Hindawi
Journal of Mathematics
Volume 2021, Article ID 4332094, 7 pages
https://doi.org/10.1155/2021/4332094

Research Article

Magnifying Elements in Semigroups of Fixed Point Set Transformations Restricted by an Equivalence Relation

Thananya Kaewnoi, Ronnason Chinram, and Montakarn Petapirak

Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand

Correspondence should be addressed to Montakarn Petapirak; montakarn.p@psu.ac.th

Received 3 October 2021; Accepted 11 December 2021; Published 26 December 2021

Academic Editor: Feng Feng

Let X be a nonempty set and ρ be an equivalence relation on X. For a nonempty subset S of X, we denote the semigroup of transformations restricted by an equivalence relation ρ fixing S pointwise by $E_{F(S)}(X,\rho)$. In this paper, magnifying elements in $E_{F(S)}(X,\rho)$ will be investigated. Moreover, we will give the necessary and sufficient conditions for elements in $E_{F(S)}(X,\rho)$ to be right or left magnifying elements.

1. Introduction

Magnifying elements of a semigroup were first mentioned in 1963 by Ljapin in [1]. An element a of a semigroup S is called a right (left) magnifying element if there exists a proper subset M of S such that $Ma = S$ ($aM = S$). Many studies on magnifying elements in semigroups were conducted by many authors in various aspects between 1971 and 2003. In 1971, Migliorini [2] introduced the new notion of a minimal subset M relative to a magnifying element a of S. By means of this, he constructed an infinite chain of minimal subsets properly contained in the preceding ones, which gives rise to an infinite number of magnifying elements of the form a^n, where $a \in S$ is a magnifying element and n is any positive integer. Migliorini also investigated the structure of a semigroup S with minimal subsets in [3]. According to Tolo [4], if the proper subset M of S relative to a magnifying element a is a subsemigroup of S, then a is called a strong magnifying element. It was shown that if a semigroup S contains a strong magnifying element, then S is factorizable, i.e., $S = AB$ for some proper subsemigroups A, B of S. In other words, the existence of strong magnifying elements plays an important role in factorizing a semigroup. In 1992, Catino and Migliorini [5] provided the example of semigroups with nonstrong magnifying elements which are factorizable and determined the existence of magnifying elements in simple, bisimple, and regular semigroups by improving Tolo's results. Moreover, they proved that the magnifying elements of the natural partial-order semigroups are maximal. Two years later, Magill [6] provided the necessary and sufficient conditions for elements in a semigroup with identity to be left or right magnifying elements and applied the result to the semigroup of linear transformations of a vector space and the semigroup of all continuous self-maps of a topological space. In 1996, Gutan [7] constructed the semigroup containing both strong and nonstrong magnifying elements, which turned out to be a positive answer to the question, posed in [5, 6], of whether or not there is a semigroup containing both strong and nonstrong magnifying elements. A year later, he showed in [8] the necessary and sufficient conditions for a semigroup containing a magnifying element to be factorizable. The results point out that every semigroup containing magnifying elements, which is not necessarily strong, is factorizable; and this improved the results by Tolo, and Catino and Migliorini. In 1999, Gutan characterized semigroups containing left strong magnifying elements with a minimal subsemigroup and proposed the method for obtaining such a semigroup in [9]. In 2000, Gutan [10] introduced the definition of very good magnifying elements in a semigroup. If such a set M relative to a magnifying element a is a ring ideal, then a is a very good magnifying element.

Furthermore, he established a characterization of semigroups in which all the left magnifying elements are very good. In 2003, Gutan and Kisielewicz [11] presented the notion of primitive semigroups and further constructed semigroups having both good and bad magnifying elements. In addition, some general properties of semigroups containing magnifying elements with its minimal subsemigroup were established. At that time, many researchers focused on the minimal subsets relative to magnifying elements. Recently, some researchers have paid attention to magnifying elements in various transformation semigroups. For instance, Luangchaisri et al. [12] generalized Magill's results in partial transformation semigroups, and Prakitsri [13] investigated magnifying elements in linear transformation semigroups with infinite nullity and in those with infinite co-rank. He showed that linear transformation semigroups with infinite nullity have no right magnifying elements. However, all left magnifying elements in these semigroups are strong magnifying elements. Contrarily, linear transformation semigroups with infinite co-rank have no left magnifying elements but all right magnifying elements are strong. In [14], the conditions for elements in the semigroup of transformations with a fixed point set to be magnifying elements have been established by Petapirak, Kaewnoi, and Chinram. In this paper, efforts have been made to extend the results obtained in [14] by showing the necessary and sufficient conditions for elements in the semigroup of transformations restricted by an equivalence relation with a fixed point set to be right or left magnifying elements.

2. Preliminaries

In this section, we first provide the reader with some basic but essential definitions.

A *semigroup* is a system $(S,\cdot)$ consisting of a nonempty set S together with the binary associative operation $\cdot$, i.e., $a\cdot b$ belongs to S and $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ for all elements $a,b,c\in S$. For convenience, we write S instead of $(S,\cdot)$ and let ab stand for $a\cdot b$ for any $a,b\in S$. A subset T of a semigroup S is called a *subsemigroup* of S if T is a semigroup under the operation of S. A nonempty set T of a semigroup S is a subsemigroup of S if $ab\in T$ for all $a,b\in T$. The intersection of any set of subsemigroups of S is either an empty set or a subsemigroup of S.

The next finding is an initial factor in this study.

