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Research Article

On r-Generalized Fuzzy ℓ-Closed Sets: Properties and Applications

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In the present study, we introduce and characterize the class of r-generalized fuzzy ℓ-closed sets in a fuzzy ideal topological space (X, τ, ℓ) in Šostak sense. Also, we show that r-generalized fuzzy closed set by Kim and Park (2002) $\Longrightarrow$ r-generalized fuzzy ℓ-closed set, but the converse need not be true. Moreover, if we take $\ell = \ell_0$, the r-generalized fuzzy ℓ-closed set and r-generalized fuzzy closed set are equivalent. After that, we define fuzzy upper (lower) generalized ℓ-continuous multifunctions, and some properties of these multifunctions along with their mutual relationships are studied with the help of examples. Finally, some separation axioms of r-generalized fuzzy ℓ-closed sets are introduced and studied. Also, the notion of r-fuzzy G^*-connected sets is defined and studied with help of r-generalized fuzzy ℓ-closed sets.

1. Introduction

Zadeh [1] introduced the basic idea of a fuzzy set as an extension of classical set theory. The theory of fuzzy sets provides a framework for mathematical modeling of those real-world situations, which involve an element of imprecision, uncertainty, or vagueness in their description. This theory has found wide applications in engineering, information sciences, etc.; for details, the reader is referred to [2, 3].

A fuzzy multifunction is a fuzzy set valued function [4–7]. The biggest difference between fuzzy functions and fuzzy multifunctions has to do with the definition of an inverse image. For a fuzzy multifunction, there are two types of inverses, upper and lower. These two definitions of the inverse then lead to two definitions of continuity; for more details, the reader is referred to [8, 9].

In this work, we introduce and characterize the class of r-generalized fuzzy ℓ-closed sets in a fuzzy ideal topological space in Šostak sense. Also, we show that r-generalized fuzzy closed set $\Longrightarrow$ r-generalized fuzzy ℓ-closed set, but the converse need not be true. Moreover, if we take $\ell = \ell_0$, the r-generalized fuzzy ℓ-closed set and r-generalized fuzzy closed set are equivalent. After that, we define fuzzy upper (lower) generalized ℓ-continuous multifunctions, and some properties of these multifunctions along with their mutual relationships are discussed with the help of examples. In the end, some separation axioms of r-generalized fuzzy ℓ-closed sets are introduced and studied. Also, the notion of r-fuzzy G^*-connected sets is defined and studied with help of r-generalized fuzzy ℓ-closed sets.

2. Preliminary Assertions

In this section, we present the basic definitions and results which we need in the next sections. Throughout this work, X refers to an initial universe, I^X is the set of all fuzzy sets on X, and for $\mu \in I^X$, $\mu^c(a) = 1 - \mu(a)$ for all $a \in X$ (where $I_\circ = (0,1]$ and $I = [0,1]$). For $s \in I$, $\underline{s}(a) = s$ for all $a \in X$. The difference between $\mu, \lambda \in I^X$ [9] is defined as follows:

$$\mu \overline{\wedge} \lambda = \begin{cases} \underline{0}, & \text{if } \mu \leq \lambda, \\ \mu \wedge \lambda^c, & \text{otherwise.} \end{cases} \quad (1)$$

A fuzzy ideal ℓ on Y [10] is a map ℓ: $I^Y \longrightarrow I$ that satisfies the following:

(i) $\forall \mu, \lambda \in I^Y$ and $\mu \leq \lambda \Longrightarrow \ell(\lambda) \leq \ell(\mu)$
(ii) $\forall \mu, \lambda \in I^Y \Longrightarrow \ell(\mu \vee \lambda) \geq \ell(\mu) \wedge \ell(\lambda)$

The simplest fuzzy ideals on Y, ℓ_0, and ℓ_1 are defined as follows:

$$\ell_0(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \\ 0, & \text{otherwise} \end{cases} \text{ and } \ell_1(\mu) = 1 \quad \forall \mu \in I^Y. \tag{2}$$

