Research Article

A Novel Method for Developing Efficient Probability Distributions with Applications to Engineering and Life Science Data

Alamgir Khalil,¹ Abdullah Ali H. Ahmadini²,³ Muhammad Ali,¹ Wali Khan Mashwani³,⁴ Shokrya S. Alshqaq,² and Zabidin Salleh⁴

¹Department of Statistics, University of Peshawar, Peshawar, Khyber Pakhtunkhwa, Pakistan

²Department of Mathematics, College of Science, Jazan University, Jazan, Saudi Arabia

³Institute of Numerical Sciences, Kohat University of Science & Technology, Kohat, Pakistan

⁴Department of Mathematics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, Kuala Nerus 21030, Terengganu, Malaysia

Correspondence should be addressed to Wali Khan Mashwani; mashwanigr8@gmail.com

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In this paper, a new approach for deriving continuous probability distributions is developed by incorporating an extra parameter to the existing distributions. Frechet distribution is used as a submodel for an illustration to have a new continuous probability model, termed as modified Frechet (MF) distribution. Several important statistical properties such as moments, order statistics, quantile function, stress-strength parameter, mean residual life function, and mode have been derived for the proposed distribution. In order to estimate the parameters of MF distribution, the maximum likelihood estimation (MLE) method is used. To evaluate the performance of the proposed model, two real datasets are considered. Simulation studies have been carried out to investigate the performance of the parameters' estimates. The results based on the real datasets and simulation studies provide evidence of better performance of the suggested distribution.

1. Introduction

In the last few years, the literature of distribution theory has become rich due to the induction of additional parameters in the existing distribution. The inclusion of an extra parameter has shown greater flexibility compared to competitive models. The inclusion of a new parameter can be performed either using the available generator or by developing a new technique for generating new improved distribution compared to classical baseline distribution. Azzalini [1] proposed a modified form of the normal distribution by inserting an extra parameter, known as skew normal distribution, which indicated greater flexibility over normal distribution. Mudholkar and Srivastava [2] introduced exponentiated Weibull distribution by introducing a shape parameter in

two-parameter Weibull distribution. Its cumulative distribution function is as follows:

$$G(y; \alpha, \lambda, \beta) = (1 - e^{-\lambda x^\alpha})^\beta, \quad x, \alpha, \lambda, \beta > 0. \quad (1)$$

This model provides greater flexibility compared to the base line distribution. Note that, for $\beta = 1$, the exponentiated Weibull distribution and base line distribution coincide. Later on, various researchers have introduced different forms of exponentiated distributions; see, for example, the work of Gupta et al. [3]. Marshall and Olkin [4] introduced another technique to add an extra parameter to a probability distribution. Eugene et al. [5] suggested the beta-generated technique and applied this method to beta distribution and proposed beta-generated distribution by incorporating an

extra parameter in beta distribution. Alzaatreh et al. [6] proposed a new technique and produced T-X class of continuous probability models by interchanging the probability density function of beta distribution with a probability density function, $g(t)$, of a continuous random variable and used a function $W(F(x))$ which fulfills some particular conditions. Recently, Aljarrah et al. [7] introduced T-X class of distributions using quantile functions. For more details about new techniques to produce probability distributions, see the works of Lee et al. [8] and Jones [9]. Al-Aqtash et al. [10] proposed a new class of models using the logit function as a baseline and obtained the particular case referred to as Gumbel–Weibull distribution. Alzaatreh et al. [11] studied the gamma-X class of distributions and recommended the particular case using the normal distribution as a baseline distribution. Abid and Abdulrazak [12] introduced truncated Frechet-G class of distributions. Korkmaz and Genc [13] presented a generalized two-sided class of probability distributions. Alzaghal et al. [14] worked on the T-X class of distributions. Aldeni et al. [15] used the quantile function of generalized lambda distribution and introduced a new family. For more details, see the works of Cordeiro et al. [16], Alzaatreh et al. [17], and Nasir et al. [18]. The more recent modified Weibull distributions are introduced by Abid and Abdulrazak [12], Korkmaz and Genc [13], Aldeni et al. [15], Cordeiro et al. [16], and Pe and Jurek [19].