Theorem 1 (see [1], pp. 118–119). *The following statements are true:*

(1) *No element of a semigroup is simultaneously a left and a right magnifying element.*

(2) *Finite semigroups have no magnifying elements.*

(3) *Commutative semigroups have no magnifying elements.*

(4) *Semigroups with two-sided cancellation have no magnifying elements, and hence groups do not contain magnifying elements.*

Therefore, we will focus our attention on infinite noncommutative semigroups without two-sided cancellation.

Let $T(X)$ be the set of all functions from a nonempty set X into itself. As is well known, the composition of functions $T(X)$ is closed, and the associative law holds. Therefore, $T(X)$ is a semigroup under the composition of functions. We then call $T(X)$ the *full transformation semigroup*. Throughout this paper, the identity function on X is denoted by id_X. The range of a function α is denoted by $\operatorname{ran}\alpha$ for all elements $\alpha\in T(X)$. Moreover, we write functions from the right, $x\alpha$ rather than $\alpha(x)$, and compose from the left to the right, $x\alpha\beta$ rather than $(\beta\circ\alpha)(x)$ for all elements $\alpha,\beta\in T(X)$.

Let ΔX be an identity relation on X, and let ρ be an equivalence relation on X. For each $x\in X$, we denote the equivalence class of ρ containing x by $[x]_\rho = \{y\in X \mid (x,y)\in\rho\}$ and $X/\rho = \{[x]_\rho \mid x\in X\}$. For any $\varnothing\neq S\subseteq X$, as in [14], we set $T_{F(S)}(X) = \{\alpha\in T(X)\mid x\alpha = x \text{ for all } x\in S\}$. The conditions for elements in $T_{F(S)}(X)$ to be magnifying elements are demonstrated as follows.

Theorem 2 (see [14], Theorem 2.5). *A function $\alpha\in T_{F(S)}(X)$ is a left magnifying element if and only if α is one-to-one but not onto.*

Theorem 3 (see [14], Theorem 2.14). *A function $\alpha\in T_{F(S)}(X)$ is a right magnifying element if and only if α is onto but not one-to-one.*

Denote the transformation semigroup restricted by an equivalence relation ρ by $E(X,\rho) = \{\alpha\in T(X)\mid \forall x, y\in X; (x,y)\in\rho \text{ implies } x\alpha = y\alpha\}$. It is widely known that $E(X,\rho)$ is a subsemigroup of $T(X)$. Let $E_{F(S)}(X,\rho)$ be the intersection of $E(X,\rho)$ and $T_{F(S)}(X)$. Evidently, if $E_{F(S)}(X,\rho)$ is a nonempty set, then $E_{F(S)}(X,\rho)$ is a subsemigroup of $E(X,\rho)$ and $T_{F(S)}(X)$. We then call $E_{F(S)}(X,\rho)$ the semigroup of transformations restricted by an equivalence relation ρ fixing S pointwise.

3. Propositions for $E_{F(S)}(X,\rho)$

In this section, we illustrate the following propositions which are characterizations of $E_{F(S)}(X,\rho)$.

Proposition 1. $E(X,\rho) = T(X,\rho)$ *if and only if* $\rho = \Delta X$.

Proof. Assume that $E(X,\rho) = T(X,\rho)$. Let $x,y\in X$ such that $(x,y)\in\rho$. Clearly, the identity function $id_X\in T(X,\rho)$. By assumption, $id_X\in E(X,\rho)$. Hence, $x = xid_X = yid_X = y$. This shows that $\rho = \Delta X$. Conversely, assume that $\rho = \Delta X$. Clearly, $E(X,\rho)\subseteq T(X,\rho)$. Let $\alpha\in T(X,\rho)$. Then, for all $(x,y)\in\rho$, $(x\alpha,y\alpha)\in\rho$. So $x\alpha = y\alpha$ since $\rho = \Delta X$. This shows that $\alpha\in E(X,\rho)$. Therefore, $E(X,\rho) = T(X,\rho)$. ◻

Note that if $\rho = \Delta X$ and $\alpha\in T(X)$, then for all $(x,y)\in\rho$, $(x\alpha,y\alpha)\in\rho$. Hence, $T(X,\rho) = T(X)$ if $\rho = \Delta X$. By the proof of Proposition 1, $E(X,\rho) = T(X)$ if and only if $\rho = \Delta X$.

Proposition 2. *$E_{F(S)}(X,\rho) = T_{F(S)}(X)$ if and only if $\rho = \Delta X$.*

Proof. Assume that $E_{F(S)}(X,\rho) = T_{F(S)}(X)$. Let $x, y \in X$ be such that $(x, y) \in \rho$. Clearly, the identity function $id_X \in T_{F(S)}(X)$. By assumption, $id_X \in E_{F(S)}(X,\rho)$. Hence, $x = xid_X = yid_X = y$. This shows that $\rho = \Delta X$. Conversely, assume that $\rho = \Delta X$. It is clear that $T(X,\rho) = T(X)$. By Proposition 1, we have $E(X,\rho) = T(X)$. So we have $E_{F(S)}(X,\rho) = \{\alpha \in E(X,\rho) \mid x\ \alpha = x \text{ for all } x \in\ S\} = \{\alpha \in T(X) \mid x\alpha = x\ \text{ for all } x \in S\} = T_{F(S)}(X)$. □

Proposition 3. *The identity function id_X belongs to $E_{F(S)}(X,\rho)$ if and only if $\rho = \Delta X$.*