In a fuzzy topological space (X, τ) based on the sense of Šostak [11], the closure and the interior of $\mu \in I^X$ are denoted by $C_\tau(\mu, r)$ and $I_\tau(\mu, r)$. A fuzzy set $\lambda \in I^X$ is called a r-generalized fuzzy closed set [12] if $C_\tau(\lambda, r) \le \mu$ whenever $\lambda \le \mu$ and $\tau(\mu) \ge r$. Also, (X, τ) is called r-GF-regular [12] iff $x_t \overline{q} \lambda$ for each r-generalized fuzzy closed set $\lambda \in I^X$ implies that there exist $\mu_i \in I^X$ with $\tau(\mu_i) \ge r$ for $i \in \{1, 2\}$ such that $x_t \in \mu_1$, $\lambda \le \mu_2$, and $\mu_1 \overline{q} \mu_2$. Moreover, (X, τ) is called r-GF-normal [12] iff $\lambda_1 \overline{q} \lambda_2$ for each of the r-generalized fuzzy closed sets $\lambda_i \in I^X$ for $i \in \{1, 2\}$ implies that there exist $\mu_i \in I^X$ with $\tau(\mu_i) \ge r$ such that $\lambda_i \le \mu_i$ and $\mu_1 \overline{q} \mu_2$.

Definition 1 (see [9]). Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$, and $r \in {}_{I^\circ}$. Then, the r-fuzzy local function λ_r^* of λ is defined as follows: $\lambda_r^* = \wedge\{\mu \in I^X\colon \ell(\lambda \overline{\wedge} \mu) \ge r, \tau(\mu^c) \ge r\}$.

Remark 1 (see [9]).

(i) If we take $\ell = \ell_0$, for each $\lambda \in I^X$, we have

$$\lambda_r^* = \wedge\left\{\mu \in I^X\colon \lambda \le \mu, \tau(\mu^c) \ge r\right\} = C_\tau(\lambda, r). \tag{3}$$

(ii) If we take $\ell = \ell_1$ (resp., $\ell(\lambda) \ge r$), for each $\lambda \in I^X$, we have $\lambda_r^* = \underline{0}$.

Definition 2 (see [9]). Let (X, τ, ℓ) be a fuzzy ideal topological space. Then, for each $\lambda \in I^X$ and $r \in {}_{I^\circ}$, we define an operator $C_\tau^*\colon I^X \times {}_{I^\circ} \longrightarrow I^X$ as follows:

$$C_\tau^*(\lambda, r) = \lambda \vee \lambda_r^*. \tag{4}$$

Now, if $\ell = \ell_0$, then $C_\tau^*(\lambda, r) = \lambda \vee \lambda_r^* = \lambda \vee C_\tau(\lambda, r) = C_\tau(\lambda, r)$ for each $\lambda \in I^X$.

Theorem 1 (see [9]). *Let (X, τ, ℓ) be a fuzzy ideal topological space. Then, for any fuzzy sets λ, $\nu \in I^X$ and $r \in {}_{I^\circ}$, the operator $C_\tau^*\colon I^X \times {}_{I^\circ} \longrightarrow I^X$ satisfies the following properties:*

(i) $C_\tau^*(\underline{0}, r) = \underline{0}$

(ii) $\lambda \le C_\tau^*(\lambda, r) \le C_\tau(\lambda, r)$

(iii) If $\lambda \le \nu$, *then* $C_\tau^*(\lambda, r) \le C_\tau^*(\nu, r)$

(iv) $C_\tau^*(\lambda \vee \nu, r) = C_\tau^*(\lambda, r) \vee C_\tau^*(\nu, r)$

(v) $C_\tau^*(\lambda \wedge \nu, r) \le C_\tau^*(\lambda, r) \wedge C_\tau^*(\nu, r)$

(vi) $C_\tau^*(C_\tau^*(\lambda, r), r) = C_\tau^*(\lambda, r)$

Definition 3 (see [13, 14]). Let (X, τ, ℓ) be a fuzzy ideal topological space, $\lambda \in I^X$, and $r \in {}_{I^\circ}$. Then, λ is said to be