Pearson [20] used the system of differential equation technique and produced new probability distributions. Burr [21] also proposed a new method by using the differential equation method, which may take on a wide variety of forms of the continuous distributions. Since 1980, methodologies of suggesting new models moved to the inclusion of extra parameters to an existing family of distributions to increase the level of flexibility. These include Weibull-G presented by Bourguignon et al. [22], Garhy-G proposed by Elgarhy et al. [23], Kumaraswamy (Kw-G) proposed by Cordeiro and de Castro [24], Type II half-logistic-G by Hassan et al. [25], exponentiated extended-G suggested by Elgarhy et al. [26], the Kumaraswamy–Weibull introduced by Hassan and Elgarhy [27], exponentiated Weibull by Hassan and Elgarhy [28], odd Frechet-G introduced by Haq and Elgarhy [29], and Muth-G by Almarashi and Elgarhy [30]. For a short review, one can study the work of Kotz and Vicari [31]. Recently, Mahdavi and Kundu [32] developed a new technique for proposing probability distributions which is referred as alpha power transformation (APT) technique, defined by the cumulative distribution function (CDF) as follows:

$$F_{\text{APT}}(x) = \begin{cases} \frac{\alpha^{F(x)} - 1}{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1, \\ F(x), & \text{if } \alpha = 1. \end{cases} \quad (2)$$

The core purpose of introducing new family of distributions is to overcome the difficulties that are present in the existing probability distributions. In this study, we suggest a new method for obtaining new continuous probability

distributions. Frechet distribution is used as a submodel to have a new probability distribution which is referred as modified Frechet (MF) distribution. Our proposed family of distributions also models monotonic and nonmonotonic hazard rate function and provides increased flexibility as compared to the already available probability distribution in the literature.

2. The Proposed Class of Distributions

The proposed class of probability distributions is termed as modified Frechet class (MFC) of distributions. The cumulative distribution function (CDF) and probability density function (PDF) of the suggested class of distributions are given by the following expressions:

$$G_{\text{MFC}}(x) = \frac{e^{-(F(x))^\alpha} - 1}{(e^{-1} - 1)}, \quad x > 0, \quad (3)$$

$$g_{\text{MFC}}(x) = \frac{\alpha f(x) (F(x))^{\alpha-1} e^{-(F(x))^\alpha}}{(1 - e^{-1})}, \quad x > 0, \quad (4)$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the baseline distribution and α is the shape parameter. The method in (3) is used to produce a new model referred as modified Frechet (MF) distribution with the aim to attain more flexibility in modeling life time data. The derivation of MF distribution is given in Section 2.1.

2.1. The Proposed Distribution. The CDF of Frechet distribution is as follows:

$$F(x) = e^{-x^{-\beta}}, \quad x > 0, \quad (5)$$

where β is the shape parameter.

This portion of the manuscript is concerned with introducing a subclass of MF class of distributions using the cumulative distribution function of Frechet distribution. The resultant distribution is what we call modified Frechet (MF) distribution.

Definition 1. A random variable X is said to have MF distribution with two parameters α and β if its PDF is given as follows:

$$f_{\text{MF}}(x) = \frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{(1 - e^{-1})}, \quad x > 0. \quad (6)$$

Its CDF is given by

$$F_{\text{MF}}(x) = \frac{e^{-(e^{-\alpha x^{-\beta}})} - 1}{(e^{-1} - 1)}, \quad x > 0. \quad (7)$$

The hazard rate function of MF distribution is as follows:

$$h_{\text{MF}}(x) = \frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}} (e^{-1} - 1)}{(e^{-1} - e^{-(e^{-\alpha x^{-\beta}})}) (1 - e^{-1})}. \quad (8)$$

The survival function of the MF model is given by

$$S_{MF}(x) = \frac{e^{-1} - e^{-(e^{-\alpha x^{-\beta}})}}{e^{-1} - 1}. \quad (9)$$

These four functions have been plotted in Figures 1 and 2

Lemma 1. If $f(x)$ is decreasing function, then $f_{MF}(x)$ is also decreasing function for $0 \leq \alpha < 1$ and $\beta > 0$.

Proof. If $f(x)$ is a differentiable function and if $f'(x) < 0$ or $(d/dx)\ln f(x) < 0$ for all X , then $f(x)$ is a decreasing function and vice versa.

Taking first derivative of $\ln f_{MF}(x)$, we have

$$\frac{d}{dx} \ln f_{MF}(x) = \frac{d}{dx} \ln \left[\frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{(1 - e^{-1})} \right], \quad (10)$$

$$\frac{d}{dx} \ln f_{MF}(x) = -\frac{\beta}{x} - \frac{1}{x} - \alpha \beta x^{-\beta-1} (e^{-\alpha x^{-\beta}} - 1).$$

$$\frac{d^2}{dx^2} \ln f_{MF}(x) = \frac{d}{dx} \left(-\frac{\beta}{x} - \frac{1}{x} - \alpha \beta x^{-\beta-1} (e^{-\alpha x^{-\beta}} - 1) \right), \quad (11)$$

$$\frac{d^2}{dx^2} \ln f_{MF}(x) = \frac{\beta}{x^2} + \frac{1}{x^2} + \alpha \beta x^{-\beta-2} \left[(-\beta - 1) (e^{-\alpha x^{-\beta}} - 1) + \alpha \beta x^{-\beta} e^{-\alpha x^{-\beta}} \right].$$

