Research Article

Weighted Composition Operators from Derivative Hardy Spaces into n -th Weighted-Type Spaces

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The boundedness, compactness, and the essential norm of weighted composition operators from derivative Hardy spaces into n -th weighted-type spaces are investigated in this paper.

1. Introduction

Let $H(\mathbb{D})$ denote the space of analytic functions on the open unit disk $\mathbb{D}$. Let $S(\mathbb{D})$ denote the set of all analytic self-maps of $\mathbb{D}$. Let $\varphi \in S(\mathbb{D})$. The composition operator C_φ with the symbol φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}). \quad (1)$$

Let $\varphi \in S(\mathbb{D})$ and $\psi \in H(\mathbb{D})$. The weighted composition operator ψC_φ is defined on $H(\mathbb{D})$ by

$$(\psi C_\varphi f)(z) = \psi(z)f(\varphi(z)), \quad f \in H(\mathbb{D}). \quad (2)$$

It is important to give function theoretic descriptions when ψ and φ induce a bounded or compact weighted composition operator on various function spaces. See references [1, 2] for more information of this research field.

For $0 < p < \infty$, the Hardy space, denoted by $\mathcal{H}^p$, consists of all functions $f \in H(\mathbb{D})$ such that (see [3])

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty. \quad (3)$$

As usual, $\mathcal{H}^\infty$ denotes the space of bounded analytic functions in $\mathbb{D}$. If $f' \in \mathcal{H}^p$, we say that f belongs to the derivative Hardy space, denoted by $\mathcal{S}^p$. When $p > 1$, the space $\mathcal{S}^p$ is a Banach space under the norm defined by

$$\|f\|_{\mathcal{S}^p}^p = |f(0)|^p + \|f'\|_{\mathcal{H}^p}^p. \quad (4)$$

When $p = 2$, $\mathcal{S}^2$ is a Hilbert space. In [4], Roan started the study of composition operators on the space $\mathcal{S}^p$. In [5], MacCluer investigated composition operators on the space $\mathcal{S}^p$ in terms of Carleson measure. The boundedness and compactness of weighted composition operators on $\mathcal{S}^p$ were studied in [6]. See references [4–7] and references therein for more study of composition operators and weighted composition operators on the space $\mathcal{S}^p$.

If μ is a radial, positive, and continuous function on $\mathbb{D}$, then μ is called a radial weight. Let μ be a radial weight and $n \in \mathbb{N}$, the set of all positive integers. Let $\mathcal{W}_\mu^n$ denote the n -th weighted space, which consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{W}_\mu^n} = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{z \in \mathbb{D}} \mu(z) |f^{(n)}(z)| < \infty. \quad (5)$$

It is a Banach space with the norm $\|\cdot\|_{\mathcal{W}_\mu^n}$. When $n = 1$ and $n = 2$, $\mathcal{W}_\mu^n$ becomes the Bloch-type space $\mathcal{B}_\mu$ and the Zygmund-type space $\mathcal{Z}_\mu$, respectively. Furthermore, when $\mu(z) = (1 - |z|^2)$, $\mathcal{B}_\mu = \mathcal{B}$ is the Bloch space and $\mathcal{Z}_\mu = \mathcal{Z}$ is the Zygmund space. For some results on the space $\mathcal{W}_\mu^n$, see references [2, 8–13].

Let $n, k \in \mathbb{N}_0$ with $k \leq n$. Recall that the partial Bell polynomials are defined as follows:

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \dots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \dots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}, \quad (6)$$

where the sum taken over all sequences $j_1, j_2, \dots, j_{n-k+1}$ of nonnegative integers such that the following two conditions hold:

$$\begin{aligned} j_1 + j_2 + \dots + j_{n-k+1} &= k \\ j_1 + 2j_2 + \dots + (n-k+1)j_{n-k+1} &= n. \end{aligned} \quad (7)$$

See reference [14] for more information about Bell polynomials.