Proof. Assume that the identity function id_X belongs to $E_{F(S)}(X,\rho)$. Let $x, y \in X$ such that $(x, y) \in \rho$. Then, $x = xid_X = yid_X = y$. This shows that for all $x, y \in X$, if $(x, y) \in \rho$, then $x = y$, which implies that ρ is an identity relation on X. Conversely, assume that $\rho = \Delta X$. By Proposition 2, $E_{F(S)}(X,\rho) = T_{F(S)}(X)$. It is clear that $id_X \in T_{F(S)}(X)$. So $id_X \in E_{F(S)}(X,\rho)$. □

Clearly, $E_{F(S)}(X,\rho)$ is a proper subset of $E(X,\rho)$. It is easy to verify that if $\rho = X \times X$, then $E(X,\rho)$ is the set of all constant functions, and if $\rho = \Delta X$ and $S = X$, then $E_{F(S)}(X,\rho) = \{id_X\}$.

Proposition 4. *If $\rho = X \times X$ and $|S| > 1$, then $E_{F(S)}(X,\rho)$ is empty.*

Proof. Assume that $\rho = X \times X$ and $|S| > 1$. Then there are distinct elements $x, y \in S$ such that $(x, y) \in \rho$. Suppose that there exists an element $\alpha \in E_{F(S)}(X,\rho)$. Then $x = x\alpha = y\alpha = y$, which is a contradiction. Therefore, $E_{F(S)}(X,\rho)$ is empty. □

Proposition 5. *If $\rho = X \times X$ and $|S| = 1$, then $|E_{F(S)}(X,\rho)| = 1$.*

Proof. Assume that $\rho = X \times X$ and $|S| = 1$. Then there is only one element $s \in S$. Hence, there is a function $\alpha \in E_{F(S)}(X,\rho)$ defined by $x\alpha = s$ for all $x \in X$. Suppose that $|E_{F(S)}(X,\rho)| > 1$, and let $\beta \in E_{F(S)}(X,\rho)$. By assumption, $(x, s) \in \rho$ for all $x \in X$, and hence, $x\alpha = s = s\beta = x\beta$ for all $x \in X$. This shows that $\alpha = \beta$. Therefore, $|E_{F(S)}(X,\rho)| = 1$. □

By Propositions 4 and 5, left and right magnifying elements do not exist in $E_{F(S)}(X,\rho)$ if $\rho = X \times X$.

4. Main Results

In this section, we will give the necessary and sufficient conditions for elements in $E_{F(S)}(X,\rho)$ to be right or left magnifying elements.

4.1. Right Magnifying Elements. By Proposition 2, we obtain the next theorem.

Theorem 4 (see [14]). *Suppose that $\rho = \Delta X$ and $S \neq X$. A function α is right magnifying in $E_{F(S)}(X,\rho)$ if and only if α is onto but not one-to-one.*

Our next two results are related to the existence of magnifying elements in $E_{F(S)}(X,\rho)$.

Lemma 1. *If $|S \cap [x]_\rho| > 1$ for some $x \in X$, then $E_{F(S)}(X,\rho)$ is empty.*

Proof. Let $x \in X$. Assume that $|S \cap [x]_\rho| > 1$. Suppose that there is an element α belonging to $E_{F(S)}(X,\rho)$. By assumption, there are two distinct elements $s_1, s_2 \in S \cap [x]_\rho$. Then $(s_1, s_2) \in \rho$. So $s_1 = s_1\alpha = s_2\alpha = s_2$, which is a contradiction. □

Lemma 2. *If $|S \cap [x]_\rho| = 1$ for all $x \in X$, then $|E_{F(S)}(X,\rho)| = 1$.*

Proof. If $\rho = \Delta X$ and $|S \cap [x]_\rho| = 1$ for all $x \in X$, then $X = S$, and hence, $E_{F(S)}(X,\rho) = \{id_X\}$. So $|E_{F(S)}(X,\rho)| = 1$. Next, assume that $\rho \neq \Delta X$ and $|S \cap [x]_\rho| = 1$ for all $x \in X$. Then $S \neq X$, and hence for each $x \in X \smallsetminus S$, there is a unique element $s_x \in S$ such that $(x, s_x) \in \rho$. Define a function α by

$$x\alpha = \begin{cases} x, & \text{if } x \in S, \\ s_x, & \text{if } x \in X \smallsetminus S, \end{cases} \tag{1}$$

for all $x \in X$. Clearly, $\alpha \in E_{F(S)}(X,\rho)$. Next, we will show that α is the only element in $E_{F(S)}(X,\rho)$. Let β be a function in $E_{F(S)}(X,\rho)$. For all $s \in S$, $s\alpha = s = s\beta$. By assumption, for each $x \in X \smallsetminus S$, there is a unique $s_x \in S$ such that $(x, s_x) \in \rho$, and hence, $x\beta = s_x\beta = s_x = x\alpha$. Therefore, $\alpha = \beta$ and $|E_{F(S)}(X,\rho)| = 1$. □

By Lemmas 1 and 2, if $\rho \neq \Delta X$ and either $|S \cap [x]_\rho| = 1$ for all $x \in X$ or $|S \cap [x]_\rho| > 1$ for some $x \in X$, then there exists no magnifying element in $E_{F(S)}(X,\rho)$.

Example 1. Consider $X = \mathbb{N}$, $S = \{1, 2\}$, and $(x, y) \in \rho$ if and only if $x \equiv y \mod 2$.