When $0 \leq \alpha < 1$ and $\beta > 0$, then $(d^2/dx^2)\ln f_{MF}(x) > 0$. Therefore, for $0 < \alpha < 1$, $f_{MF}(x)$ is log-convex [33]. $\square$

2.1.1. Quantile Function. Let $X \sim MF(\alpha, \beta)$; then, the quantile function is as follows:

$$F(X) = U \implies X = F^{-1}(U), \quad (12)$$

where U is uniformly distributed random variable. The quantile function of the MF model is given as

$$X_p = \left[-\frac{1}{\alpha} \ln \{ -\ln(u(e^{-1} - 1) + 1) \} \right]^{(-1/\beta)}. \quad (13)$$

2.1.2. Median. Median of MF distribution is obtained by substituting $u = 1/2$ in equation (13), that is,

$$\text{Median} = \left[-\frac{1}{\alpha} \ln \left\{ -\ln \left(\frac{1}{2} (e^{-1} + 1) \right) \right\} \right]^{(-1/\beta)}. \quad (14)$$

2.1.3. Mode. Mode of MF is obtained by solving the following equation for x .

Thus, for $0 \leq \alpha < 1$ and $\beta > 0$, $(d/dx)\ln f_{MF}(x) < 0$. This concludes the lemma. $\square$

Lemma 2. For $\alpha < 1$, if $f(x)$ is log-convex and decreasing function, then $h_{MF}(x)$ is a decreasing function.

Proof. If the second-order differential of $f(x)$ exists and $(d^2/dx^2)\ln f(x) > 0$, then $f(x)$ is said to be log-convex. Taking second-order derivative of equation (10), we obtain

$$\begin{aligned} \frac{d}{dx} f_{MF}(x) &= 0, \quad \text{i.e.} \quad \frac{d}{dx} \left(\frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{(1 - e^{-1})} \right) = 0, \\ x^{-\beta} (e^{-\alpha x^{-\beta}} - 1) &= \frac{-(\beta + 1)}{\alpha \beta}. \end{aligned} \quad (15)$$

Mode of the distribution satisfies the above equation.

2.2. r^{th} Moment of MF Distribution. Let $X \sim MF(\alpha, \beta)$; and the r^{th} moment of X is as follows:

$$\mu_r = E(X^r) = \int_0^\infty x^r \frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{(1 - e^{-1})} dx. \quad (16)$$

Using $x^{-\beta} = y$ and then $e^{-\alpha y} = z$ in (16), the expression will take the following form:

$$\mu_r = E(X^r) = -\frac{(-1/\alpha)^m}{(1 - e^{-1})} \int_0^1 (\ln z)^m e^{-z} dz, \quad (17)$$

where $m = (-r/\beta)$.

Again substituting $\log z = u$ in (17) and after some simplification, the expression becomes

FIGURE 1: Graph of CDF and PDF of the MF model.

FIGURE 2: Graph of survival function and hazard function of MF distribution.

$$\mu_r / = E(X^r) = \frac{(-1/\alpha)^m}{(e^{-1} - 1)} \int_{-\infty}^0 u^m e^{-e^u} e^u dz. \quad (18)$$

Using series notation $e^{-e^u} = \sum_{r=0}^{\infty} ((-e^u)^r / r!)$ in (18), we obtain

$$\begin{aligned} \mu_r / = E(X^r) &= \frac{(-1/\alpha)^m}{(e^{-1} - 1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{b \rightarrow -\infty} -it \int_b^0 u^m e^{u(k+1)} dz, \\ \mu_r / = E(X^r) &= \frac{(-1/\alpha)^m}{(e^{-1} - 1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(m+1, -bk-b) - \Gamma(m+1, 0)}{(-k-1)^m (k+1)} \right], \end{aligned} \quad (19)$$

where $b > 0$.

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{(1 - e^{-1})} dx. \quad (20)$$

2.3. Moment Generating Function. Let $X \sim \text{MF}(\alpha, \beta)$; then, the moment generating function is given by

Using series $e^{tx} = \sum_{r=0}^{\infty} (t^r x^r / r!)$ in (20) and simplifying, we have

$$M_x(t) = \frac{(-1/\alpha)^m}{(e^{-1} - 1)} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r (-1)^k}{r! k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(m+1, -bk-b) - \Gamma(m+1, 0)}{(-k-1)^m (k+1)} \right], \quad (21)$$

where $m = (-r/\beta)$ and $b > 0$.

