{1 - x_1}{1 - \kappa} \right] \times \left[\frac{1 - y_2}{1 - \lambda}, \frac{1 - y_1}{1 - \lambda} \right] \right) \right) \geq 0 \quad (28)$$

since $1 - \kappa, \lambda \in [0, 1]$ and C is double-increasing as a copula. Hence, $\mathfrak{C}$ is double-increasing in the rectangle $[\kappa, 1] \times [0, \lambda]$. Finally, $\mathfrak{C}$ is double-increasing in the rectangle $[0, 1]^2$. As a result, $\mathfrak{C}$ is a copula, and the proof is completed. $\square$

$\lambda \leq y_1 \leq y_2 \leq 1$, and with the usage of the second branch of $\mathfrak{C}$, we obtain

$0 \leq y_1 \leq y_2 \leq \lambda$, and with the usage of the third branch of $\mathfrak{C}$, we obtain

Remark 4. Let C be the product copula Π in Theorem 1. Then, $\forall x, y \in [0, 1]$, $\mathfrak{C}(x, y) = \Pi(x, y)$.

The flowchart in Figure 1 illustrates the novel copula construction process.

The following example illustrates the construction of a copula, with the usage of Theorem 1.

Example 2. Let $\kappa = 2/3$, $\lambda = 3/4$, and $C_1(x, y) = (xy/(x + y - xy))$, which is a member of the Ali-Mikhail-Haq family of copulas [12]. According to Theorem 1, the following copula is constructed:

$$\mathfrak{C}_1(x, y) = \begin{cases} \frac{6xy}{9x + 8y - 12xy}, & (x, y) \in \left[0, \frac{2}{3}\right] \times \left[0, \frac{3}{4}\right], \\ \frac{12x^2y + 6x - 6xy - 9x^2}{12xy + 8 - 8y - 9x}, & (x, y) \in \left[0, \frac{2}{3}\right] \times \left[\frac{3}{4}, 1\right], \\ \frac{12xy^2 - 8y^2 + 6y - 6xy}{-8y - 9x + 9 + 12xy}, & (x, y) \in \left[\frac{2}{3}, 1\right] \times \left[0, \frac{3}{4}\right], \\ \frac{9x^2 + 8y^2 - 12x^2y - 12xy^2 + 30xy - 15x - 14y + 6}{9x + 8y - 5 - 12xy}, & (x, y) \in \left[\frac{2}{3}, 1\right] \times \left[\frac{3}{4}, 1\right]. \end{cases} \quad (29)$$

The produced copula of Example 1, is presented in Figure 2.

Fréchet [29] and Hoeffding [30] introduced the Fréchet–Hoeffding [12] bounds of copulas for any $x, y \in [0, 1]$, $\max\{x + y - 1, 0\} \leq C(x, y) \leq \min\{x, y\}$. In

order to present the bounds of the produced copulas $\mathfrak{C}$, we provide the following theorem.

Theorem 2. Let κ and λ be fixed in $[0, 1]$ and C and $\mathfrak{C}$ be two copulas of Theorem 1. Then, the functions $\mathfrak{C}_-$ and $\mathfrak{C}_+$ with

FIGURE 1: Flowchart of copula construction process.

FIGURE 2: Produced copula of C_1 with $\kappa = 2/3$ and $\lambda = 3/4$ (left axis represents y , right axis represents x , and vertical axis represents $C_1(x, y)$).

$\text{Dom}\mathfrak{C}_\sim = \text{Dom}\mathfrak{C}^\sim = [0, 1]^2$ and $\text{Ran}\mathfrak{C}_\sim = \text{Ran}\mathfrak{C}^\sim = [0, 1]$, given by

$$\mathfrak{C}_\sim(x, y) = \begin{cases} \max\{0, \lambda x + \kappa y - \kappa\lambda\}, & (x, y) \in [0, \kappa] \times [0, \lambda], \\ x - \min\{(1 - \lambda)x, \kappa(1 - y)\}, & (x, y) \in [0, \kappa] \times [\lambda, 1], \\ y - \min\{(1 - \kappa)y, \lambda(1 - x)\}, & (x, y) \in [\kappa, 1] \times [0, \lambda], \\ x + y - 1 + \max\{0, (1 - \lambda)(1 - x) + (1 - \kappa)(1 - y) - (1 - \kappa)(1 - \lambda)\}, & (x, y) \in [\kappa, 1] \times [\lambda, 1], \end{cases} \quad (30)$$

$$\mathfrak{C}^\sim(x, y) = \begin{cases} \min\{\lambda x, \kappa y\}, & (x, y) \in [0, \kappa] \times [0, \lambda], \\ x - \max\{0, (1 - \lambda)x + \kappa(1 - y) - \kappa(1 - \lambda)\}, & (x, y) \in [0, \kappa] \times [\lambda, 1], \\ y - \max\{0, \lambda(1 - x) + (1 - \kappa)y - (1 - \kappa)\lambda\}, & (x, y) \in [\kappa, 1] \times [0, \lambda], \\ x + y - 1 + \min\{(1 - \lambda)(1 - x)(1 - \kappa)(1 - y)\}, & (x, y) \in [\kappa, 1] \times [\lambda, 1], \end{cases}$$

respectively, are copulas, and $\mathfrak{C}_\sim(x, y) \leq \mathfrak{C}(x, y) \leq \mathfrak{C}^\sim(x, y)$, for every copula, $\forall x, y \in [0, 1]$.