(i) r-fuzzy ℓ-open iff $\lambda \le I_\tau(\lambda_r^*, r)$

(ii) r-fuzzy semi-ℓ-open iff $\lambda \le C_\tau^*(I_\tau(\lambda, r), r)$

(iii) r-fuzzy pre-ℓ-open iff $\lambda \le I_\tau(C_\tau^*(\lambda, r), r)$

(iv) r-fuzzy α-ℓ-open iff $\lambda \le I_\tau(C_\tau^*(I_\tau(\lambda, r), r), r)$

(v) r-fuzzy β-ℓ-open iff $\lambda \le C_\tau^*(I_\tau(C_\tau^*(\lambda, r), r), r)$

(vi) r-fuzzy δ-ℓ-open iff $I_\tau(C_\tau^*(\lambda, r), r) \le C_\tau^*(I_\tau(\lambda, r), r)$

The complement of r-fuzzy ℓ-open (resp., semi-ℓ-open, pre-ℓ-open, α-ℓ-open, β-ℓ-open, and δ-ℓ-open) set is a r-fuzzy ℓ-closed (resp., semi-ℓ-closed, pre-ℓ-closed, α-ℓ-closed, β-ℓ-closed, and δ-ℓ-closed) set.

Definition 4 (see [15, 16]). Let $H\colon X \multimap Y$. Then, H is called a fuzzy multifunction iff $H(x) \in I^Y$ for each $x \in X$. The degree of membership of y in $H(x)$ is denoted by $H(x)(y) = G_H(x, y)$ for any $(x, y) \in X \times Y$. The domain of H is denoted by $\mathrm{dom}(H)$ and the range of H is denoted by $\mathrm{rng}(H)$, for any $x \in X$ and $y \in Y$: $\mathrm{dom}(H)(x) = \vee_{y \in Y} G_H(x, y)$ and $\mathrm{rng}(H)(y) = \vee_{x \in X} G_H(x, y)$.

Definition 5 (see [15]). Let $H\colon X \multimap Y$ be a fuzzy multifunction. Then, H is called

(i) Normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_H(x, y_0) = 1$.

(ii) A crisp iff $G_H(x, y) = 1$ for each $x \in X$ and $y \in Y$.

Definition 6 (see [15]). Let $H\colon X \multimap Y$ be a fuzzy multifunction. Then, we have the following:

(i) The image of $\lambda \in I^X$ is a fuzzy set $H(\lambda) \in I^Y$ and defined by

$$H(\lambda)(y) = \bigvee_{x \in X} [G_H(x, y) \wedge \lambda(x)]. \tag{5}$$

(ii) The lower inverse of $\mu \in I^Y$ is a fuzzy set $H^l(\mu) \in I^X$ and defined by

$$H^l(\mu)(x) = \bigvee_{y \in Y} [G_H(x, y) \wedge \mu(y)]. \tag{6}$$

(iii) The upper inverse of $\mu \in I^Y$ is a fuzzy set $H^u(\mu) \in I^X$ and defined by

$$H^u(\mu)(x) = \bigwedge_{y \in Y} [G_H^c(x, y) \vee \mu(y)]. \tag{7}$$

All properties of image, lower and upper, are found in [15].

3. On r-Generalized Fuzzy ℓ-Closed Sets

In this section, we introduce and characterize the class of r-generalized fuzzy ℓ-closed sets in a fuzzy ideal topological space (X, τ, ℓ) in Šostak sense. Also, we show that r-generalized fuzzy closed set $\Longrightarrow$ r-generalized fuzzy ℓ-closed set, but the converse need not be true. Moreover, if we take $\ell = \ell_0$, the r-generalized fuzzy ℓ-closed set and r-generalized fuzzy closed set are equivalent. Our fundamental definition is the following.

Definition 7. Let $\mu, \lambda \in I^X$ and $r \in {}_{I^\circ}$. In (X, τ, ℓ), μ is called a r-generalized fuzzy ℓ-closed set if $C_\tau^*(\mu, r) \le \lambda$ whenever

$\mu \leq \lambda$ and $\tau(\lambda) \geq r$. The complement of r-generalized fuzzy ℓ-closed set is called r-generalized fuzzy ℓ-open.