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) [F(x)]^{i-1} [1 - F(x)]^{(n-i)}. \quad (22)$$

2.4. Order Statistics. Let $X_1, X_2, \dots, X_n$ be a random sample taken from MF distribution, and let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ denote the order statistics. Then, the probability density function of $X_{i:n}$ is given by

Substitute PDF and CDF of MF in equation (22), and we obtain distribution of i^{th} order statistic as

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} \left(\frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{(1 - e^{-1})} \right) \left[\frac{e^{-(\alpha x^{-\beta})} - 1}{e^{-1} - 1} \right]^{i-1} \left[1 - \left(\frac{e^{-(\alpha x^{-\beta})} - 1}{e^{-1} - 1} \right) \right]^{(n-i)}, \\ f_{i:n}(x) &= \frac{n!}{(1 - e^{-1})(i-1)!(n-i)!} \frac{1}{(e^{-1} - 1)^{n-1}} \left[e^{-(\alpha x^{-\beta})} - 1 \right]^{(i-1)} \alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}} \left[e^{-1} - e^{-(\alpha x^{-\beta})} \right]^{(n-i)}. \end{aligned} \quad (23)$$

Lemma 3. The Renyi entropy of $X \sim \text{MF}(\alpha, \beta)$ is given as

$$\text{RE}_X(\nu) = \frac{1}{(1-\nu)} \log \left[\frac{\alpha^{((1-\nu)/\beta)} \beta^{\nu-1} (-1)^l}{(1 - e^{-1})^\nu} \sum_{k=0}^{\infty} \frac{(-\nu)^k}{k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(l+1, -b\nu - bk + b) - \Gamma(l+1, 0)}{(-\nu - k + 1)^l (\nu + k - 1)} \right] \right], \quad (24)$$

where $l = ((\beta+1)(\nu-1)/\beta)$ and $b > 0$.

Proof. The Renyi entropy of MF is given by

$$\text{RE}_X(\nu) = \frac{1}{1-\nu} \log \left\{ \int_{-\infty}^{+\infty} f(x)^\nu dx \right\} = \frac{1}{1-\nu} \log \left[\int_0^{\infty} \left(\frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha x^{-\beta} - e^{-\alpha x^{-\beta}}}}{(1 - e^{-1})} \right)^\nu dx \right]. \quad (25)$$

Put $e^{-\alpha x^{-\beta}} = y$ in (25); the expression will take the form

$$RE_X(\nu) = \frac{1}{(1-\nu)} \log \left[\frac{\beta^{\nu-1}}{(1-e^{-1})^\nu} \alpha^{((1-\nu)/\beta)} (-1)^l \int_0^1 (\ln y)^l y^{\nu-1} e^{-\nu y} dy \right], \quad (26)$$

where $l = ((\beta + 1)(\nu - 1)/\beta)$. Using series notation $e^{-\nu y} = \sum_{k=0}^{\infty} ((\nu)^k (-y)^k / k!)$ in (26), the expression will take the following form:

$$RE_X(\nu) = \frac{1}{(1-\nu)} \log \left[\frac{\alpha^{((1-\nu)/\beta)} \beta^{\nu-1} (-1)^l}{(1-e^{-1})^\nu} \sum_{k=0}^{\infty} \frac{(-\nu)^k}{k!} \int_0^1 (\ln y)^l y^{\nu-1+k} dy \right]. \quad (27)$$

Again substituting $\ln y = z$ in (27) and simplifying, we obtain

$$RE_X(\nu) = \frac{1}{(1-\nu)} \log \left[\frac{\alpha^{((1-\nu)/\beta)} \beta^{\nu-1} (-1)^l}{(1-e^{-1})^\nu} \sum_{k=0}^{\infty} \frac{(-\nu)^k}{k!} \lim_{b \rightarrow -\infty} -it \int_b^0 z^l e^{z(\nu-1+k)} dz \right], \quad (28)$$

$$RE_X(\nu) = \frac{1}{(1-\nu)} \log \left[\frac{\alpha^{((1-\nu)/\beta)} \beta^{\nu-1} (-1)^l}{(1-e^{-1})^\nu} \sum_{k=0}^{\infty} \frac{(-\nu)^k}{k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(l+1, -b\nu - bk + b) - \Gamma(l+1, 0)}{(-\nu - k + 1)^l (\nu + k - 1)} \right] \right],$$

where $l = ((\beta + 1)(\nu - 1)/\beta)$ and $b > 0$. $\square$

2.5. Mean Residual Life Function. Let X be the lifetime of an object having MF distribution. The mean residual life function is the average remaining life span that a component has survived until time t . The mean residual life function, say, $\mu(t)$, has the following expression:

$$\mu(t) = \frac{1}{P(X > t)} \int_t^{\infty} P(X > x) dx, \quad t \geq 0, \quad (29)$$

$$\mu(t) = \frac{1}{S(t)} \left(E(t) - \int_0^t x f(x) dx \right) - t, \quad t \geq 0.$$