Proof. The proof that $\mathfrak{C}_\sim$ and $\mathfrak{C}^\sim$ have $\text{Dom}\mathfrak{C}_\sim = \text{Dom}\mathfrak{C}^\sim = [0, 1]^2$ and $\text{Ran}\mathfrak{C}_\sim = \text{Ran}\mathfrak{C}^\sim = [0, 1]$, and that satisfy the boundary conditions of copulas is straightforward. In addition, for the proof that $\mathfrak{C}_\sim$ and $\mathfrak{C}^\sim$ are double-increasing, we have to examine this in each one of the four rectangles of their domain. For the function $\mathfrak{C}^\sim$, we have that $\forall x_1, x_2 \in [0, \kappa]$ and $\forall y_1, y_2 \in [0, \lambda]$, with $x_1 \leq x_2$ and $y_1 \leq y_2$:

$$\kappa\lambda \left(V_M \left(\left[\frac{x_1}{\kappa}, \frac{x_2}{\kappa} \right] \times \left[\frac{y_1}{\lambda}, \frac{y_2}{\lambda} \right] \right) \right) \geq 0 \quad (31)$$

since $\kappa, \lambda \in [0, 1]$, and M is double-increasing as a copula. Hence, $\mathfrak{C}^\sim$ is double-increasing in $[0, \kappa] \times [0, \lambda]$. Next, $\forall x_1, x_2 \in [0, \kappa]$ and $\forall y_1, y_2 \in [\lambda, 1]$, with $0 \leq (x_1/\kappa) \leq (x_2/\kappa) \leq 1$ and $\lambda \leq y_1 \leq y_2 \leq 1 \Rightarrow 0 \leq ((1 - y_2)/(1 - \lambda)) \leq ((1 - y_1)/(1 - \lambda)) \leq 1$:

$$\kappa(1 - \lambda) \left(V_W \left(\left[\frac{x_1}{\kappa}, \frac{x_2}{\kappa} \right] \times \left[\frac{1 - y_2}{1 - \lambda}, \frac{1 - y_1}{1 - \lambda} \right] \right) \right) \geq 0 \quad (32)$$

since $\kappa, 1 - \lambda \in [0, 1]$, and W is double-increasing as a copula. Hence, $\mathfrak{C}^\sim$ is double-increasing in $[0, \kappa] \times [\lambda, 1]$. Next, $\forall x_1, x_2 \in [\kappa, 1]$ and $\forall y_1, y_2 \in [0, \lambda]$, with $\kappa \leq x_1 \leq x_2 \leq 1 \Rightarrow 0 \leq ((1 - x_2)/(1 - \kappa)) \leq ((1 - x_1)/(1 - \kappa)) \leq 1$ and $0 \leq (y_1/\lambda) \leq (y_2/\lambda) \leq 1$:

$$(1 - \kappa)\lambda \left(V_W \left(\left[\frac{1 - x_2}{1 - \kappa}, \frac{1 - x_1}{1 - \kappa} \right] \times \left[\frac{y_1}{\lambda}, \frac{y_2}{\lambda} \right] \right) \right) \geq 0 \quad (33)$$

since $1 - \kappa, \lambda \in [0, 1]$, and W is double-increasing as a copula. Hence, $\mathfrak{C}^\sim$ is double-increasing in $[\kappa, 1] \times [0, \lambda]$. Next, $\forall x_1, x_2 \in [\kappa, 1]$ and $\forall y_1, y_2 \in [\lambda, 1]$, with $\kappa \leq x_1 \leq x_2 \leq 1 \Rightarrow 0 \leq ((1 - x_2)/(1 - \kappa)) \leq ((1 - x_1)/(1 - \kappa)) \leq 1$ and $\lambda \leq y_1 \leq y_2 \leq 1 \Rightarrow 0 \leq ((1 - y_2)/(1 - \lambda)) \leq ((1 - y_1)/(1 - \lambda)) \leq 1$

$$(1 - \kappa)(1 - \lambda) \left(V_M \left(\left[\frac{1 - x_2}{1 - \kappa}, \frac{1 - x_1}{1 - \kappa} \right] \times \left[\frac{1 - y_2}{1 - \lambda}, \frac{1 - y_1}{1 - \lambda} \right] \right) \right) \geq 0 \quad (34)$$

since $1 - \kappa, 1 - \lambda \in [0, 1]$ and M is double-increasing as a copula. Hence, $\mathfrak{C}^\sim$ is double-increasing in $[\kappa, 1] \times [\lambda, 1]$. As a result, $\mathfrak{C}^\sim$ is a copula. The proof that $\mathfrak{C}_\sim$ is a copula can be considered in the same manner. For the case of the first rectangle, we have $0 \leq x \leq \kappa$ and $0 \leq y \leq \lambda$, and with the usage of the first branch of copula $\mathfrak{C}$ and the Fréchet–Hoeffding [12] bounds of copulas, we obtain

$$\max\{0, x + y - 1\} \leq C(x, y) \leq \min\{x, y\} \Leftrightarrow$$

$$\kappa\lambda \max\left\{0, \frac{x}{\kappa} + \frac{y}{\lambda} - 1\right\} \leq \kappa\lambda C\left(\frac{x}{\kappa}, \frac{y}{\lambda}\right) \leq \kappa\lambda \min\left\{\frac{x}{\kappa}, \frac{y}{\lambda}\right\} \Leftrightarrow$$

$$\max\{0, \lambda x + \kappa y - \kappa\lambda\} \leq \mathfrak{C}(x, y) \leq \min\{\lambda x, \kappa y\}. \quad (35)$$

For the cases of the other three rectangles, with the same approach as the case of the first rectangle, we obtain the desirable equations. Hence, $\mathfrak{C}_\sim$ and $\mathfrak{C}^\sim$ are copulas, and the bounds of every copula $\mathfrak{C}$, and as a result, the proof is completed. $\square$

The present construction method of copulas that can be achieved through Theorem 1, and based on the fact that $\mathfrak{C}_\sim \neq \mathfrak{C}^\sim$, of Theorem 2, can lead us in the result that there exists an infinite number of copulas that can be constructed via this transformation. The disadvantage of this method is that, for the construction of copula $\mathfrak{C}$, we are using for each of the four branches only the copula C .