Clearly, ρ is an equivalence relation on X and $X/\rho = \{\{1, 3, 5, \ldots\}, \{2, 4, 6, \ldots\}\}$. Let α be a function in $E_{F(S)}(X,\rho)$ defined by, for all $x \in X$,

$$x\alpha = \begin{cases} 1, & \text{if } x \text{ is odd}, \\ 2, & \text{if } x \text{ is even}. \end{cases} \tag{2}$$

By Lemma 2, the function α is the only function in $E_{F(S)}(X,\rho)$. So $E_{F(S)}(X,\rho)$ has no nonempty proper subset, and hence, there exists no magnifying element in $E_{F(S)}(X,\rho)$.

For the rest of this section, we focus on nontrivial cases, i.e., $\rho \neq \Delta X$ and $\rho \neq X \times X$, and establish the existence of right magnifying elements in $E_{F(S)}(X,\rho)$. We thus assume now that $E_{F(S)}(X,\rho) \neq \varnothing$.

Lemma 3. *Suppose that X is countably infinite. If there are infinite $[x]_\rho \in X/\rho$ such that $S\cap[x]_\rho=\varnothing$, then there is a surjective function in $E_{F(S)}(X,\rho)$. Consequently, if $\alpha\in E_{F(S)}(X,\rho)$ is a right magnifying element, then α is onto.*

Proof. Suppose that there are infinite $[x]_\rho \in X/\rho$ such that $S\cap[x]_\rho=\varnothing$. Hence, $X\backslash S$ is infinite. Let $A=\{[x]_\rho \mid S\cap[x]_\rho=\varnothing\}$. By assumption, A is infinite. Then there is a bijective function σ from A to $X\backslash S$. For each $x\in X\backslash S$ such that $S\cap[x]_\rho\neq\varnothing$, there exists a unique $s_x\in S$ such that $(x,s_x)\in\rho$, by Lemma 1. Define a function η by

$$x\eta=\begin{cases}[x]_\rho\sigma, & \text{if } x\in X\backslash S \text{ and } [x]_\rho\in A,\\ s_x, & \text{if } x\in X\backslash S \text{ and } [x]_\rho\notin A,\\ x, & \text{if } x\in S.\end{cases} \tag{3}$$

Clearly, $\eta\in E_{F(S)}(X,\rho)$. Moreover, η is onto. Let α be a right magnifying element in $E_{F(S)}(X,\rho)$. There is a proper subset M of $E_{F(S)}(X,\rho)$ such that $M\alpha=E_{F(S)}(X,\rho)$. Then $\beta\alpha=\eta$ for some $\beta\in M$. This implies that α is onto. □

Suppose that X is uncountably infinite and $A=\{[x]_\rho \mid S\cap[x]_\rho=\varnothing\}$. It is easy to prove that if X and A have the same cardinality, then $X\backslash S$ and A have the same cardinality as well. The proof of our next result is similar to the proof of Lemma 3 and so will be omitted.

Lemma 4. *Suppose that X is uncountably infinite and $A=\{[x]_\rho \mid S\cap[x]_\rho=\varnothing\}$. If X and A have the same cardinality, then there is a surjective function in $E_{F(S)}(X,\rho)$. Consequently, if $\alpha\in E_{F(S)}(X,\rho)$ is a right magnifying elements, then α is onto.*

Theorem 5. *Suppose that $\rho\neq\Delta X$ and $\rho\neq X\times X$. If $|S\cap[x]_\rho|\le 1$ for all $x\in X$ and there are infinite $[x]_\rho\in X/\rho$ such that $S\cap[x]_\rho=\varnothing$, then a function $\alpha\in E_{F(S)}(X,\rho)$ is a right magnifying element if and only if α is onto.*

Proof. By Lemmas 3 and 4, the necessity is clear. Conversely, assume that $\alpha\in E_{F(S)}(X,\rho)$ is onto. Let $M=\{\beta\in E_{F(S)}(X,\rho)\mid \beta \text{ is not onto}\}$. Clearly, M is a proper subset of $E_{F(S)}(X,\rho)$ since $\alpha\notin M$. Let γ be a function in $E_{F(S)}(X,\rho)$. Since α is onto, for each $x\in X$, there exists an element $y_x\in X$ such that $y_x\alpha=x\gamma$. For each $[x]_\rho\in X/\rho$, if $S\cap[x]_\rho\neq\varnothing$, then there is a unique element $s_x\in S$ such that $(x,s_x)\in\rho$; and if $S\cap[x]_\rho=\varnothing$, we choose only one y_x to define the function β as

$$a\beta=\begin{cases} s_x, & \text{if } a\in[x]_\rho \text{ and } S\cap[x]_\rho\neq\varnothing,\\ y_x, & \text{if } a\in[x]_\rho \text{ and } S\cap[x]_\rho=\varnothing,\end{cases} \tag{4}$$

for all $a\in X$. It is readily seen that β fixes every element in S. Next, we let $a,b\in X$ such that $(a,b)\in\rho$. Then $a,b\in[x]_\rho$ for some $[x]_\rho\in X/\rho$. If $S\cap[x]_\rho\neq\varnothing$, then $a\beta=s_x=b\beta$. If $S\cap[x]_\rho=\varnothing$, then $a\beta=y_x=b\beta$. Therefore, $\beta\in E_{F(S)}(X,\rho)$. Since $\rho\neq\Delta X$ and $\alpha\in E_{F(S)}(X,\rho)$, the function α is not one-to-one. Then there are distinct elements $y_a,y_b\in X$ such that $y_a\alpha=y_b\alpha$. So at least one of y_a and y_b does not belong to $\operatorname{ran}\beta$, and hence, β is not onto. So $\beta\in M$. For all $x\in X$, $x\beta\alpha=x\gamma$. This shows that $\beta\alpha=\gamma$. Hence, $M\alpha=E_{F(S)}(X,\rho)$, which implies that α is a right magnifying element. □