Lemma 1. *Every r-generalized fuzzy closed set [12] is r-generalized fuzzy ℓ-closed.*

Proof. It follows from Theorem 1 (2).

In general, the converse of Lemma 1 is not true as shown by Example 1. □

Example 1. Define $\tau, \ell: I^X \longrightarrow I$ as follows:

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \mu = \underline{0.4}, \\ 0, & \text{otherwise,} \end{cases}$$
$$\ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{3}, & \text{if } \underline{0} < \nu \leq \underline{0.4}, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Then, $\underline{0.35}$ is 1/3-generalized fuzzy ℓ-closed set but it is not 1/3-generalized fuzzy closed.

Lemma 2. *If $\ell = \ell_0$, the r-generalized fuzzy ℓ-closed set and r-generalized fuzzy closed set are equivalent.*

Proof. It follows from Definition 2.

The following implications are obtained:

$$\begin{gathered} r - \text{fuzzy closed} \\ \Downarrow \\ r - \text{generalized fuzzy closed} \\ \Downarrow \\ r - \text{generalized fuzzy } \ell - \text{closed.} \end{gathered} \quad (9)$$

□

Remark 2. r-generalized fuzzy ℓ-closed and r-fuzzy α-ℓ-closed [13] are independent notions as shown by Examples 2 and 3.

Example 2. Let $X = \{a, b, c, d\}$ be a set and $\lambda_1, \lambda_2 \in I^X$ be defined as follows: $\lambda_1 = \{a/0.9, b/0.9, c/0.4, d/0.5\}$ and $\lambda_2 = \{a/0.1, b/0.1, c/0.1, d/0.5\}$. Define $\tau, \ell: I^X \longrightarrow I$ as follows:

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if } \mu = \lambda_1, \\ 0, & \text{otherwise,} \end{cases}$$
$$\ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{2}{3}, & \text{if } \underline{0} < \nu < \underline{0.3}, \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

Then, λ_2 is a 2/3-fuzzy α-ℓ-closed set, but it is not 2/3-generalized fuzzy ℓ-closed.

Example 3. Define $\tau, \ell: I^X \longrightarrow I$ as follows:

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \mu = \underline{0.2}, \\ \frac{2}{3}, & \text{if } \mu = \underline{0.8}, \\ 0, & \text{otherwise,} \end{cases}$$
$$\ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{2}{3}, & \text{if } \underline{0} < \nu < \underline{0.4}, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Then, $\underline{0.5}$ is a 1/3-generalized fuzzy ℓ-closed set, but it is not 1/3-fuzzy α-ℓ-closed.

Remark 3. r-generalized fuzzy ℓ-closed and r-fuzzy δ-ℓ-closed [14] are independent notions as shown by Examples 4 and 5.

Example 4. Let $X = \{a, b, c, d\}$ be a set and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ be defined as follows: $\lambda_1 = \{a/1.0, b/0.0, c/0.0, d/0.0\}$, $\lambda_2 = \{a/0.0, b/1.0, c/0.0, d/0.0\}$, $\lambda_3 = \{a/0.0, b/0.0, c/1.0,$

$d/0.0\}$, and $\lambda_4 = \{a/0.0, b/0.0, c/0.0, d/1.0\}$. Define $\tau, \ell\colon I^X \longrightarrow I$ as follows:

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \mu = \lambda_4, \\ \frac{2}{3}, & \text{if } \mu = \lambda_1 \vee \lambda_2 \vee \lambda_3, \\ 0, & \text{otherwise,} \end{cases}$$
$$\ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{3}, & \text{if } \nu = \lambda_3, \\ \frac{2}{3}, & \text{if } \underline{0} < \nu < \lambda_3, \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

Then, $(\lambda_3 \vee \lambda_4)^c$ is a 1/3-fuzzy δ-ℓ-closed set, but it is not 1/3-generalized fuzzy ℓ-closed.