Example 3. Let $\kappa = 7/8$, $\lambda = \sqrt{2}/2$, and copulas $\mathfrak{C}_\sim$ and $\mathfrak{C}^\sim$ of Theorem 2. Then, the produced copulas for those specific values of κ, λ are given by

$$\begin{aligned}
\mathfrak{C}_\sim(x, y) = & \begin{cases} \max\left\{0, \frac{\sqrt{2}x}{2} + \frac{7y}{8} - \frac{7\sqrt{2}}{16}\right\}, & (x, y) \in \left[0, \frac{7}{8}\right] \times \left[0, \frac{\sqrt{2}}{2}\right], \\ x - \min\left\{\frac{(2 - \sqrt{2})x}{2}, \frac{7(1 - y)}{8}\right\}, & (x, y) \in \left[0, \frac{7}{8}\right] \times \left[\frac{\sqrt{2}}{2}, 1\right], \\ y - \min\left\{\frac{y}{8}, \frac{\sqrt{2}(1 - x)}{2}\right\}, & (x, y) \in \left[\frac{7}{8}, 1\right] \times \left[0, \frac{\sqrt{2}}{2}\right], \\ x + y - 1 + \\ + \max\left\{0, \frac{16 - 7\sqrt{2} - 8(2 - \sqrt{2})x - 2y}{16}\right\}, & (x, y) \in \left[\frac{7}{8}, 1\right] \times \left[\frac{\sqrt{2}}{2}, 1\right], \end{cases} \\
\mathfrak{C}^\sim(x, y) = & \begin{cases} \min\left\{\frac{\sqrt{2}x}{2}, \frac{7y}{8}\right\}, & (x, y) \in \left[0, \frac{7}{8}\right] \times \left[0, \frac{\sqrt{2}}{2}\right], \\ x - \max\left\{0, \frac{(2 - \sqrt{2})x}{2} - \frac{7y}{8} + \frac{7\sqrt{2}}{16}\right\}, & (x, y) \in \left[0, \frac{7}{8}\right] \times \left[\frac{\sqrt{2}}{2}, 1\right], \\ y - \max\left\{0, \frac{3\sqrt{2}}{8} - \frac{\sqrt{2}x}{2} + \frac{y}{8}\right\}, & (x, y) \in \left[\frac{7}{8}, 1\right] \times \left[0, \frac{\sqrt{2}}{2}\right], \\ x + y - 1 + \\ + \min\left\{\frac{(2 - \sqrt{2})(1 - x)}{2}, \frac{1 - y}{8}\right\}, & (x, y) \in \left[\frac{7}{8}, 1\right] \times \left[\frac{\sqrt{2}}{2}, 1\right]. \end{cases}
\end{aligned} \tag{36}$$

The plots of $\mathfrak{C}_\sim$ and $\mathfrak{C}^\sim$ are illustrated in Figures 3 and 4, respectively.

variables $\tilde{X}$ and $\tilde{Y}$ at $(x, y) \in [0, 1] \times [0, 1]$, with the following α -cuts:

4. The Novel Construction Method of Copulas for Fuzzy Random Variables

The extension of copulas was achieved in [15], through the notion of fuzzy copula functions of two fuzzy random

$$[\tilde{C}(x, y)]_{[\alpha]} = \left[F\left(\left(\tilde{F}_X^{-1}(x)\right)_{\alpha/2}, \left(\tilde{F}_Y^{-1}(y)\right)_{\alpha/2}\right), F\left(\left(\tilde{F}_X^{-1}(x)\right)_{1-(\alpha/2)}, \left(\tilde{F}_Y^{-1}(y)\right)_{1-(\alpha/2)}\right) \right]. \tag{37}$$

As a result, $\tilde{C}$ is a joint fuzzy distribution function [15]. The following proposition examines the properties of the fuzzy copula.

Proposition 1 (see [15]). *For a fuzzy copula $\tilde{C}(x, y)$, the following conditions hold:*

$$\begin{aligned}
\tilde{C}(x, 0) &= \tilde{C}(0, y) = \tilde{0}, \\
\tilde{C}(x, 1) &= \tilde{x}, \\
\tilde{C}(1, y) &= \tilde{y},
\end{aligned} \tag{38}$$

$$\tilde{C}(x_2, y_2) \oplus \tilde{C}(x_1, y_1) \succ \tilde{C}(x_1, y_2) \oplus \tilde{C}(x_2, y_1),$$

where $x, y, x_1, x_2, y_1, y_2 \in [0, 1]$, with $x_1 < x_2$ and $y_1 < y_2$.

FIGURE 3: $\mathfrak{G}_\sim$ with $\kappa = 7/8$ and $\lambda = \sqrt{2}/2$ (left axis represents y , right axis represents x , and vertical axis represents $\mathfrak{G}_\sim(x, y)$).

FIGURE 4: $\mathfrak{G}_\sim$ with $\kappa = 7/8$ and $\lambda = \sqrt{2}/2$ (left axis represents y , right axis represents x , and vertical axis represents $\mathfrak{G}_\sim(x, y)$).

The proof of this proposition can be found in [15]. Next, inspired by the fuzzy copula for fuzzy random variables, we propose a novel method for the construction of fuzzy copulas. This is achieved through the following theorem.

Theorem 3. Let κ and λ be fixed in $[0, 1]$, $\tilde{C}: [0, 1] \times [0, 1] \rightarrow [0, 1]$ be any fuzzy copula and $\mathfrak{G}: [0, 1] \times [0, 1] \rightarrow [0, 1]$ be produced by any copula $C: [0, 1] \times [0, 1] \rightarrow [0, 1]$ with the usage of Theorem 1. Then, $\mathfrak{G}_\sim(x, y)$ at $(x, y) \in [0, 1] \times [0, 1]$, defined by

$$\mathfrak{G}_\sim(x, y) = \begin{cases} \tilde{C}\left(\frac{x}{\kappa}, \frac{y}{\lambda}\right) \otimes (\kappa\lambda), & (x, y) \in [0, \kappa] \times [0, \lambda], \\ \tilde{x} \oplus \tilde{C}\left(\frac{x}{\kappa}, \frac{1-y}{1-\lambda}\right) \otimes (\kappa(\lambda-1)), & (x, y) \in [0, \kappa] \times [\lambda, 1], \\ \tilde{y} \oplus \tilde{C}\left(\frac{1-x}{1-\kappa}, \frac{y}{\lambda}\right) \otimes ((\kappa-1)\lambda), & (x, y) \in [\kappa, 1] \times [0, \lambda], \\ \tilde{x} \oplus \tilde{y} \oplus (-\tilde{1}) \oplus \tilde{C}\left(\frac{1-x}{1-\kappa}, \frac{1-y}{1-\lambda}\right) \otimes \\ \otimes ((1-\kappa)(1-\lambda)), & (x, y) \in [\kappa, 1] \times [\lambda, 1], \end{cases} \quad (39)$$

is the fuzzy copula of the fuzzy random variables $\tilde{X}$ and $\tilde{Y}$.