Example 2. Consider $X=\mathbb{Z}$ and $S=\{0\}$. Let ρ be an equivalence relation on X such that $X/\rho=\{\{0,-1,-2,-3,\ldots\},\{1\},\{2\},\{3\},\ldots\}$. Assume that α is a function in $E_{F(S)}(X,\rho)$ defined by, for all $x\in X$,

$$x\alpha=\begin{cases}\dfrac{x}{2}, & \text{if } x\in\mathbb{Z}^+ \text{ and } 2\mid x,\\ -n, & \text{if } x\in\mathbb{Z}^+ \text{ and } x=2n-1 \text{ for some } n\in\mathbb{N},\\ 0, & \text{otherwise.}\end{cases} \tag{5}$$

That is,

$$\alpha=\begin{pmatrix}\cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots\\ \cdots & 0 & 0 & 0 & -1 & 1 & -2 & 2 & -3 & \cdots\end{pmatrix}. \tag{6}$$

Clearly, the function α is onto. By Theorem 5, the function α is a right magnifying element. Let $M=\{\beta\in E_{F(S)}(X,\rho)\mid\beta \text{ is not onto}\}$. Then there is a proper subset M of $E_{F(S)}(X,\rho)$ such that $M\alpha=E_{F(S)}(X,\rho)$. Let γ be a function in $E_{F(S)}(X,\rho)$ defined by, for all $x\in X$,

$$x\gamma=\begin{cases}-x, & \text{if } x\in\mathbb{Z}^+,\\ 0, & \text{if } x\in\mathbb{Z}^-\cup\{0\}.\end{cases} \tag{7}$$

That is,

$$\gamma=\begin{pmatrix}\cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots\\ \cdots & 0 & 0 & 0 & -1 & -2 & -3 & -4 & -5 & \cdots\end{pmatrix}. \tag{8}$$

Then there is a function $\beta\in M$ such that $\beta\alpha=\gamma$. For more details, define a function $\beta\in E_{F(S)}(X,\rho)$ by, for all $x\in X$,

$$x\beta=\begin{cases}2x-1, & \text{if } x\in\mathbb{Z}^+,\\ 0, & \text{if } x\in\mathbb{Z}^-\cup\{0\}.\end{cases} \tag{9}$$

That is,

$$\beta=\begin{pmatrix}\cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots\\ \cdots & 0 & 0 & 0 & 1 & 3 & 5 & 7 & 9 & \cdots\end{pmatrix}. \tag{10}$$

We obtain that $\beta\in M$ and $\beta\alpha=\gamma$, as required.

Corollary 1. *The following statements hold:*

(1) *Suppose that $\rho=\Delta X$ and $S\neq X$. Then $\alpha\in E_{F(S)}(X,\rho)$ is a right magnifying element if and only if α is one-to-one but not onto.*

(2) *Suppose that $\rho\neq\Delta X$ and $\rho\neq X\times X$. Let X be infinite and $A=\{[x]_\rho\mid S\cap[x]_\rho=\varnothing\}$. If X and A have the same cardinality, and for all $x\in X$, $|S\cap[x]_\rho|\le 1$, then $\alpha\in E_{F(S)}(X,\rho)$ is a right magnifying element if and only if α is onto.*

4.2. Left Magnifying Elements. By Proposition 2, we obtain the next theorem.

Theorem 6 (see [14]). *Suppose that $\rho = \Delta X$, $S \neq X$, and there are infinite $[x]_\rho \in X/\rho$ such that $S \cap [x]_\rho = \varnothing$. The function α is a left magnifying element in $E_{F(S)}(X, \rho)$ if and only if α is one-to-one but not onto.*

Next, we will consider the case in which the equivalence relation $\rho \neq \Delta X$ and $\rho \neq X \times X$.

Lemma 5. *Suppose that $\rho \neq \Delta X$ and $\rho \neq X \times X$. Let α be a function in $E_{F(S)}(X, \rho)$. Then α is a left magnifying element if the following conditions hold:*

(1) For each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap ran\, \alpha \neq \varnothing$, there exists a unique element $y_x \in [x]_\rho \cap ran\, \alpha$,

(2) For all $y \in ran\, \alpha$, there is a unique $[z]_\rho \in X/\rho$ such that $x\alpha = y$ for all $x \in [z]_\rho$.

Proof. Let α be a function in $E_{F(S)}(X, \rho)$ satisfying

(1) For each $[x]_\rho \in X/\rho$ with $[x]_\rho \cap \operatorname{ran} \alpha \neq \varnothing$, there exists a unique element $y_x \in [x]_\rho \cap \operatorname{ran} \alpha$,

(2) For all $y \in \operatorname{ran} \alpha$, there is a unique $[z]_\rho \in X/\rho$ such that $x\alpha = y$ for all $x \in [z]_\rho$.