Example 5. Let $X = \{a, b, c, d\}$ be a set and $\lambda_1, \lambda_2, \lambda_3 \in I^X$ be defined as follows: $\lambda_1 = \{a/1.0, b/0.0, c/0.0, d/0.0\}$, $\lambda_2 = \{a/0.0, b/1.0, c/0.0, d/0.0\}$, and $\lambda_3 = \{a/0.0, b/0.0, c/1.0, d/0.0\}$. Define $\tau, \ell\colon I^X \longrightarrow I$ as follows:

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if } \mu = \lambda_1 \vee \lambda_2, \\ 0, & \text{otherwise,} \end{cases}$$
$$\ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ 0, & \text{otherwise.} \end{cases} \quad (13)$$

Then, $(\lambda_1 \vee \lambda_3)^c$ is a 2/3-generalized fuzzy ℓ-closed set, but it is not 2/3-fuzzy δ-ℓ-closed.

Definition 8. In (X, τ, ℓ), for any $\mu \in I^X$ and $r \in {}_{I^\circ}$, we define $GC_\tau^*\colon I^X \times {}_{I^\circ} \longrightarrow I^X$ as follows: $GC_\tau^*(\mu, r) = \wedge\{\lambda \in I^X\colon \mu \leq \lambda, \quad \lambda \text{ is } r - \text{generalized fuzzy } \ell - \text{closed}\}$.

Theorem 2. *In (X, τ, ℓ), for any $\mu, \nu \in I^X$, the operator $GC_\tau^*\colon I^X \times {}_{I^\circ} \longrightarrow I^X$ satisfies the following:*

(i) $GC_\tau^*(\underline{0}, r) = \underline{0}$
(ii) $\mu \leq GC_\tau^*(\mu, r)$
(iii) If $\mu \leq \nu$, then $GC_\tau^*(\mu, r) \leq GC_\tau^*(\nu, r)$
(iv) $GC_\tau^*(GC_\tau^*(\mu, r), r) = GC_\tau^*(\mu, r)$
(v) $GC_\tau^*(\mu, r) \vee GC_\tau^*(\nu, r) \leq GC_\tau^*(\mu \vee \nu, r)$
(vi) If μ is r-generalized fuzzy ℓ-closed, then $GC_\tau^*(\mu, r) = \mu$

Proof. (i), (ii), (iii), and (vi) are easily proved from the definition of GC_τ^*.

(iv) From (ii) and (iii), $GC_\tau^*(\mu, r) \leq GC_\tau^*(GC_\tau^*(\mu, r), r)$. Now, we show that $GC_\tau^*(\mu, r) \geq GC_\tau^*(GC_\tau^*(\mu, r), r)$. Suppose that $GC_\tau^*(\mu, r) \ngeq GC_\tau^*(GC_\tau^*(\mu, r), r)$. There exist $x \in X$ and $t \in (0, 1)$ such that

$$GC_\tau^*(\mu, r)(x) < t < GC_\tau^*(GC_\tau^*(\mu, r), r)(x). \ (A). \quad (14)$$

Since $GC_\tau^*(\mu, r)(x) < t$, by the definition of GC_τ^*, there exists r − generalized fuzzy ℓ − closed λ_1 with $\mu \leq \lambda_1$ such that $GC_\tau^*(\mu, r)(x) \leq \lambda_1(x) < t$. Since $\mu \leq \lambda_1$, we have $GC_\tau^*(\mu, r) \leq \lambda_1$. Again, by the definition of GC_τ^*, we have $GC_\tau^*(GC_\tau^*(\mu, r), r) \leq \lambda_1$. Hence, $GC_\tau^*(GC_\tau^*(\mu, r), r)(x) \leq \lambda_1(x) < t$; it is a contradiction for (A). Thus, $GC_\tau^*(\mu, r) \geq GC_\tau^*(GC_\tau^*(\mu, r), r)$. Then, $GC_\tau^*(GC_\tau^*(\mu, r), r) = GC_\tau^*(\mu, r)$.