Proof. For the first branch of $\tilde{\mathfrak{C}}$, we have that, for $x = 0$ and $\forall y \in [0, \lambda]$,

$$\begin{aligned}
 [\tilde{\mathfrak{C}}(0, y)]_{[\alpha]} &= \left[\tilde{C}\left(\frac{0}{\kappa}, \frac{y}{\lambda}\right) \kappa \lambda \right]_{[\alpha]} \\
 &= \left[F\left(\left(\tilde{F}_X^{-1}(0)\right)_{\alpha/2}, \left(\tilde{F}_Y^{-1}\left(\frac{y}{\lambda}\right)\right)_{\alpha/2}\right) \kappa \lambda, \right. \\
 &\quad \left. F\left(\left(\tilde{F}_X^{-1}(0)\right)_{1-(\alpha/2)}, \left(\tilde{F}_Y^{-1}\left(\frac{y}{\lambda}\right)\right)_{1-(\alpha/2)}\right) \kappa \lambda \right] \\
 &= \left[F\left(\left(\tilde{F}_{\tilde{X}_{1-(\alpha/2)}}^{-1}(0)\right)^{-1}, \left(\tilde{F}_Y^{-1}\left(\frac{y}{\lambda}\right)\right)_{\alpha/2}\right) \kappa \lambda, \right. \\
 &\quad \left. F\left(\left(\tilde{F}_{\tilde{X}_{\alpha/2}}^{-1}(0)\right)^{-1}, \left(\tilde{F}_Y^{-1}\left(\frac{y}{\lambda}\right)\right)_{1-(\alpha/2)}\right) \kappa \lambda \right] \\
 &= \left[F\left(-\infty, \left(\tilde{F}_Y^{-1}\left(\frac{y}{\lambda}\right)\right)_{\alpha/2}\right) \kappa \lambda, F\left(-\infty, \left(\tilde{F}_Y^{-1}\left(\frac{y}{\lambda}\right)\right)_{1-(\alpha/2)}\right) \kappa \lambda \right] \\
 &= [0, 0] = \tilde{0}_{[\alpha]}, \quad \forall \alpha \in (0, 1].
 \end{aligned} \tag{40}$$

As a result, $\tilde{\mathfrak{C}}(0, y) = \tilde{0}$. The proof that $\tilde{\mathfrak{C}}(x, 0) = \tilde{0}$ can be considered in the same manner. The next step is to examine the second condition that holds in the special case of $\kappa = 1$ and $\lambda = 1$. In this case, $\tilde{\mathfrak{C}}$ becomes $\tilde{C}$, and the proof is straightforward. In the last step, we have that $\forall x_1, x_2 \in [0, \kappa]$ and $\forall y_1, y_2 \in [0, \lambda]$, where $x_1 < x_2$ and $y_1 < y_2$ and

$$\tilde{C}(x_2, y_2) \oplus \tilde{C}(x_1, y_1) \succ \tilde{C}(x_1, y_2) \oplus \tilde{C}(x_2, y_1). \tag{41}$$

Since $\kappa, \lambda \geq 0$, $\forall \alpha \in (0, 1]$, we have that

$$\begin{aligned}
 &(\kappa \lambda \otimes (\tilde{C}(x_2, y_2) \oplus \tilde{C}(x_1, y_1)))_{\alpha} \\
 &\quad > (\kappa \lambda \otimes (\tilde{C}(x_1, y_2) \oplus \tilde{C}(x_2, y_1)))_{\alpha} \Leftrightarrow \\
 &\quad \tilde{\mathfrak{C}}(x_2, y_2) \oplus \tilde{\mathfrak{C}}(x_1, y_1) \succ \tilde{\mathfrak{C}}(x_1, y_2) \oplus \tilde{\mathfrak{C}}(x_2, y_1).
 \end{aligned} \tag{42}$$

As a result, the proof for the first branch is completed. Next, for the case of the second branch, of $\tilde{\mathfrak{C}}$, we have that $\forall x \in [0, \kappa]$ and for $y = 1$,

$$\begin{aligned}
 &\left[\tilde{x} \oplus \tilde{C}\left(\frac{x}{\kappa}, \frac{0}{1-\lambda}\right) \otimes (\kappa(\lambda-1)) \right]_{\alpha} = \\
 &= \left[\tilde{x}_{\alpha/2} - F\left(\left(\tilde{F}_X^{-1}\left(\frac{x}{\kappa}\right)\right)_{1-(\alpha/2)}, \left(\tilde{F}_Y^{-1}(0)\right)_{1-(\alpha/2)}\right) \kappa(1-\lambda), \right. \\
 &\quad \left. \tilde{x}_{1-(\alpha/2)} - F\left(\left(\tilde{F}_X^{-1}\left(\frac{x}{\kappa}\right)\right)_{\alpha/2}, \left(\tilde{F}_Y^{-1}(0)\right)_{\alpha/2}\right) \kappa(1-\lambda) \right] \\
 &= \left[\tilde{x}_{\alpha/2} - F\left(\left(\tilde{F}_X^{-1}\left(\frac{x}{\kappa}\right)\right)_{1-(\alpha/2)}, \left(\tilde{F}_{\tilde{Y}_{\alpha/2}}^{-1}(0)\right)^{-1}\right) \kappa(1-\lambda), \right. \\
 &\quad \left. \tilde{x}_{1-(\alpha/2)} - F\left(\left(\tilde{F}_X^{-1}\left(\frac{x}{\kappa}\right)\right)_{\alpha/2}, \left(\tilde{F}_{\tilde{Y}_{1-(\alpha/2)}}^{-1}(0)\right)^{-1}\right) \kappa(1-\lambda) \right] \\
 &= \left[\tilde{x}_{\alpha/2} - F\left(\left(\tilde{F}_X^{-1}\left(\frac{x}{\kappa}\right)\right)_{1-(\alpha/2)}, -\infty\right) \kappa(1-\lambda), \right. \\
 &\quad \left. \tilde{x}_{1-(\alpha/2)} - F\left(\left(\tilde{F}_X^{-1}\left(\frac{x}{\kappa}\right)\right)_{\alpha/2}, -\infty\right) \kappa(1-\lambda) \right].
 \end{aligned} \tag{43}$$