In [15], Stević studied the boundedness and compactness of the composition operator from A_α^p to $\mathcal{W}_\mu^n$ on the unit disk. In [12], Stević studied the boundedness and compactness of weighted composition operators from $\mathcal{H}^\infty$ and the Bloch space to $\mathcal{W}_\mu^n$. See references [8–10] for more characterizations for weighted composition operators from $\mathcal{H}^\infty$ and the Bloch space to $\mathcal{W}_\mu^n$. In [16], Zhu and Du studied the boundedness, compactness, and essential norm of weighted composition operators from weighted Bergman spaces with doubling weight A_ω^p to $\mathcal{W}_\mu^n$. Recall that the essential norm of a bounded linear operator $T: X \rightarrow Y$ is its distance to the set of compact operators K mapping X into Y , that is,

$$\|T\|_{e, X \rightarrow Y} = \inf \{ \|T - K\|_{X \rightarrow Y} : K \text{ is a compact operator} \}. \quad (8)$$

Here, X and Y are Banach spaces and $\|\cdot\|_{X \rightarrow Y}$ denotes the operator norm. In [17], Colonna and Tjani studied the boundedness and compactness of weighted composition operators from derivative Hardy spaces $\mathcal{S}^p$ to Bloch-type space $\mathcal{B}_\mu$.

In this paper, we use Bell polynomials to study the boundedness and compactness of weighted composition operators from derivative Hardy spaces $\mathcal{S}^p$ to n -th weighted-type spaces $\mathcal{W}_\mu^n$. Moreover, we give an estimate for the essential norm of weighted composition operators from $\mathcal{S}^p$ to $\mathcal{W}_\mu^n$.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. In addition, we say that $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Boundedness

In this section, the boundedness of weighted composition operators from $\mathcal{S}^p$ to $\mathcal{W}_\mu^n$ is characterized.

Lemma 1. *Suppose $1 < p < \infty$ and $k \in \mathbb{N}$. Then, there exists a positive constant C such that*

$$\begin{aligned} \|f\|_\infty &\leq C \|f\|_{\mathcal{S}^p}, \\ |f^{(k)}(z)| &\leq \frac{C \|f\|_{\mathcal{S}^p}}{(1 - |z|^2)^{k-1+(1/p)}}, \end{aligned} \quad (9)$$

for every $f \in \mathcal{S}^p$.

Proof. The first inequality follows from the fact that $\mathcal{S}^p$ are contained in the disk algebra for $p > 1$. In addition, it is well known that for every $f \in \mathcal{H}^p$, there exists a positive constant C such that

$$|f^{(k)}(z)| \leq \frac{C \|f\|_{\mathcal{H}^p}}{(1 - |z|^2)^{k+(1/p)}}, \quad (10)$$

which implies the second inequality.

For any $a \in \mathbb{D}$, $1 < p < \infty$, and $j \in \{1, 2, \dots, n+1\}$, set

$$f_{j,a}(z) = \frac{(1 - |a|^2)^j}{(1 - \bar{a}z)^{j+(1/p)-1}}, \quad z \in \mathbb{D}. \quad (11)$$

After a calculation, for each $j \in \{1, 2, \dots, n+1\}$,

$$\begin{aligned} f_{j,a} &\in \mathcal{S}^p \\ \sup_{a \in \mathbb{D}} \|f_{j,a}\|_{\mathcal{S}^p} &< \infty. \end{aligned} \quad (12) \quad \square$$

Lemma 2. *Let $1 < p < \infty$ and $0 \neq a \in \mathbb{D}$. For any $i \in \{1, \dots, n\}$, there exist constants $c_2^i, \dots, c_{n+1}^i$ such that*

$$\begin{aligned} v_{i,a} &= f_{1,a} + \sum_{j=2}^{n+1} c_j^i f_{j,a} \in \mathcal{S}^p, \\ v_{i,a}(a) &= 0, \\ v_{i,a}^{(k)}(a) &= \begin{cases} \frac{\bar{a}^i}{(1 - |a|^2)^{i+(1/p)-1}}, & k = i, \\ 0, & k \neq i. \end{cases} \end{aligned} \quad (13)$$

Moreover, $v_{i,a}$ converges to 0 uniformly in $\bar{\mathbb{D}}$.

Proof. The proof is similar to the proof of Theorem 1 in [16]. Hence, we omit the details of the proof.