(v) However, μ and $\nu \leq \mu \vee \nu$ imply $GC_\tau^*(\mu, r) \leq GC_\tau^*(\mu \vee \nu, r)$ and $GC_\tau^*(\nu, r) \leq GC_\tau^*(\mu \vee \nu, r)$. Thus, $GC_\tau^*(\mu, r) \vee GC_\tau^*(\nu, r) \leq GC_\tau^*(\mu \vee \nu, r)$. Hence, the proof is complete.

In a similar way, one can prove the following theorem. □

Theorem 3. *In (X, τ, ℓ), for any $\mu \in I^X$ and $r \in {}_{I^\circ}$, we define $GI_\tau^*\colon I^X \times {}_{I^\circ} \longrightarrow I^X$ as follows: $GI_\tau^*(\mu, r) = \vee\{\lambda \in I^X\colon \lambda \leq \mu, \quad \lambda \text{ is } r - \text{generalized fuzzy } \ell - \text{open}\}$. For each μ, $\nu \in I^X$, the operator GI_τ^* satisfies the following:*

(i) $GI_\tau^*(\underline{1}, r) = \underline{1}$
(ii) $GI_\tau^*(\mu, r) \leq \mu$
(iii) $GI_\tau^*(\mu^c, r) = (GC_\tau^*(\mu, r))^c$
(iv) $GI_\tau^*(GI_\tau^*(\mu, r), r) = GI_\tau^*(\mu, r)$
(v) $GI_\tau^*(\mu, r) \wedge GI_\tau^*(\nu, r) \geq GI_\tau^*(\mu \wedge \nu, r)$
(vi) If μ is r-generalized fuzzy ℓ-open, then $GI_\tau^*(\mu, r) = \mu$

4. Fuzzy Upper and Lower Generalized ℓ-Continuous Multifunctions

In this section, we define fuzzy upper (resp., lower) generalized ℓ-continuous multifunctions, and some properties of these multifunctions along with their mutual relationships are discussed with the help of examples. Moreover, we show that the notions of fuzzy upper (resp., lower) semi-ℓ-continuous multifunctions [17] and fuzzy upper (resp., lower) generalized ℓ-continuous multifunctions are independent.

Definition 9. A fuzzy multifunction $H\colon (X, \tau, \ell) \multimap (Y, \eta)$ is called

(i) Fuzzy upper generalized ℓ-continuous at a fuzzy point $x_t \in \text{dom}(H)$ iff $x_t \in H^u(\mu)$ for each $\mu \in I^Y$ with $\eta(\mu) \geq r$, there exists a r-generalized fuzzy

ℓ-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \wedge \text{dom}(H) \leq H^u(\mu)$

(ii) Fuzzy lower generalized ℓ-continuous at a fuzzy point $x_t \in \text{dom}(H)$ iff $x_t \in H^l(\mu)$ for each $\mu \in I^Y$ with $\eta(\mu) \geq r$, there exists a r-generalized fuzzy ℓ-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq H^l(\mu)$

(iii) Fuzzy upper (resp., lower) generalized ℓ-continuous iff it is fuzzy upper (resp., lower) generalized ℓ-continuous at every $x_t \in \text{dom}(H)$

Remark 4. If H is normalized, then H is fuzzy upper generalized ℓ-continuous at a fuzzy point $x_t \in \text{dom}(H)$ iff $x_t \in H^u(\mu)$ for each $\mu \in I^Y$ with $\eta(\mu) \geq r$, there exists r-generalized fuzzy ℓ-open set $\lambda \in I^X$ and $x_t \in \lambda$ such that $\lambda \leq H^u(\mu)$.

Theorem 4. *A fuzzy multifunction H: $(X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy lower generalized ℓ-continuous if $H^l(\mu)$ (resp., $H^u(\mu)$) is r-generalized fuzzy ℓ-open (resp., r-generalized fuzzy ℓ-closed) for each $\mu \in I^Y$ with $\eta(\mu) \geq r$ (resp., $\mu \in I^Y$ with $\eta(\mu^c) \geq r$), $r \in I_\circ$.*

Proof. Let $x_t \in \text{dom}(H)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$, and $x_t \in H^l(\mu)$. Then, we have $H^l(\mu)$ as r-generalized fuzzy ℓ-open. Thus, H is fuzzy lower generalized ℓ-continuous. The other case follows similar lines and from $H^u(\mu^c) = (H^l(\mu))^c$.