Hence, we obtain

$$\left[\tilde{x} \oplus \tilde{C}\left(\frac{x}{\kappa}, \frac{0}{1-\lambda}\right) \otimes (\kappa(\lambda-1)) \right]_{\alpha} = [\tilde{x}_{\alpha/2}, \tilde{x}_{1-(\alpha/2)}] = \tilde{x}_{\alpha}, \quad \forall \alpha \in (0, 1] \Rightarrow$$

$$\tilde{\mathfrak{C}}(x, 0) = \tilde{x}. \quad (44)$$

For the examination of the third condition, we have that $\forall x_1, x_2 \in [0, \kappa]$ and $\forall y_1, y_2 \in [\lambda, 1]$, where $x_1 < x_2$ and $y_1 < y_2$ and

$$\begin{aligned} & (\tilde{x}_2)_{\alpha} - \left(\tilde{C}\left(\frac{x_2}{\kappa}, \frac{1-y_2}{1-\lambda}\right) \kappa(1-\lambda) \right)_{1-\alpha} + (\tilde{x}_1)_{\alpha} - \left(\tilde{C}\left(\frac{x_1}{\kappa}, \frac{1-y_1}{1-\lambda}\right) \kappa(1-\lambda) \right)_{1-\alpha} \\ & > (\tilde{x}_2)_{\alpha} - \left(\tilde{C}\left(\frac{x_2}{\kappa}, \frac{1-y_1}{1-\lambda}\right) \kappa(1-\lambda) \right)_{1-\alpha} + (\tilde{x}_1)_{\alpha} - \left(\tilde{C}\left(\frac{x_1}{\kappa}, \frac{1-y_2}{1-\lambda}\right) \kappa(1-\lambda) \right)_{1-\alpha} \\ & \Leftrightarrow \left(\tilde{C}\left(\frac{x_2}{\kappa}, \frac{1-y_2}{1-\lambda}\right) \kappa(1-\lambda) \right)_{\alpha} + \left(\tilde{C}\left(\frac{x_1}{\kappa}, \frac{1-y_1}{1-\lambda}\right) \kappa(1-\lambda) \right)_{\alpha} \\ & < \left(\tilde{C}\left(\frac{x_2}{\kappa}, \frac{1-y_1}{1-\lambda}\right) \kappa(1-\lambda) \right)_{\alpha} + \left(\tilde{C}\left(\frac{x_1}{\kappa}, \frac{1-y_2}{1-\lambda}\right) \kappa(1-\lambda) \right)_{\alpha}. \end{aligned} \quad (45)$$

This holds if we consider that since $\lambda \leq y_1 < y_2 \leq 1$, then $0 \leq ((1-y_2)/(1-\lambda)) < ((1-y_1)/(1-\lambda)) \leq 1$. Hence, the proof of the second branch is completed. The cases of the third and the fourth branches of $\tilde{\mathfrak{C}}$ can be examined in a similar way as the first and the second branches of $\tilde{\mathfrak{C}}$. Hence, the proof that $\tilde{\mathfrak{C}}(x, y)$ is a fuzzy copula is completed. $\square$

The flowchart of Figure 5 illustrates the novel fuzzy copula construction process.

Now, in order to illustrate the novel construction of fuzzy copula, the following example is provided, based on an example that may be found in [15].

Example 4. Let X and Y be random variables that have joint distribution function, given by

$$F_{XY}(x, y) = \frac{(x+1)(1-e^{-y})}{(x-1)e^{-y}+2}, \quad (x, y) \in [-1, 1] \times [0, \infty]. \quad (46)$$

As a result, $X \sim U[-1, 1]$ and $Y \sim \text{Weibull}(1, 1)$. Let $\tilde{\Theta} = (2; 1, 4)_T$ and $\tilde{\Gamma} = (8; 5, 9)_T$ be two nonsymmetric triangular fuzzy numbers and let $\tilde{X} = \tilde{\Theta} + X$ and $\tilde{Y} = \tilde{\Gamma} + Y$. Hence, we have that $\tilde{X} = (X+2; 1, 4)_T$ and $\tilde{Y} = (Y+8; 5, 9)_T$. Therefore, we obtain the α -pessimistic values of $\tilde{X}$ and $\tilde{Y}$, respectively, by

$$\tilde{X}_{\alpha} = \begin{cases} X + 2\alpha + 1, & 0 < \alpha \leq \frac{1}{2}, \\ X + 4\alpha, & \frac{1}{2} \leq \alpha \leq 1, \end{cases} \quad (47)$$

$$\tilde{Y}_{\alpha} = \begin{cases} Y + 6\alpha + 5, & 0 < \alpha \leq \frac{1}{2}, \\ Y + 2\alpha + 7, & \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

Based on the fact that $\forall \alpha \in (0, 1]$, $\tilde{X}_{\alpha}$ has uniform distribution and $\tilde{Y}_{\alpha}$ has Weibull distribution. Hence,

$$\tilde{X}_{\alpha} \sim \begin{cases} U(2\alpha, 4\alpha - 1), & 0 < \alpha \leq \frac{1}{2}, \\ U(4\alpha - 1, \alpha + 1), & \frac{1}{2} \leq \alpha \leq 1, \end{cases} \quad (48)$$

$$\tilde{Y}_{\alpha} \sim \begin{cases} \text{Weibull}(6\alpha + 5, 1, 1), & 0 < \alpha \leq \frac{1}{2}, \\ \text{Weibull}(2\alpha + 7, 1, 1), & \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

Also, we have the α -pessimistic values of $\tilde{\Theta}$ and $\tilde{\Gamma}$ given as follows:

FIGURE 5: Flowchart of fuzzy copula construction process.