For the simplicity of this paper, we define

$$I_i^n(z) = \sum_{l=i}^n \binom{n}{l} \psi^{(n-l)}(z) B_{l,i}(\varphi'(z), \varphi''(z), \dots, \varphi^{(l-i+1)}(z)). \quad (14)$$

From the definition, we see that, for example,

$$\begin{aligned} B_{0,0}(x_1) &= 1, \\ B_{1,0}(x_1, x_2) &= 0, \\ B_{1,1}(x_1) &= x_1, \\ B_{2,0}(x_1, x_2, x_3) &= 0, \\ B_{2,1}(x_1, x_2) &= x_2, \\ B_{2,2}(x_1) &= x_1^2. \end{aligned} \quad (15)$$

This fact will be used in the proof of the following theorem.

Now, we are in a position to state and prove the first result in this paper. $\square$

Theorem 1. Let $n \in \mathbb{N}$, $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and μ be a weight. Then, the operator $\psi C_\varphi: \mathcal{S}^p \longrightarrow \mathcal{W}_\mu^n$ is bounded if and only if $\psi \in \mathcal{W}_\mu^n$ and

$$\sum_{i=1}^n \sup_{z \in \mathbb{D}} \frac{\mu(z) \left| \sum_{l=i}^n \binom{n}{l} \psi^{(n-l)}(z) B_{l,i}(\varphi'(z), \varphi''(z), \dots, \varphi^{(l-i+1)}(z)) \right|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}} < \infty. \quad (16)$$

Proof (sufficiency). Let $f \in \mathcal{S}^p$. After a calculation, we have (see, e.g., [12])

$$(\psi C_\varphi f)^{(n)}(z) = \sum_{i=0}^n f^{(i)}(\varphi(z)) \sum_{l=i}^n \binom{n}{l} \psi^{(n-l)}(z) B_{l,i}(\varphi'(z), \dots, \varphi^{(l-i+1)}(z)). \quad (17)$$

Since $B_{0,0}(\varphi'(z)) = 1$ and $B_{l,0}(\varphi'(z), \dots, \varphi^{(l+1)}(z)) = 0$ ($l \in \mathbb{N}$), we see that $I_0^n(z) = \psi^{(n)}(z)$. Hence, by (14) and Lemma 1, we get

$$\begin{aligned} \mu(z) \left| (\psi C_\varphi f)^{(n)}(z) \right| \\ \leq \mu(z) + \mu(z) \sum_{i=1}^n \left| f^{(i)}(\varphi(z)) \right| |I_i^n(z)|, \end{aligned} \quad (18)$$

$$\leq \|f\|_{\mathcal{S}^p} \|\psi\|_{\mathcal{W}_\mu^n} + \|f\|_{\mathcal{S}^p} \sum_{i=1}^n \sup_{z \in \mathbb{D}} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}}. \quad (19)$$

From (18), for each $j \in \{0, 1, \dots, n-1\}$,

$$\begin{aligned} \left| (\psi C_\varphi f)^{(j)}(0) \right| &\leq \left| f(\varphi(0)) \right| \left| \psi^{(j)}(0) \right| + \sum_{i=1}^j \left| f^{(i)}(\varphi(0)) \right| |I_i^j(0)| \\ &\leq \|f\|_{\mathcal{S}^p} \left| \psi^{(j)}(0) \right| + \|f\|_{\mathcal{S}^p} \sum_{i=1}^j \frac{|I_i^j(0)|}{(1 - |\varphi(0)|^2)^{i+(1/p)-1}}. \end{aligned} \quad (20)$$

Therefore, from (19) and (20) and the fact that $\psi \in \mathcal{W}_\mu^n$, we get that $\psi C_\varphi: \mathcal{S}^p \longrightarrow \mathcal{W}_\mu^n$ is bounded.

Necessity. Assume that $\psi C_\varphi: \mathcal{S}^p \longrightarrow \mathcal{W}_\mu^n$ is bounded. It is clear that $\psi \in \mathcal{W}_\mu^n$, and for each $j \in \{1, \dots, n+1\}$ and $a \in \mathbb{D}$,

$$\sup_{a \in \mathbb{D}} \|\psi C_\varphi f_{j,a}\|_{\mathcal{W}_\mu^n} < \infty, \quad (21)$$

by the boundedness of $\psi C_\varphi: \mathcal{S}^p \longrightarrow \mathcal{W}_\mu^n$.