In a similar way, one can prove the following theorem. □

Theorem 5. *A normalized fuzzy multifunction H: $(X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy upper generalized ℓ-continuous if $H^u(\mu)$ (resp., $H^l(\mu)$) is a r-generalized fuzzy ℓ-open (resp., r-generalized fuzzy ℓ-closed) set for each $\mu \in I^Y$ with $\eta(\mu) \geq r$ (resp., $\mu \in I^Y$ with $\eta(\mu^c) \geq r$) and $r \in I_\circ$.*

Lemma 3. *Every fuzzy lower (resp., upper) generalized continuous multifunction [18] is a fuzzy lower (resp., upper) generalized ℓ-continuous multifunction.*

Proof. It follows from Lemma 1.

In general, the converse of Lemma 3 is not true as shown by Example 6. □

Example 6. Let $X = \{a_1, a_2\}$, $Y = \{b_1, b_2, b_3\}$, and H: $X \multimap Y$ be a fuzzy multifunction defined by $G_H(a_1, b_1) = 0.2, G_H(a_1, b_2) = 0.8, G_H(a_1, b_3) = 0.3, G_H(a_2, b_1) = 0.5, G_H(a_2, b_2) = 0.3$, and $G_H(a_2, b_3) = 0.7$. Define τ, ℓ: $I^X \longrightarrow I$ and η: $I^Y \longrightarrow I$ as follows:

$$\tau(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \mu = \underline{0.4}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if } \lambda = \underline{0.65}, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

$$\ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{3}, & \text{if } \underline{0} < \nu \leq \underline{0.4}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, H: $(X, \tau, \ell) \multimap (Y, \eta)$ is fuzzy lower (resp., upper) generalized ℓ-continuous but not fuzzy lower (resp., upper) generalized continuous.

Lemma 4. *Let H: $(X, \tau, \ell) \multimap (Y, \eta)$ be a fuzzy multifunction (resp., normalized fuzzy multifunction). If $\ell = \ell_0$, fuzzy lower (resp., upper) generalized ℓ-continuous multifunction and fuzzy lower (resp., upper) generalized continuous multifunction are equivalent.*

Proof. It follows from Lemma 2. □

Remark 5. Fuzzy lower (resp., upper) semi-ℓ-continuous multifunctions [17] and fuzzy lower (resp., upper) generalized ℓ-continuous multifunctions are independent notions as shown by Examples 7 and 8.

Example 7. Let $X = \{a_1, a_2\}$, $Y = \{b_1, b_2, b_3\}$, and H: $X \multimap Y$ be a fuzzy multifunction defined by $G_H(a_1, b_1) = 0.1, G_H(a_1, b_2) = 1.0, G_H(a_1, b_3) = 0.3$, $G_H(a_2, b_1) = 0.5, G_H(a_2, b_2) = 0.1$, and $G_H(a_2, b_3) = 1.0$. Define τ, ℓ: $I^X \longrightarrow I$ and η: $I^Y \longrightarrow I$ as follows:

$$\tau(\mu)=\begin{cases}1, & \text{if } \mu\in\{\underline{0},\underline{1}\},\\ \frac{1}{3}, & \text{if } \mu=\underline{0.2},\\ \frac{2}{3}, & \text{if } \mu=\underline{0.8},\\ 0, & \text{otherwise,}\end{cases}$$

$$\ell(\nu)=\begin{cases}1, & \text{if } \nu=\underline{0},\\ \frac{1}{3}, & \text{if } \underline{0}<\nu<\underline{0.4},\\ 0, & \text{otherwise,}\end{cases} \tag{16}$$

$$\eta(\lambda)=\begin{cases}1, & \text{if } \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{3}, & \text{if } \lambda=\underline{0.5},\\ 0, & \text{otherwise.}\end{cases}$$

Then, $H\colon (X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy lower (resp., upper) generalized ℓ-continuous, but it is not fuzzy lower (resp., upper) semi-ℓ-continuous.