$$\tilde{\Theta}_\alpha = \begin{cases} 2\alpha + 1, & 0 < \alpha \leq \frac{1}{2}, \\ 4\alpha, & \frac{1}{2} \leq \alpha \leq 1, \end{cases} \quad (49)$$

$$\tilde{\Gamma}_\alpha = \begin{cases} 6\alpha + 5, & 0 < \alpha \leq \frac{1}{2}, \\ 2\alpha + 7, & \frac{1}{2} \leq \alpha \leq 1. \end{cases}$$

Hence, we obtain the next α -pessimistic values:

$$\begin{aligned} (\tilde{F}_X(x))_\alpha &= F_{\tilde{X}_{1-\alpha}}(x) = \mathbb{P}(\tilde{X}_{1-\alpha} \leq x) = \mathbb{P}(X \leq x - \tilde{\Theta}_{1-\alpha}) \Leftrightarrow \\ (\tilde{F}_X(x))_\alpha &= \frac{x - \tilde{\Theta}_{1-\alpha} + 1}{2}, \quad x \in [\tilde{\Theta}_{1-\alpha} - 1, \tilde{\Theta}_{1-\alpha} + 1], \\ (\tilde{F}_Y(y))_\alpha &= F_{\tilde{Y}_{1-\alpha}}(y) = \mathbb{P}(\tilde{Y}_{1-\alpha} \leq y) = \mathbb{P}(Y \leq y - \tilde{\Gamma}_{1-\alpha}) \Leftrightarrow \\ (\tilde{F}_Y(y))_\alpha &= 1 - e^{\tilde{\Gamma}_{1-\alpha} - y}, \quad y \in [\tilde{\Gamma}_{1-\alpha}, \infty]. \end{aligned} \quad (50)$$

Now, $\forall \alpha \in (0, 1]$, and we get the inverse functions of $(\tilde{F}_X(x))_\alpha$ and $(\tilde{F}_Y(y))_\alpha$ by

$$\begin{aligned} ((\tilde{F}_X(u))_\alpha)^{-1} &= 2u + \tilde{\Theta}_{1-\alpha} - 1, \\ ((\tilde{F}_Y(v))_\alpha)^{-1} &= \tilde{\Gamma}_{1-\alpha} - \ln(1 - v). \end{aligned} \quad (51)$$

Also, we have that

$$\begin{aligned} (\tilde{F}_X^{-1}(u))_\alpha &= 2u + \tilde{\Theta}_\alpha - 1, \\ (\tilde{F}_Y^{-1}(v))_\alpha &= \tilde{\Gamma}_\alpha - \ln(1 - v). \end{aligned} \quad (52)$$

Thus, we obtain the left and right parts of the α -cuts of the fuzzy copula by

$$\begin{aligned} \tilde{C}_\alpha^L(u, v) &= F_{XY}((\tilde{F}_X^{-1}(u))_{\alpha/2}, (\tilde{F}_Y^{-1}(v))_{\alpha/2}) \\ &= \frac{(2u + \tilde{\Theta}_{\alpha/2})(1 - e^{-\tilde{\Gamma}_{\alpha/2}}(1 - v))}{(2u + \tilde{\Theta}_{\alpha/2} - 2)e^{-\tilde{\Gamma}_{\alpha/2}}(1 - v) + 2}, \\ \tilde{C}_\alpha^R(u, v) &= F_{XY}((\tilde{F}_X^{-1}(u))_{1-(\alpha/2)}, (\tilde{F}_Y^{-1}(v))_{1-(\alpha/2)}) \\ &= \frac{(2u + \tilde{\Theta}_{1-(\alpha/2)})(1 - e^{-\tilde{\Gamma}_{1-(\alpha/2)}}(1 - v))}{(2u + \tilde{\Theta}_{1-(\alpha/2)} - 2)e^{-\tilde{\Gamma}_{1-(\alpha/2)}}(1 - v) + 2}, \end{aligned} \quad (53)$$

where $[\tilde{C}(u, v)]_{[\alpha]} = [\tilde{C}_\alpha^L(u, v), \tilde{C}_\alpha^R(u, v)]$. Now, based on Theorem 3, we have that, for the first branch of $\tilde{\mathfrak{C}}$,

$$[\tilde{\mathfrak{C}}(u, v)]_{[\alpha]} = \left[\tilde{\mathfrak{C}}_{\alpha}^L(u, v), \tilde{\mathfrak{C}}_{\alpha}^R(u, v) \right], \quad (54) \quad \text{where}$$

$$\begin{aligned} \tilde{\mathfrak{C}}_{\alpha}^L(u, v) &= \tilde{C}_{\alpha}^L\left(\frac{u}{\kappa}, \frac{v}{\lambda}\right) \kappa \lambda = \frac{\left((2u/\kappa) + \tilde{\Theta}_{\alpha/2}\right) \left(1 - e^{-\tilde{\Gamma}_{\alpha/2}} (1 - (v/\lambda))\right)}{\left((2u/\kappa) + \tilde{\Theta}_{\alpha/2} - 2\right) e^{-\tilde{\Gamma}_{\alpha/2}} (1 - (v/\lambda)) + 2} \kappa \lambda, \\ \tilde{\mathfrak{C}}_{\alpha}^R(u, v) &= \tilde{C}_{\alpha}^R\left(\frac{u}{\kappa}, \frac{v}{\lambda}\right) \kappa \lambda = \frac{\left((2u/\kappa) + \tilde{\Theta}_{1-(\alpha/2)}\right) \left(1 - e^{-\tilde{\Gamma}_{\alpha/2}} (1 - (v/\lambda))\right)}{\left((2u/\kappa) + \tilde{\Theta}_{1-(\alpha/2)} - 2\right) e^{-\tilde{\Gamma}_{\alpha/2}} (1 - (v/\lambda)) + 2} \kappa \lambda, \end{aligned} \quad (55)$$