Since $y_x \in \operatorname{ran} \alpha$, there is a unique $[x']_\rho \in X/\rho$ such that $x'\alpha = y_x$. Fix an element $y_0 \in X$ and let $M = \{\beta \in E_{F(S)}(X, \rho) \mid x\beta = y_0 \text{ for all } x \in X \text{ such that } [x]_\rho \cap \operatorname{ran} \alpha = \varnothing\}$. Clearly, M is a proper subset of $E_{F(S)}(X, \rho)$. Let γ be a function in $E_{F(S)}(X, \rho)$. We then define a function β by, for all $x \in X$,

$$x\beta = \begin{cases} x'\gamma, & \text{if } [x]_\rho \cap \operatorname{ran} \alpha \neq \varnothing, \\ y_0, & \text{if } [x]_\rho \cap \operatorname{ran} \alpha = \varnothing. \end{cases} \tag{11}$$

Clearly, $\beta \in M$, and for all $x \in X$, $x\alpha\beta = x\gamma$. Hence, $\alpha\beta = \gamma$, and so $\alpha M = E_{F(S)}(X, \rho)$. Therefore, α is a left magnifying element. □

Lemma 6. *Suppose that $\rho \neq \Delta X$ and $\rho \neq X \times X$. If $\alpha \in E_{F(S)}(X, \rho)$ is a left magnifying element, then the following conditions hold:*

(1) For each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap ran\, \alpha \neq \varnothing$, there exists a unique element $y_x \in [x]_\rho \cap ran\, \alpha$,

(2) For all $y \in ran\, \alpha$, there is a unique $[z]_\rho \in X/\rho$ such that $x\alpha = y$ for all $x \in [z]_\rho$.

Proof. Since $\rho \neq \Delta X$ and $\rho \neq X \times X$, $|X| > 2$. Let $\alpha \in E_{F(S)}(X, \rho)$ be a left magnifying element. Then there exists a proper subset M of $E_{F(S)}(X, \rho)$ such that $\alpha M = E_{F(S)}(X, \rho)$. Suppose that there exists $[x]_\rho \in X/\rho$ such that there are two distinct elements $y_1, y_2 \in [x]_\rho \cap \operatorname{ran} \alpha$. Since $y_1, y_2 \in \operatorname{ran} \alpha$ and $y_1 \neq y_2$, there are two distinct equivalence classes $[x_1]_\rho, [x_2]_\rho \in X/\rho$ such that $x\alpha = y_1$ for all $x \in [x_1]_\rho$ and $x\alpha = y_2$ for all $x \in [x_2]_\rho$.

Case 1 ($S \cap [x_1]_\rho \neq \varnothing$ and $S \cap [x_2]_\rho \neq \varnothing$): Then $y_1, y_2 \in S$. This implies $y_1, y_2 \in S \cap [x]_\rho$, which is impossible, by Lemma 1.

Case 2 ($S \cap [x_1]_\rho = \varnothing$ and $S \cap [x_2]_\rho = \varnothing$): Note that, for any $x \in X$, by Lemma 1, if $S \cap [x]_\rho \neq \varnothing$, then there exists a unique $s_x \in X$ such that $s_x \in S \cap [x]_\rho$. Let $a, b, c \in X$ be distinct elements and let γ be a function in $E_{F(S)}(X, \rho)$ defined by, for all $x \in X$,

$$x\gamma = \begin{cases} a, & \text{if } x \in [x_1]_\rho, \\ b, & \text{if } x \in [x_2]_\rho, \\ s_x, & \text{if } S \cap [x]_\rho \neq \varnothing, \\ c, & \text{otherwise}. \end{cases} \tag{12}$$

Then there is a function $\beta \in M$ such that $\alpha\beta = \gamma$. Hence, $a = x_1\gamma = x_1\alpha\beta = y_1\beta = y_2\beta = x_2\alpha\beta = x_2\gamma = b$, which is a contradiction.

Case 3 (WLOG, $S \cap [x_1]_\rho \neq \varnothing$ and $S \cap [x_2]_\rho = \varnothing$): By Lemma 1, if $S \cap [x_1]_\rho \neq \varnothing$, then there exists a unique $s_{x_1} \in X$ such that $s_{x_1} \in S \cap [x]_\rho$, and hence, $y_1 = s_{x_1}$. Let $a \in X$ such that $(s_{x_1}, a) \notin \rho$, and let γ be a function in $E_{F(S)}(X, \rho)$ defined by, for all $x \in X$,

$$x\gamma = \begin{cases} s_x, & \text{if } S \cap [x]_\rho \neq \varnothing, \\ a, & \text{if } S \cap [x]_\rho = \varnothing. \end{cases} \tag{13}$$

Then there is a function $\beta \in M$ such that $\alpha\beta = \gamma$. Since $y_1, y_2 \in [x]_\rho$ and $\beta \in E_{F(S)}(X, \rho)$, we obtain $s_{x_1} = x_1\gamma = x_1\alpha\beta = y_1\beta = y_2\beta = x_2\alpha\beta = x_2\gamma = a$, which is a contradiction. Therefore, for each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap \operatorname{ran} \alpha \neq \varnothing$, there exists a unique element $y_x \in [x]_\rho \cap \operatorname{ran} \alpha$.

Let $y \in \operatorname{ran} \alpha$. Suppose that there are two distinct equivalence classes $[a]_\rho, [b]_\rho \in X/\rho$ such that $x\alpha = y$ for all $x \in [a]_\rho \cup [b]_\rho$.

Case 1 ($S \cap [a]_\rho \neq \varnothing$ and $S \cap [b]_\rho \neq \varnothing$): Then we have $y \in S \cap [a]_\rho$ and $y \in S \cap [b]_\rho$. Hence, $y \in [a]_\rho$ and $y \in [b]_\rho$, which is a contradiction.