Next, we show that for $i = 1, 2, \dots, n$,

$$\sup_{z \in \mathbb{D}} \mu(z) |I_i^n(z)| < \infty. \quad (22)$$

Applying the operator ψC_φ for $h_1(z) = z$, by (17) and (14), we obtain

$$\begin{aligned} \sup_{z \in \mathbb{D}} \mu(z) |I_0^n(z) \varphi(z) + I_1^n(z)| &= \sup_{z \in \mathbb{D}} \mu(z) |(\psi C_\varphi h_1)^{(n)}(z)| \\ &\leq \|\psi C_\varphi h_1\|_{\mathcal{W}_\mu^n} < \infty. \end{aligned} \quad (23)$$

Therefore, by the triangle inequality and the boundedness of φ , we obtain

$$\sup_{z \in \mathbb{D}} \mu(z) |I_1^n(z)| < \infty. \quad (24)$$

Now, we assume that for $1 \leq i \leq j-1$ ($j \leq n$), $\sup_{z \in \mathbb{D}} \mu(z) |I_i^n(z)| < \infty$. To get the desired result, we only need to show that

$$\sup_{z \in \mathbb{D}} \mu(z) |I_j^n(z)| < \infty. \quad (25)$$

Applying the operator ψC_φ for $h_j(z) = z^j$, we obtain

$$\begin{aligned} \sup_{z \in \mathbb{D}} \mu(z) \left| \varphi^j(z) I_0^n(z) + \sum_{k=1}^j j(j-1), \dots, (j-k+1) (\varphi(z))^{j-k}, I_k^n(z) \right| \\ \leq \|\psi C_\varphi h_j\|_{\mathcal{W}_\mu^n} < \infty. \end{aligned} \quad (26)$$

Hence, from the boundedness of φ and triangle inequality again, we get the desired result.

For any $i \in \{1, \dots, n\}$ and $\varphi(a) \neq 0$, by Lemma 2,

$$\frac{\mu(a) |\varphi(a)|^i |I_i^n(a)|}{(1 - |\varphi(a)|^2)^{i+(1/p)-1}} \leq \sup_{a \in \mathbb{D}} \|\psi C_\varphi v_{i,\varphi(a)}\|_{\mathcal{W}_\mu^n} \quad (27)$$

$$\leq \sup_{a \in \mathbb{D}} \|\psi C_\varphi f_{1,a}\|_{\mathcal{W}_\mu^n} + \sum_{j=2}^{n+1} |c_j^i| \sup_{a \in \mathbb{D}} \|\psi C_\varphi f_{j,a}\|_{\mathcal{W}_\mu^n} < \infty,$$

where c_j^i are independent of the choice of a . From the last inequality and (22), for any $i \in \{1, \dots, n\}$, we get

$$\begin{aligned} \sup_{|\varphi(a)| > (1/2)} \frac{\mu(a) |I_i^n(a)|}{(1 - |\varphi(a)|^2)^{i+(1/p)-1}} \\ \leq \sup_{a \in \mathbb{D}} \|\psi C_\varphi f_{1,a}\|_{\mathcal{W}_\mu^n} + \sum_{j=2}^{n+1} |c_j^i| \sup_{a \in \mathbb{D}} \|\psi C_\varphi f_{j,a}\|_{\mathcal{W}_\mu^n} < \infty, \\ \sup_{|\varphi(a)| \leq (1/2)} \frac{\mu(a) |I_i^n(a)|}{(1 - |\varphi(a)|^2)^{i+(1/p)-1}} \leq \sup_{|\varphi(a)| \leq (1/2)} \mu(a) |I_i^n(a)| < \infty. \end{aligned} \quad (28)$$

Therefore,

$$\sum_{i=1}^n \sup_{z \in \mathbb{D}} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}} < \infty. \quad (29)$$

The proof is complete.

Let $n = 1$. We get the following result, which was first obtained in [17]. $\square$

Corollary 1. Let $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and μ be a weight. Assume that $\psi C_\varphi: \mathcal{S}^p \longrightarrow \mathcal{B}_\mu$ is bounded if and only if $\psi \in \mathcal{B}_\mu$ and

$$\sup_{z \in \mathbb{D}} \frac{\mu(z) |\psi(z) \varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1/p)}} < \infty. \quad (30)$$

Let $n = 2$. We get the following result.