Note that

$$\int_0^t x f(x) dx = \frac{1}{(1-e^{-1})} \left(\frac{-1}{\alpha} \right)^l \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(l+1, ((\alpha k + \alpha)/t^\beta)) (-\alpha/t^\beta)^l}{(k+1)((\alpha k + \alpha)/t^\beta)^l} - \frac{b^l \Gamma(l+1, (-bk - b))}{(-bk - b)^l (k+1)} \right], \quad (30)$$

$$E(t) = \frac{\alpha^{-l} (-1)^l}{(1-e^{-1})} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(l+1, -bk - b) - \Gamma(l+1, 0)}{(-k - 1)^l (k+1)} \right], \quad (31)$$

where $l = (-1/\beta)$. Put equation (9), (30), and (31) in (29), and we obtain

$$\mu(t) = \frac{(e^{-1} - 1)}{(e^{-1} - e^{-(e^{-\alpha x^{-\beta}})})} \frac{1}{(1 - e^{-1})} \left(\frac{-1}{\alpha}\right)^l \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{b \rightarrow -\infty} -it * \quad (32)$$

$$\left[\frac{\Gamma(l+1, -bk-b) - \Gamma(l+1, 0)}{(-k-1)^l (k+1)} \right] - \left[\frac{\Gamma(l+1, ((\alpha k + \alpha)/t^\beta)) (-\alpha/t^\beta)^l}{(k+1)((\alpha k + \alpha)/t^\beta)^l} - \frac{b^l \Gamma(l+1, (-bk-b))}{(-bk-b)^l (k+1)} \right] - t.$$

This is the final expression of mean residual life function.

$$R = \int_{-\infty}^{+\infty} f_1(x) F_2(x) dx. \quad (33)$$

2.6. Stress-Strength Parameter. Let X_1 and X_2 be two independently and identically distributed variables such that $X_1 \sim \text{MF}(\alpha_1, \beta)$ and $X_2 \sim \text{MF}(\alpha_2, \beta)$. Then, the stress-strength parameter is defined by

Using equation PDF and CDF of MF in the above expression, the stress-strength parameter is given as

$$R = \int_0^{\infty} \left(\frac{\alpha \beta x^{-(\beta+1)} e^{-\alpha_1 x^{-\beta}} - e^{-\alpha_1 x^{-\beta}}}{(1 - e^{-1})} \right) \left(\frac{e^{-(e^{-\alpha_2 x^{-\beta}})} - 1}{(e^{-1} - 1)} \right) dx, \quad (34)$$

$$R = \frac{1}{(e^{-1} - 1)(1 - e^{-1})} \int_0^{\infty} \alpha_1 \beta x^{-\beta-1} e^{-\alpha_1 x^{-\beta}} - e^{-\alpha_1 x^{-\beta}} e^{-(e^{-\alpha_2 x^{-\beta}})} dx - \frac{1}{(e^{-1} - 1)}.$$

Substituting $x^{-\beta} = y$ in (34), we obtain

$$R = \frac{1}{(e^{-1} - 1)(1 - e^{-1})} \int_0^{\infty} \alpha_1 e^{-\alpha_1 y} e^{-e^{-\alpha_1 y}} e^{-(e^{-\alpha_2 y})} dy - \frac{1}{(e^{-1} - 1)}. \quad (35)$$

Again putting $e^{-\alpha_1 y} = z$ in (35), it will take the following form:

$$R = \frac{1}{(e^{-1} - 1)^2} \int_0^1 e^{-z} e^{-(e^{-\alpha_2 (-\ln z/\alpha_1)})} dz - \frac{1}{(e^{-1} - 1)}. \quad (36)$$

Using series representation $e^{-z} = \sum_{k=0}^{\infty} ((-z)^k/k!)$ and $e^{-(e^{-\alpha_2 (-\ln z/\alpha_1)})} = \sum_{m=0}^{\infty} ((-e^{\alpha_2 \ln z/\alpha_1})^m/m!)$ in (36), we get the expression as follows:

$$R = \frac{1}{(e^{-1} - 1)^2} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{k!m!} \int_0^1 z^k e^{n \ln z} dz - \frac{1}{(e^{-1} - 1)}, \quad (37)$$

where $n = m(\alpha_2/\alpha_1)$. Using series representation $e^{n \ln z} = \sum_{i=0}^{\infty} ((n \ln z)^i/i!)$ in (52), after simplification, we get the following expression:

$$R = \frac{1}{(e^{-1} - 1)^2} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{k+m} n^i}{k!m!i!} \int_0^1 z^k (\ln z)^i dz - \frac{1}{(e^{-1} - 1)}. \quad (38)$$