Case 2 ($S \cap [a]_\rho = \varnothing$ and $S \cap [b]_\rho = \varnothing$): For any $x \in X$, by Lemma 1, if $S \cap [x]_\rho \neq \varnothing$, then there exists a unique $s_x \in X$ such that $s_x \in S \cap [x]_\rho$. Let $c \in X$ and define a function γ in $E_{F(S)}(X, \rho)$ by

$$x\gamma = \begin{cases} a, & \text{if } x \in [a]_\rho, \\ b, & \text{if } x \in [b]_\rho, \\ s_x, & \text{if } S \cap [x]_\rho \neq \varnothing, \\ c, & \text{otherwise}, \end{cases} \tag{14}$$

for all $x \in X$. Then there is a function $\beta \in M$ such that $\alpha\beta = \gamma$. Hence, $a = a\gamma = a\alpha\beta = y\beta = b\alpha\beta = b\gamma = b$, which is a contradiction.

Case 3 (WLOG, $S \cap [a]_\rho \neq \varnothing$ and $S \cap [b]_\rho = \varnothing$): By Lemma 1, if $S \cap [x]_\rho \neq \varnothing$, then there exists a unique $s_x \in X$

such that $s_x \in S\cap [x]_\rho$. Since $S\cap [a]_\rho \neq \varnothing$ and $x\alpha = y$ for all $x \in [a]_\rho$, there exists a unique element $s_a \in S\cap [a]_\rho$ such that $s_a = s_a\alpha = y$. Define a function γ in $E_{F(S)}(X,\rho)$ by

$$x\gamma = \begin{cases} s_x, & \text{if } S\cap [x]_\rho \neq \varnothing, \\ b, & \text{otherwise,} \end{cases} \tag{15}$$

for all $x \in X$. Then there is a function $\beta \in M$ such that $\alpha\beta = \gamma$. Hence, $y = s_a = a\gamma = a\alpha\beta = y\beta = b\alpha\beta = b\gamma = b$. Then $y \in [b]_\rho$, and hence $S\cap [b]_\rho \neq \varnothing$, which is a contradiction. All the cases show that there is a unique $[z]_\rho \in X/\rho$ such that $x\alpha = y$ for all $x \in [z]_\rho$. Therefore, the proof is complete. □

By Lemmas 5 and 6, we obtain the following theorem.

Theorem 7. *Suppose that $\rho \neq \Delta X$ and $\rho \neq X\times X$. A function α is a left magnifying element in $E_{F(S)}(X,\rho)$ if and only if the following conditions hold:*

(1) *For each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap \operatorname{ran}\alpha \neq \varnothing$, there exists a unique element $y_x \in [x]_\rho \cap \operatorname{ran}\alpha$,*

(2) *For all $y \in \operatorname{ran}\alpha$, there is a unique $[z]_\rho \in X/\rho$ such that $x\alpha = y$ for all $x \in [z]_\rho$.*

Example 3. Consider $X = \mathbb{N}$ and $S = \{1,5,9,13,\ldots\}$. Let ρ be an equivalence relation on X such that $X/\rho = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\ldots\}$. Assume that α is a function in $E_{F(S)}(X,\rho)$ defined by, for all $x \in X$,

$$x\alpha = \begin{cases} x, & \text{if } x \equiv 1 \bmod 4, \\ x-1, & \text{if } x \equiv 2 \bmod 4, \\ x+4, & \text{if } x \equiv 3 \bmod 4, \\ x+3, & \text{if } x \equiv 0 \bmod 4. \end{cases} \tag{16}$$

That is,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 1 & 1 & 7 & 7 & 5 & 5 & 11 & 11 & 9 & \cdots \end{pmatrix}. \tag{17}$$

Clearly, $X/\rho = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\ldots\} = \{[1]_\rho, [3]_\rho, [5]_\rho, [7]_\rho, \ldots\}$ and $\operatorname{ran}\alpha = \{1,5,9,13,\ldots\} \cup \{7,11,15,19,\ldots\}$. We can see that $1 \in [1]_\rho \cap \operatorname{ran}\alpha$, $5 \in [5]_\rho \cap \operatorname{ran}\alpha$, $7 \in [7]_\rho \cap \operatorname{ran}\alpha$, $9 \in [9]_\rho \cap \operatorname{ran}\alpha$, and $11 \in [11]_\rho \cap \operatorname{ran}\alpha, \ldots$. So for each $[x]_\rho \in X/\rho$ such that $[x]_\rho \cap \operatorname{ran}\alpha \neq \varnothing$, there exists a unique element $y_x \in [x]_\rho \cap \operatorname{ran}\alpha$. Moreover, for all $y \in \operatorname{ran}\alpha$, there is a unique $[z]_\rho \in X/\rho$ such that $x\alpha = y$ for all $x \in [z]_\rho$. By Theorem 7, α is a left magnifying element. Let $M = \{\beta \in E_{F(S)}(X,\rho) \mid x\beta = 1 \text{ for all } x \in X \text{ such that } [x]_\rho \cap \operatorname{ran}\alpha = \varnothing\}$. Then there is a proper subset M of $E_{F(S)}(X,\rho)$ such that $\alpha M = E_{F(S)}(X,\rho)$. Let γ be a function in $E_{F(S)}(X,\rho)$ defined by, for all $x \in X$,

$$x\gamma = \begin{cases} x, & \text{if } x \equiv 1 \bmod 4, \\ x-1, & \text{if } x \equiv 2,3 \bmod 4, \\ x-2, & \text{if } x \equiv 0 \bmod 4. \end{cases} \tag{18}$$