Corollary 2. Let $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and μ be a weight. Then, the operator $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{X}_\mu$ is bounded if and only if $\psi \in \mathcal{X}_\mu$,

$$\sup_{z \in \mathbb{D}} \frac{\mu(z) |\psi(z) \varphi''(z) + 2\psi'(z) \varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1/p)}} < \infty$$

$$\sup_{z \in \mathbb{D}} \frac{\mu(z) |\psi(z) \varphi'^2(z)|}{(1 - |\varphi(z)|^2)^{1+(1/p)}} < \infty. \quad (31)$$

3. Essential Norm

In this section, we obtain some estimates for the essential norm of the weighted composition operator $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n$, and we need the following lemma.

Lemma 3 (see [17]). Let X be a Banach space that is continuously contained in the disk algebra, and let Y be any Banach space of analytic functions on $\mathbb{D}$. Suppose that

- (1) The point evaluation functionals on Y are continuous
- (2) For every sequence $\{f_n\}$ in the unit ball of X , there exists $f \in X$ and a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \rightarrow f$ uniformly on $\overline{\mathbb{D}}$
- (3) The operator $T: X \rightarrow Y$ is continuous if X has the supremum norm and Y is given the topology of uniform convergence on compact sets

Then, T is a compact operator if and only if, given a bounded sequence $\{f_n\}$ in X such that $f_n \rightarrow 0$ uniformly on $\overline{\mathbb{D}}$, then $\|Tf_n\|_Y \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 4 (see [17]). Let $1 < p < \infty$. Every sequence in $\mathcal{S}^p$ bounded in norm has a subsequence which converges uniformly in $\overline{\mathbb{D}}$ to a function in $\mathcal{S}^p$.

The following result is a direct consequence of Lemma 3 and Lemma 4.

Lemma 5. Let $n \in \mathbb{N}$, $1 < p < \infty$, and μ be a weight. If T is a bounded linear operator from $\mathcal{S}^p$ into $\mathcal{W}_\mu^n$, then T is compact if and only if $\|Tf_n\|_{\mathcal{W}_\mu^n} \rightarrow 0$ as $n \rightarrow \infty$ for any sequence $\{f_n\}$ in $\mathcal{S}^p$ bounded in norm which converges to 0 uniformly in $\overline{\mathbb{D}}$.

Theorem 2. Let $n \in \mathbb{N}$, $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and μ be a weight such that $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n$ is bounded. Then,

$$\|\psi C_\varphi\|_{e, \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n} \approx \sum_{i=1}^n \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}}. \quad (32)$$

Proof. First, we prove that

$$\sum_{i=1}^n \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}} \leq \|\psi C_\varphi\|_{e, \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n}. \quad (33)$$

If $\sup_{z \in \mathbb{D}} |\varphi(z)| < 1$, there is a number $\delta \in (0, 1)$ such that $\sup_{z \in \mathbb{D}} |\varphi(z)| < \delta$. In this case, (32) is vacuously satisfied and the desired result follows.

Assume that $\sup_{z \in \mathbb{D}} |\varphi(z)| = 1$. Let $\{z_j\}_{j \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \rightarrow 1$ as $j \rightarrow \infty$. Since $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n$ is bounded, for any compact operator $K: \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n$ and $i \in \{1, \dots, n\}$, by using Lemma 2 and Lemma 5, we obtain

$$\begin{aligned} \|\psi C_\varphi - K\|_{\mathcal{S}^p \rightarrow \mathcal{W}_\mu^n} &\geq \limsup_{j \rightarrow \infty} \|\psi C_\varphi v_{i, \varphi}(z_j)\|_{\mathcal{W}_\mu^n} - \limsup_{j \rightarrow \infty} \|K v_{i, \varphi}(z_j)\|_{\mathcal{W}_\mu^n} \\ &\geq \limsup_{j \rightarrow \infty} \frac{\mu(z_j) |\varphi(z_j)|^i |I_i^n(z_j)|}{(1 - |\varphi(z_j)|^2)^{i+(1/p)-1}}. \end{aligned} \quad (34)$$

Hence,

$$\|\psi C_\varphi\|_{e, \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n} \geq \limsup_{j \rightarrow \infty} \frac{\mu(z_j) |\varphi(z_j)|^i |I_i^n(z_j)|}{(1 - |\varphi(z_j)|^2)^{i+(1/p)-1}}, \quad (35)$$

which implies that

$$\|\psi C_\varphi\|_{e, \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n} \geq \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}}, \quad (36)$$

as desired.