$\forall u \in [0, \kappa]$, and $v \in [0, \lambda]$. Next, for the second branch of $\tilde{\mathfrak{C}}$, we have that $\forall u \in [0, \kappa]$ and $v \in [\lambda, 1]$,

$$[\tilde{\mathfrak{C}}(u, v)]_{[\alpha]} = \left[\tilde{\mathfrak{C}}_{\alpha}^L(u, v), \tilde{\mathfrak{C}}_{\alpha}^R(u, v) \right], \quad (56)$$

where

$$\begin{aligned} \tilde{\mathfrak{C}}_{\alpha}^L(u, v) &= \tilde{u}_{\alpha/2} + \frac{\left((2u/\kappa) + \tilde{\Theta}_{\alpha/2}\right) \left(1 - e^{-\tilde{\Gamma}_{\alpha/2}} (1 - ((1-v)/(1-\lambda)))\right)}{\left((2u/\kappa) + \tilde{\Theta}_{\alpha/2} - 2\right) e^{-\tilde{\Gamma}_{\alpha/2}} (1 - ((1-v)/(1-\lambda))) + 2} \kappa (\lambda - 1), \\ \tilde{\mathfrak{C}}_{\alpha}^R(u, v) &= \tilde{u}_{1-(\alpha/2)} + \frac{\left((2u/\kappa) + \tilde{\Theta}_{1-(\alpha/2)}\right) \left(1 - e^{-\tilde{\Gamma}_{1-(\alpha/2)}} (1 - ((1-v)/(1-\lambda)))\right)}{\left((2u/\kappa) + \tilde{\Theta}_{1-(\alpha/2)} - 2\right) e^{-\tilde{\Gamma}_{1-(\alpha/2)}} (1 - ((1-v)/(1-\lambda))) + 2} \kappa (\lambda - 1). \end{aligned} \quad (57)$$

For the third branch of $\tilde{\mathfrak{C}}$, we have that $\forall u \in [\kappa, 1]$ and $v \in [0, \lambda]$,

$$[\tilde{\mathfrak{C}}(u, v)]_{[\alpha]} = \left[\tilde{\mathfrak{C}}_{\alpha}^L(u, v), \tilde{\mathfrak{C}}_{\alpha}^R(u, v) \right], \quad (58)$$

where

$$\begin{aligned} \tilde{\mathfrak{C}}_{\alpha}^L(u, v) &= \tilde{v}_{\alpha/2} + \frac{\left(2((1-u)/(1-\kappa)) + \tilde{\Theta}_{\alpha/2}\right) \left(1 - e^{-\tilde{\Gamma}_{\alpha/2}} (1 - (v/\lambda))\right)}{\left(2((1-u)/(1-\kappa)) + \tilde{\Theta}_{\alpha/2} - 2\right) e^{-\tilde{\Gamma}_{\alpha/2}} (1 - (v/\lambda)) + 2} (\kappa - 1) \lambda, \\ \tilde{\mathfrak{C}}_{\alpha}^R(u, v) &= \tilde{v}_{1-(\alpha/2)} + \frac{\left(2((1-u)/(1-\kappa)) + \tilde{\Theta}_{1-(\alpha/2)}\right) \left(1 - e^{-\tilde{\Gamma}_{1-(\alpha/2)}} (1 - (v/\lambda))\right)}{\left(2((1-u)/(1-\kappa)) + \tilde{\Theta}_{1-(\alpha/2)} - 2\right) e^{-\tilde{\Gamma}_{1-(\alpha/2)}} (1 - (v/\lambda)) + 2} (\kappa - 1) \lambda. \end{aligned} \quad (59)$$

Finally, for the case of the fourth branch of $\tilde{\mathfrak{C}}$, we have that $\forall u \in [\kappa, 1]$ and $v \in [\lambda, 1]$,

$$[\tilde{\mathfrak{C}}(u, v)]_{[\alpha]} = \left[\tilde{\mathfrak{C}}_{\alpha}^L(u, v), \tilde{\mathfrak{C}}_{\alpha}^R(u, v) \right], \quad (60)$$

where

$$\begin{aligned} \tilde{\mathfrak{C}}_{\alpha}^L(u, v) &= \tilde{u}_{\alpha/2} + \tilde{v}_{\alpha/2} - 1 + \frac{\left(2((1-u)/(1-\kappa)) + \tilde{\Theta}_{\alpha/2}\right) \left(1 - e^{-\tilde{\Gamma}_{\alpha/2}} (1 - ((1-v)/(1-\lambda)))\right)}{\left(2((1-u)/(1-\kappa)) + \tilde{\Theta}_{\alpha/2} - 2\right) e^{-\tilde{\Gamma}_{\alpha/2}} (1 - ((1-v)/(1-\lambda))) + 2} (1 - \kappa) (1 - \lambda), \\ \tilde{\mathfrak{C}}_{\alpha}^R(u, v) &= \tilde{u}_{1-(\alpha/2)} + \tilde{v}_{1-(\alpha/2)} - 1 + \frac{\left(2((1-u)/(1-\kappa)) + \tilde{\Theta}_{1-(\alpha/2)}\right) \left(1 - e^{-\tilde{\Gamma}_{1-(\alpha/2)}} (1 - ((1-v)/(1-\lambda)))\right)}{\left(2((1-u)/(1-\kappa)) + \tilde{\Theta}_{1-(\alpha/2)} - 2\right) e^{-\tilde{\Gamma}_{1-(\alpha/2)}} (1 - ((1-v)/(1-\lambda))) + 2} (1 - \kappa) (1 - \lambda). \end{aligned} \quad (61)$$

In Example 3 of [15], the production of a fuzzy copula $\tilde{C}$ was established. In the illustrative Example 4 of the present paper, the generalization of this result was presented with the usage of (α, β) transformation, in four different subrectangles of the unit square.