That is,

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 1 & 1 & 2 & 2 & 5 & 5 & 6 & 6 & 9 & \cdots \end{pmatrix}. \tag{19}$$

Then there is a function $\beta \in M$ such that $\alpha\beta = \gamma$. Define a function $\beta \in E_{F(S)}(X,\rho)$ by, for all $x \in X$,

$$x\beta = \begin{cases} x, & \text{if } x \equiv 1 \bmod 4, \\ x-1, & \text{if } x \equiv 2 \bmod 4, \\ x-5, & \text{if } x \equiv 3 \bmod 4, \text{ and } x \neq 3, \\ x-6, & \text{if } x \equiv 0 \bmod 4, \text{ and } x \neq 4, \\ 1, & \text{if } x = 3,4. \end{cases} \tag{20}$$

That is,

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 1 & 1 & 1 & 1 & 5 & 5 & 2 & 2 & 9 & \cdots \end{pmatrix}. \tag{21}$$

So $\beta \in M$, and hence,

$$\begin{aligned} \alpha\beta &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 1 & 1 & 7 & 7 & 5 & 5 & 11 & 11 & 9 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 1 & 1 & 1 & 1 & 5 & 5 & 2 & 2 & 9 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 1 & 1 & 2 & 2 & 5 & 5 & 6 & 6 & 9 & \cdots \end{pmatrix} = \gamma. \end{aligned} \tag{22}$$

5. Conclusion

In this paper, some particular properties of $E_{F(S)}(X,\rho)$ are investigated in Section 3. The main task of this paper is to establish the necessary and sufficient conditions for an elements $\alpha \in E_{F(S)}(X,\rho)$ to be right or left magnifying elements, which are summarized as follows:

If $\rho = \Delta X$ and $S \neq X$, the following statements are true:

(1) The function α is a right magnifying element if and only if α is onto but not one-to-one.

(2) Suppose that there are infinite equivalence classes $[x]_\rho$ such that $S \cap [x]_\rho = \varnothing$. Then α is a left magnifying element if and only if α is one-to-one but not onto.

If $\rho \neq \Delta X$ and $\rho \neq X \times X$, the following statements are true:

(1) If the set X and the set of all equivalence classes $[x]_\rho$ such that $S \cap [x]_\rho = \varnothing$ have the same cardinality, and for all $x \in X$, $|S \cap [x]_\rho| \leq 1$, then α is a right magnifying element if and only if α is onto.

(2) The function α is a left magnifying element if and only if

(i) If $[x]_\rho \cap \operatorname{ran} \alpha$ is nonempty, then it is singleton,
(ii) For all $y \in \operatorname{ran} \alpha$, there is a unique equivalence class $[x]_\rho$ whose members are all mapped to y.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Faculty of Science Research Fund, Prince of Songkla University (Contract No. 264006).

References

[1] E. S. Ljapin, "Semigroups," in *Translations of Mathematical Monographs*vol. 3, Providence, RI, USA, American Mathematical Society, 1963.
[2] F. Migliorini, "Some researches on semigroups with magnifying elements," *Periodica Mathematica Hungarica*, vol. 1, no. 4, pp. 279–286, 1971.
[3] F. Migliorini, "Magnifying elements and minimal subsemigroups in semigroups," *Periodica Mathematica Hungarica*, vol. 5, no. 4, pp. 279–288, 1974.
[4] K. Tolo, "Factorizable semigroups," *Pacific Journal of Mathematics*, vol. 31, no. 2, pp. 523–535, 1969.
[5] F. Catino and F. Migliorini, "Magnifying elements in semigroups," *Semigroup Forum*, vol. 44, no. 1, pp. 314–319, 1992.
[6] K. D. Magill Jr., "Magnifying elements of transformation semigroups," *Semigroup Forum*, vol. 48, no. 1, pp. 119–126, 1994.
[7] M. Gutan, "Semigroups with strong and nonstrong magnifying elements," *Semigroup Forum*, vol. 53, no. 1, pp. 384–386, 1996.
[8] M. Gutan, "Semigroups which contain magnifying elements are factorizable," *Communications in Algebra*, vol. 25, no. 2, pp. 3953–3963, 1997.
[9] M. Gutan, "Semigroups with magnifiers admitting minimal subsemigroups," *Communications in Algebra*, vol. 27, no. 4, pp. 1975–1996, 1999.
[10] M. Gutan, "Good and very good magnifiers," *Bollettino dell'Unione Matematica Italiana*, vol. 3-B, no. 3, pp. 793–810, 2000.
[11] M. Gutan and A. Kisielewicz, "Semigroups with good and bad magnifiers," *Journal of Algebra*, vol. 267, no. 2, pp. 587–607, 2003.
[12] P. Luangchaisri, T. Changphas, and C. Phanlert, "Left (right) magnifying elements of a partial transformation semigroup," *Asian-European Journal of Mathematics*, vol. 13, no. 1, 7 pages, Article ID 2050016, 2020.
[13] P. Prakitsri, "Left and right magnifying elements in certain linear transformation semigroups," *Thai Journal of Mathematics*, vol. 2019, pp. 285–291, 2020, Special Issue (2020): Annual Meeting in Mathematics.
[14] M. Petapirak, T. Kaewnoi, and R. Chinram, "On magnifying elements in transformation semigroups with a fixed point set," *Thai Journal of Mathematics*, vol. 18, no. 3, pp. 1041–1049, 2020.