Again substituting $\ln z = u$ in (38) and simplifying, we obtain

$$R = \frac{1}{(e^{-1} - 1)^2} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{k+m} n^i}{k!m!i!} \int_{-\infty}^0 u^i e^{u(k+1)} du - \frac{1}{(e^{-1} - 1)},$$

$$R = \frac{1}{(e^{-1} - 1)^2} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{k+m} n^i}{k!m!i!} \lim_{b \rightarrow -\infty} -it \int_b^0 u^i e^{u(k+1)} du - \frac{1}{(e^{-1} - 1)}, \quad (39)$$

$$R = \frac{1}{(e^{-1} - 1)^2} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{k+m} n^i}{k!m!i!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(i+1, -bk-b) - \Gamma(i+1)}{(-k-1)^i (k+1)} \right] - \frac{1}{(e^{-1} - 1)},$$

where $n = m(\alpha_2/\alpha_1)$ and $b > 0$.

Lemma 4. The mean waiting time, say $\bar{\mu}(t)$, of MF distribution is given by

$$\bar{\mu}(t) = t + \frac{1}{\left(e^{-(e^{-at-\beta})} - 1\right)} \left(\frac{-1}{\alpha}\right)^l \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(l+1, ((\alpha k + \alpha)/t^\beta))(-\alpha/t^\beta)^l}{(k+1)((\alpha k + \alpha)/t^\beta)^l} - \frac{b^l \Gamma(l+1, (-bk-b))}{(-bk-b)^l (k+1)} \right]. \quad (40)$$

Proof. By definition, the mean waiting time of MF distribution is

$$\bar{\mu}(t) = t - \frac{1}{F(t)} \int_0^t x f(x) dx. \quad (41)$$

The final expression of mean waiting time of MF distribution is obtained. By substituting

$$\int_0^t x f(x) dx = \frac{1}{(1 - e^{-1})} \left(\frac{-1}{\alpha}\right)^l \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(l+1, ((\alpha k + \alpha)/t^\beta))(-\alpha/t^\beta)^l}{(k+1)((\alpha k + \alpha)/t^\beta)^l} - \frac{b^l \Gamma(l+1, (-bk-b))}{(-bk-b)^l (k+1)} \right], \quad (42)$$

and $F(t) = ((e^{-(e^{-at-\beta})} - 1)/(e^{-1} - 1))$, in (41), we obtain

$$\bar{\mu}(t) = t + \frac{1}{\left(e^{-(e^{-at-\beta})} - 1\right)} \left(\frac{-1}{\alpha}\right)^l \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lim_{b \rightarrow -\infty} -it \left[\frac{\Gamma(l+1, ((\alpha k + \alpha)/t^\beta))(-\alpha/t^\beta)^l}{(k+1)((\alpha k + \alpha)/t^\beta)^l} - \frac{b^l \Gamma(l+1, (-bk-b))}{(-bk-b)^l (k+1)} \right]. \quad (43)$$

□

2.7. Parameters' Estimation. Let $X_1, X_2, \dots, X_n$ be a random sample of size n selected from $MF(\alpha, \beta)$; then, the log-likelihood function of MF distribution is given as

$$\ln l(\alpha, \beta) = n \ln(\alpha\beta) - n \ln(1 - e^{-1}) - \beta \sum_{i=1}^n \log x_i - \sum_{i=1}^n \log x_i - \alpha \sum_{i=1}^n x_i^{-\beta} - e^{-\alpha} \sum_{i=1}^n x_i^{-\beta}. \quad (44)$$

Differentiating equation (44) with respect to α and β and equating them to 0, we obtain

$$\frac{\partial \ln l(\alpha, \beta)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n x_i^{-\beta} + e^{-\alpha} \sum_{i=1}^n x_i^{-\beta} - \sum_{i=1}^n x_i^{-\beta} = 0, \quad (45)$$

$$\frac{\partial \ln l(\alpha, \beta)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^n \log x_i - \alpha \sum_{i=1}^n x_i^{-\beta} \log x_i + \alpha e^{-\alpha} \sum_{i=1}^n x_i^{-\beta} \log x_i = 0. \quad (46)$$

Solving (45) and (46) together, we get the estimates of α and β . The Newton-Raphson method or the bisection method is used to get solution of the above equations as an analytical solution which is not possible. The maximum

likelihood estimators (MLE) are asymptotically normally distributed, that is, $\sqrt{n}(\hat{\alpha} - \alpha, \hat{\beta} - \beta) \sim N_2(0, \Sigma)$, where Σ is variance covariance matrix and can be obtained by inverting the observed Fisher information matrix F given below:

$$F = \begin{pmatrix} \frac{\partial^2 \log l}{\partial \alpha^2} & \frac{\partial^2 \log l}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \log l}{\partial \alpha \partial \beta} & \frac{\partial^2 \log l}{\partial \beta^2} \end{pmatrix}. \quad (47)$$