Now, we show that

$$\| \psi C_\varphi \|_{e, \mathcal{S}^p} \longrightarrow \mathcal{W}_\mu^n \lesssim \sum_{i=1}^n \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}}. \quad (37)$$

Let $r \in [0, 1)$ and define $K_r f(z) = f_r(z) = f(rz)$. Then, $K_r: \mathcal{S}^p \rightarrow \mathcal{S}^p$ is compact and $\|K_r\|_{\mathcal{S}^p \rightarrow \mathcal{S}^p} \leq 1$. It is clear that $f_r \rightarrow f$ uniformly on compact subsets of $\mathbb{D}$ as $r \rightarrow 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \rightarrow 1$ as

$j \rightarrow \infty$. Then, for any positive integer j , the operator $\psi C_\varphi K_{r_j}: \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n$ is compact. By the definition of the essential norm, we get

$$\| \psi C_\varphi \|_{e, \mathcal{S}^p} \longrightarrow \mathcal{W}_\mu^n \leq \limsup_{j \rightarrow \infty} \| \psi C_\varphi - \psi C_\varphi K_{r_j} \|_{\mathcal{S}^p \rightarrow \mathcal{W}_\mu^n}. \quad (38)$$

Hence, it is sufficient to show that

$$\limsup_{j \rightarrow \infty} \| \psi C_\varphi - \psi C_\varphi K_{r_j} \|_{\mathcal{S}^p \rightarrow \mathcal{W}_\mu^n} \lesssim \sum_{i=1}^n \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}}. \quad (39)$$

For any $f \in \mathcal{S}^p$ such that $\|f\|_{\mathcal{S}^p} \leq 1$,

$$\begin{aligned} & \| (\psi C_\varphi - \psi C_\varphi K_{r_j}) f \|_{\mathcal{W}_\mu^n} \\ & \leq \sum_{t=0}^{n-1} \sum_{i=0}^t \left| (f - f_{r_j})^{(i)}(\varphi(0)) \right| |I_i^t(0)| + \sup_{z \in \mathbb{D}} \mu(z) \sum_{i=0}^n \left| (f - f_{r_j})^{(i)}(\varphi(z)) \right| |I_i^n(z)| \\ & \leq \underbrace{\sum_{t=0}^{n-1} \sum_{i=0}^t \left| (f - f_{r_j})^{(i)}(\varphi(0)) \right| |I_i^t(0)|}_{\Omega_0} + \underbrace{\sup_{z \in \mathbb{D}} \mu(z) \left| (f - f_{r_j})(z) \right| |I_0^n(z)|}_{\Omega_1} \\ & \quad + \underbrace{\sup_{|\varphi(z)| \leq r_N} \mu(z) \sum_{i=1}^n \left| (f - f_{r_j})^{(i)}(\varphi(z)) \right| |I_i^n(z)|}_{\Omega_2} \\ & \quad + \underbrace{\sup_{|\varphi(z)| > r_N} \mu(z) \sum_{i=1}^n \left| (f - f_{r_j})^{(i)}(\varphi(z)) \right| |I_i^n(z)|}_{\Omega_3}, \end{aligned} \quad (40)$$

where $N \in \mathbb{N}$ such that $r_j \geq (2/3)$ for all $j \geq N$. Since for any nonnegative integer s , $(f - f_{r_j})^{(s)} \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \rightarrow \infty$. It is clear that

$$\begin{aligned} \limsup_{j \rightarrow \infty} \Omega_0 &= 0 \\ \limsup_{j \rightarrow \infty} \Omega_2 &= 0. \end{aligned} \quad (41)$$