5. Conclusions

We provided a method for the construction of copulas in four subrectangles of the unit square, via the (α, β) transformation, and we introduced the upper and lower bounds of the produced copulas of this method. Also, we developed a method to construct fuzzy copulas for fuzzy random variables. With the usage of those methods, we can conclude that the construction of an infinite number of copulas and fuzzy copulas can be achieved. Each one of the produced copulas and fuzzy copulas could be applied based on their adequacy in real-life problems. It may also be of interest considering these construction methods in the case of n -dimensional copulas $n > 2$. On the other hand, the case in which these construction methods could possibly be defined in more than four subrectangles of the unit square may be examined. In addition, the extension of those methods for Intuitionistic fuzzy sets [16, 18] and Pythagorean fuzzy sets [31] could be possibly achieved, in order to develop aggregation operators for multiple attribute decision making algorithms. These topics are the basis for our future investigations.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] A. Sklar, "Fonctions de répartition à n dimensions et leurs marges," *Publications de l'Institut Statistique de l'Université de Paris*, vol. 8, pp. 229–231, 1959.
- [2] E. Liebscher, "Construction of asymmetric multivariate copulas," *Journal of Multivariate Analysis*, vol. 99, no. 10, pp. 2234–2250, 2008.
- [3] K. F. Siburg and P. A. Stoimenov, "Gluing copulas," *Communications in Statistics—Theory and Methods*, vol. 37, no. 19, pp. 3124–3134, 2008.
- [4] S. Saminger-Platz, M. Dibala, E. P. Klement, and R. Mesiar, "Ordinal sums of binary conjunctive operations based on the product," *Publicationes Mathematicae-Debrecen*, vol. 91, no. 1–2, pp. 63–80, 2017.
- [5] R. Mesiar, V. Jagr, M. Juránová, and M. Komorníková, "Univariate conditioning of copulas," *Kybernetika*, vol. 44, no. 6, pp. 807–816, 2008.
- [6] F. Durante, S. Saminger-Platz, and P. Sarkoci, "Rectangular patchwork for bivariate copulas and tail dependence," *Communications in Statistics—Theory and Methods*, vol. 38, no. 15, pp. 2515–2527, 2009.
- [7] F. Durante, S. Samingerplatz, and P. Sarkoci, "On representations of 2-increasing binary aggregation functions," *Information Sciences*, vol. 178, no. 23, pp. 4534–4541, 2008.
- [8] B. De Baets and H. De Meyer, "Orthogonal grid constructions of copulas," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 6, pp. 1053–1062, 2007.
- [9] E. P. Klement, A. Kolesárová, R. Mesiar, and C. Sempì, "Copulas constructed from horizontal sections," *Communications in Statistics—Theory and Methods*, vol. 36, no. 16, pp. 2901–2911, 2007.
- [10] J. Fernández-Sánchez, J. J. Quesada-Molina, and M. Úbeda-Flores, "On (α, β) -homogeneous copulas," *Information Sciences*, vol. 221, pp. 181–191, 2013.
- [11] F. Durante, A. Kolesárová, R. Mesiar, and C. Sempì, "Copulas with given values on a horizontal and a vertical section," *Kybernetika*, vol. 43, no. 2, pp. 209–220, 2007.
- [12] R. B. Nelsen, *An Introduction to Copulas*, Springer Science & Business Media, New York, NY, USA, 2006.
- [13] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.
- [14] H. Kwakernaak, "Fuzzy random variables-I. Definitions and theorems," *Information Sciences*, vol. 15, no. 1, pp. 1–29, 1978.
- [15] V. Ranjbar and G. Hesamian, "Copula function for fuzzy random variables: applications in measuring association between two fuzzy random variables," *Statistical Papers*, vol. 61, no. 1, pp. 503–522, 2020.
- [16] S. Giakoumakis and B. Papadopoulos, "An algorithm for fuzzy negations based-intuitionistic fuzzy copula aggregation operators in multiple attribute decision making," *Algorithms*, vol. 13, no. 6, pp. 1–18, 2020.
- [17] W. Näther, "Copulas and t-norms: mathematical tools for combining probabilistic and fuzzy information, with application to error propagation and interaction," *Structural Safety*, vol. 32, no. 6, pp. 366–371, 2010.
- [18] Z. Tao, B. Han, and H. Chen, "On intuitionistic fuzzy copula aggregation operators in multiple- attribute decision making," *Cognitive Computation*, vol. 10, no. 4, pp. 610–624, 2018.
- [19] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets: Theory and Applications*, World Scientific, Singapore, 1994.
- [20] D. S. Sfiris and B. K. Papadopoulos, "Non-asymptotic fuzzy estimators based on confidence intervals," *Information Sciences*, vol. 279, pp. 446–459, 2014.
- [21] M. Ming, M. Friedman, and A. Kandel, "A new fuzzy arithmetic," *Fuzzy Sets and Systems*, vol. 108, no. 1, pp. 83–90, 1999.
- [22] B. Liu and Y. K. Liu, "Expected value of fuzzy variable and fuzzy expected value models," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 4, pp. 445–450, 2002.
- [23] J. Peng and B. Liu, "Some properties of optimistic and pessimistic values of fuzzy variables," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, vol. 2, pp. 745–750, Budapest, Hungary, July 2004.
- [24] G. Hesamian and J. Chachi, "Two-sample Kolmogorov-Smirnov fuzzy test for fuzzy random variables," *Statistical Papers*, vol. 56, no. 1, pp. 61–82, 2015.
- [25] A. Blanco-Fernández, M. R. Casals, A. Colubi et al., "Arithmetic and distance-based approach to the statistical analysis of imprecisely valued data," *Towards Advanced Data Analysis by Combining Soft Computing and Statistics*, vol. 10, pp. 1–18, 2013.
- [26] H. Kwakernaak, "Fuzzy random variables-II. Algorithms and examples for the discrete case," *Information Sciences*, vol. 17, no. 3, pp. 253–278, 1979.
- [27] R. Kruse and K. D. Meyer, *Statistics with Vague Data*, Reidel Publishing Company, Dordrecht, Netherlands, 1987.

- [28] J. Chachi, "On distribution characteristics of a fuzzy random variable," *Austrian Journal of Statistics*, vol. 47, no. 2, pp. 53–67, 2018.
- [29] M. Fréchet, "Sur les tableaux de corrélation dont les marges sont données," *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, vol. 242, pp. 2426–2428, 1956.
- [30] W. Hoeffding, "Masstabinvariante korrelationstheorie," *Schriften des Mathematischen Instituts und Instituts für Angewandte Mathematik der Universität Berlin*, vol. 5, pp. 181–233, 1940.
- [31] L. Wang and H. Garg, "Algorithm for multiple attribute decision-making with interactive archimedean norm operations under pythagorean fuzzy uncertainty," *International Journal of Computational Intelligence Systems*, vol. 14, no. 1, pp. 503–527, 2021.