The second derivative of equations (45) and (46) with respect to α and β yields (48)–(50) given as

$$\frac{\partial^2 \ln l}{\partial \alpha^2} = -\frac{n}{\alpha^2} - e^{-\alpha \sum_{i=1}^n x_i^{-\beta}} \left(\sum_{i=1}^n x_i^{-\beta} \right)^2, \quad (48)$$

$$\frac{\partial^2 \ln l}{\partial \beta^2} = -\frac{n}{\beta^2} - \alpha \sum_{i=1}^n x_i^{-\beta} \log x_i \left[\log x_i + \alpha e^{-\alpha \sum_{i=1}^n x_i^{-\beta}} - e^{-\alpha \sum_{i=1}^n x_i^{-\beta}} \log x_i \right], \quad (49)$$

and

$$\frac{\partial^2 \ln l}{\partial \alpha \partial \beta} = \sum_{i=1}^n x_i^{-\beta} \log x_i \left[e^{-\alpha \sum_{i=1}^n x_i^{-\beta}} - \alpha e^{-\alpha \sum_{i=1}^n x_i^{-\beta}} - 1 \right]. \quad (50)$$

Asymptotic $(1 - \zeta)100\%$ confidence intervals of the parameters of the proposed distribution can be obtained as

$$\begin{aligned} \hat{\alpha} \pm Z_{\zeta/2} \sqrt{\hat{\Sigma}_{11}}, \\ \hat{\beta} \pm Z_{\zeta/2} \sqrt{\hat{\Sigma}_{22}}, \end{aligned} \quad (51)$$

where Z_{ζ} is the upper ζ^{th} percentile of the standard normal distribution.

3. Simulations' Studies

In order to measure the performance of MLE of the parameters of MF distribution, their mean square error (MSE) and bias are calculated using simulation study. We consider $W = 100$ samples of sizes $n = 50, 70$, and 100 generated from the MF model. Bias and MSE are calculated using the expressions:

$$\begin{aligned} \text{Bias} &= \frac{1}{W} \sum_{i=1}^w (\hat{b}_i - b), \\ \text{MSE} &= \frac{1}{W} \sum_{i=1}^w (\hat{b}_i - b)^2, \quad \text{where } b = (\alpha, \beta). \end{aligned} \quad (52)$$

Simulation results have been obtained for different values of α and β . The MSEs and bias are presented in Table 1. The consistency behavior of MLE can be easily verified from these results as the MSEs and bias of the estimates decrease for all parameter combinations with increasing sample size. Hence, we conclude that MLE procedure executes very well in estimating the parameters of MF distribution.

4. Applications

Two practical datasets are used to assess the performance of MF distribution compared to Frechet distribution (FD),

exponential distribution (ED), Weibull distribution (WD), alpha power inverse Weibull distribution (APIWD) [34], alpha power Weibull distribution (APWD) [35], and Kumaraswamy inverse Weibull distribution (KIWD) [36].

4.1. Dataset 1. The performance of the suggested model is assessed using two datasets. The first dataset is taken from the work of Gross and Clark [37] which consists of 20 observations of patients receiving an analgesic and is given as follows:

$$1.1 \quad 1.4 \quad 1.3 \quad 1.7 \quad 1.9 \quad 1.8 \quad 1.6 \quad 2.2 \quad 1.7 \quad 2.7 \quad 4.1 \quad 1.8 \quad 1.5 \quad 1.2 \quad 1.4 \quad 3.0 \quad 1.7 \quad 2.3 \quad 1.6 \quad 2.0. \quad (53)$$

4.2. Dataset 2. The second dataset consists of 40 wind-related catastrophes used by Hogg and klugman [38]. It

includes claims of \$2,000,000. The sorted values, observed in millions, are as follows:

TABLE 1: MSE and bias of MLE.