From Lemma 4,

$$\lim_{j \rightarrow \infty} \Omega_1 \leq \| \psi \|_{\mathcal{W}_\mu^n} \lim_{j \rightarrow \infty} \sup_{z \in \mathbb{D}} \left| (f - f_{r_j})(z) \right| = 0, \quad (42)$$

while

$$\Omega_3 \leq \sum_{i=1}^n \sup_{|\varphi(z)| > r_N} \mu(z) \left| f^{(i)}(\varphi(z)) \right| |I_i^n(z)|_{P_i} + \sum_{i=1}^n \sup_{|\varphi(z)| > r_N} \mu(z) \left| r_j^i f^{(i)}(r_j \varphi(z)) \right| |I_i^n(z)|_{Q_i}. \quad (43)$$

For any $i \in \{1, \dots, n\}$, by Lemma 1 and Lemma 2,

$$\begin{aligned}
P_i &= \sup_{|\varphi(z)| > r_N} \mu(z) \frac{(1 - |\varphi(z)|^2)^{i+(1/p)-1} |f^{(i)}(\varphi(z))|}{|\varphi(z)|^i} \frac{|\varphi(z)|^i |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}} \\
&\leq \|f\|_{\mathcal{S}^p} \sup_{|\varphi(z)| > r_N} \|\psi C_\varphi \nu_{i,\varphi}(z)\|_{\mathcal{W}_\mu^n} \\
&\leq \sup_{|a| > r_N} \|\psi C_\varphi f_{1,a}\|_{\mathcal{W}_\mu^n} + \sum_{k=2}^{n+1} |c_k^i| \sup_{|a| > r_N} \|\psi C_\varphi f_{k,a}\|_{\mathcal{W}_\mu^n} \\
&\leq \sum_{k=1}^{n+1} \sup_{|a| > r_N} \|\psi C_\varphi f_{k,a}\|_{\mathcal{W}_\mu^n}.
\end{aligned} \tag{44}$$

Taking the limit as $N \rightarrow \infty$, by Lemma 1, we get

$$\limsup_{j \rightarrow \infty} P_i \leq \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}}. \tag{45}$$

By the same reason, we have

$$\limsup_{j \rightarrow \infty} Q_i \leq \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}}. \tag{46}$$

Hence, by (40)–(43), (45), and (46), we obtain

$$\begin{aligned}
&\limsup_{j \rightarrow \infty} \|\psi C_\varphi - \psi C_\varphi K_{r_j}\|_{\mathcal{S}^p \rightarrow \mathcal{W}_\mu^n} \\
&= \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{S}^p} \leq 1} \left\| (\psi C_\varphi - \psi C_\varphi K_{r_j}) f \right\|_{\mathcal{W}_\mu^n} \leq \sum_{i=1}^n \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}}.
\end{aligned} \tag{47}$$

Hence, from the last two inequalities, we obtain (39). The proof is complete.

From Theorem 2 and the well-known result that $\|T\|_{e,X \rightarrow Y} = 0$ if and only if $T: X \rightarrow Y$ is compact, we obtain the following corollary. $\square$

Corollary 3. Let $n \in \mathbb{N}$, $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and μ be a weight. Assume that $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n$ is bounded, then the operator $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{W}_\mu^n$ is compact if and only if

$$\sum_{i=1}^n \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^{i+(1/p)-1}} = 0. \tag{48}$$

Let $n = 1$. We get the following result, which was first obtained in [17].

Corollary 4. Let $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and μ be a weight such that $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{B}_\mu$ is bounded. Then, the operator $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{B}_\mu$ is compact if and only if

$$\limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |\psi(z) \varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1/p)}} = 0. \tag{49}$$

Let $n = 2$. We get the following result.

Corollary 5. Let $1 < p < \infty$, $\varphi \in S(\mathbb{D})$, $\psi \in H(\mathbb{D})$, and μ be a weight such that $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{F}_\mu$ is bounded. Then, the operator $\psi C_\varphi: \mathcal{S}^p \rightarrow \mathcal{F}_\mu$ is compact if and only if

$$\begin{aligned}
&\limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |\psi(z) \varphi''(z) + 2\psi'(z) \varphi'(z)|}{(1 - |\varphi(z)|^2)^{(1/p)}} = 0 \\
&\limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |\psi(z) \varphi'^2(z)|}{(1 - |\varphi(z)|^2)^{1+(1/p)}} = 0.
\end{aligned} \tag{50}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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