Parameters	N	MSE ($\hat{\alpha}$)	MSE ($\hat{\beta}$)	Bias ($\hat{\alpha}$)	Bias ($\hat{\beta}$)
$\alpha = 5.557197$ $\beta = 7.558804$	50	2.067219	0.9272915	0.5047117	0.2639728
	70	0.9708544	0.7031859	0.3088091	0.2101214
	100	0.7753223	0.372835	0.2578149	0.1197992
$\alpha = 6.457197$ $\beta = 3.558804$	50	2.026492	0.2259845	0.4415551	0.1637077
	70	1.550174	0.1505096	0.4327197	0.1085847
	100	0.8022496	0.1029645	0.1910671	0.03691258
$\alpha = 5.457197$ $\beta = 3.558804$	50	1.190428	0.2059653	0.4247818	2.342903
	70	0.8619716	0.1373483	0.2461313	2.341802
	100	0.4419579	0.08574337	0.04261996	2.333286
$\alpha = 6.457197$ $\beta = 2.558804$	50	2.518832	0.09526091	0.7185321	0.1320379
	70	1.728531	0.09429495	0.4747864	0.07025322
	100	0.8006676	0.04351014	0.1573213	0.01418112
$\alpha = 4.557197$ $\beta = 4.558804$	50	0.9564051	0.2857103	0.2557071	0.1324175
	70	0.3593258	0.2089236	0.08780059	0.04002267
	100	0.2750382	0.14968	0.02739781	0.01301724
$\alpha = 6.557197$ $\beta = 1.558804$	50	3.244032	0.04392251	0.5185741	0.03279788
	70	1.495698	0.02470381	0.3520218	0.02578572
	100	0.9790736	0.01922724	0.09188152	0.02093735
$\alpha = 5.557197$ $\beta = 7.558804$	50	2.067219	0.9272915	0.5047117	0.2639728
	70	0.9708544	0.7031859	0.3088091	0.2101214
	100	0.7753223	0.372835	0.2578149	0.1197992
$\alpha = 2.557197$ $\beta = 7.558804$	50	0.1693338	1.094077	0.1063671	0.3148875
	70	0.08930482	0.4333755	0.08538295	0.1227512
	100	0.06259663	0.3701105	0.02511625	0.03947369

TABLE 2: Goodness of fit results for dataset 1.

Distribution	MLE of the parameters			AIC	CAIC	BIC	HQIC	K-S	P value
MF	6.4571	3.5588		34.88	35.59	36.87	35.27	0.107	0.9759
FD	2.2255			59.16	59.39	60.16	59.36	0.473	0.0003
ED	0.5263			67.67	67.89	68.66	67.86	0.439	0.0009
WD	2.7843	2.1271		45.17	45.87	47.16	45.56	0.183	0.5104
APIWD	1.7688	4.1692	5.4473	36.79	38.29	39.77	37.37	0.124	0.9644
APWD	10.9388	2.0312	0.4230	46.58	48.08	49.57	47.16	0.162	0.6678
KIWD	1.5668	1.2318	3.7669	38.80	41.47	42.78	39.58	0.134	0.9540

The bold values indicate that the proposed distribution is more significant as compared to other existing distributions.

TABLE 3: Goodness of fit results for dataset 2.

Distribution	MLE of the parameters			AIC	CAIC	BIC	HQIC	K-S	P value
MF	6.1151	1.2558		234.65	234.99	237.98	235.85	0.190	0.1194
FD	0.7779			280.98	281.09	282.64	281.58	0.558	$5.61e - 11$
ED	0.1130			250.03	250.15	251.70	250.64	0.202	0.08195
WD	1.0013	8.8585		252.04	252.37	255.36	253.23	0.2017	0.0836
APIWD	2.0394	1.4913	4.9249	237.33	238.01	242.32	239.12	0.2011	0.1049
APWD	10.7622	0.69493	0.4071	255.18	255.86	260.17	256.97	0.2201	0.0458

The bold values indicate that the proposed distribution is more significant as compared to other existing distributions.

$$2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 3 \ 3 \ 3 \ 4 \ 4 \ 4 \ 5 \ 5 \ 5 \ 6 \ 6 \ 6 \ 6 \ 8 \ 8 \ 9 \ 15 \ 17 \ 22 \ 23 \ 24 \ 24 \ 25 \ 27 \ 32 \ 43. \quad (54)$$

In order to compare the MF model with other models, some standard model selection criteria such as Akaike's information criteria (AIC), consistent Akaike's information criteria (CAIC), Bayesian information criterion

(BIC), Hannan-Quinn information criteria (HQIC), Kolmogorov-Smirnov (K-S), and P value are used. Tables 2 and 3 demonstrate results based on dataset 1 and 2, respectively.

FIGURE 3: Plots of MF distribution for dataset 1.

FIGURE 4: Plots of MF distribution for dataset 2.

It is evident from the results in Tables 2 and 3 that the proposed MF distribution executes well as compared to other competitive distributions.

Figures 3 and 4 represent various graphs of MF distribution for dataset 1 and dataset 2.

5. Conclusion

In this paper, a new method for deriving new continuous probability distributions has been offered which we called modified Frechet Class (MFC) of distributions. Also, a new probability model has been proposed using MFC. We called it modified Frechet (MF) distribution. Several statistical properties of the said distribution were derived and investigated for MF distribution. The MLE method was adopted to estimate the parameters of the proposed distribution. Simulation results showed that these estimates were consistent. In order to check the performance of the AFF model, two real datasets. The results based on these datasets revealed promising performance of the suggested model compared to some other distributions existing in the literature.

Data Availability

The data used in this paper are freely available upon citing this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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