Example 8. Let $X=\{a_1,a_2,a_3\}$, $Y=\{b_1,b_2,b_3\}$, and $H\colon X\multimap Y$ be a fuzzy multifunction defined by $G_H(a_1,b_1)=0.8, G_H(a_1,b_2)=0.3, G_H(a_1,b_3)=0.3, G_H(a_2,b_1)=0.6, G_H(a_2,b_2)=0.1, G_H(a_2,b_3)=0.4, G_H(a_3,b_1)=0.1, G_H(a_3,b_2)=0.2$, and $G_H(a_3,b_3)=1.0$. Define $\lambda_1,\lambda_2\in I^X$ and $\lambda_3\in I^Y$ as follows: $\lambda_1=\{a_1/0.3,a_2/0.4,a_3/0.8\}$, $\lambda_2=\{a_1/0.2,a_2/0.3,a_3/0.2\}$, and $\lambda_3=\{b_1/0.3,b_2/0.4,b_3/0.8\}$. Define $\tau,\ell\colon I^X\longrightarrow I$ and $\eta\colon I^Y\longrightarrow I$ as follows:

$$\tau(\mu)=\begin{cases}1, & \text{if } \mu\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if } \mu=\lambda_2,\\ 0, & \text{otherwise.}\end{cases}$$

$$\ell(\nu)=\begin{cases}1, & \text{if } \nu=\underline{0},\\ \frac{2}{3}, & \text{if } \underline{0}<\nu<\underline{0.2},\\ 0, & \text{otherwise.}\end{cases} \tag{17}$$

$$\eta(\lambda)=\begin{cases}1, & \text{if } \lambda\in\{\underline{0},\underline{1}\},\\ \frac{3}{4}, & \text{if } \lambda=\lambda_3,\\ 0, & \text{otherwise.}\end{cases}$$

Then, $H\colon (X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy lower (resp., upper) semi-ℓ-continuous, but it is not fuzzy lower (resp., upper) generalized ℓ-continuous.

Theorem 6. *Let (X,τ,ℓ) be a fuzzy ideal topological space that satisfies the condition: for $\lambda\in I^X$, $\mu\in I^Y$, and $r\in I_\circ$, (G^*) If $\lambda=GI^*_\tau(\lambda,r)$, it implies that λ is r-generalized fuzzy ℓ-open.*

Then, $H\colon (X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy lower generalized ℓ-continuous if one of the following statements is hold:

(i) $H(\mathrm{GC}^*_\tau(\lambda,r))\le C_\eta(H(\lambda),r)$

(ii) $\mathrm{GC}^*_\tau(H^u(\mu),r)\le H^u(C_\eta(\mu,r))$

(iii) $H^l(I_\eta(\mu,r))\le \mathrm{GI}^*_\tau(H^l(\mu),r)$

Proof.

(i) $\Longrightarrow$ (ii) For each $\mu\in I^Y$ and $r\in I_\circ$, let $\lambda=H^u(\mu)$. Then, by (i), $H(\mathrm{GC}^*_\tau(H^u(\mu),r))\le C_\eta(H(H^u(\mu)),r)\le C_\eta(\mu,r)$. Hence, we have $\mathrm{GC}^*_\tau(H^u(\mu),r)\le H^u(C_\eta(\mu,r))$.

(ii) $\Longrightarrow$ (iii) Let $\mu\in I^Y$. Then, by (ii), $[\mathrm{GI}^*_\tau(H^l(\mu),r)]^c=\mathrm{GC}^*_\tau(H^u(\mu^c),r)\le H^u(C_\eta(\mu^c,r))=[H^l(I_\eta(\mu,r))]^c$. Hence, $H^l(I_\eta(\mu,r))\le \mathrm{GI}^*_\tau(H^l(\mu),r)$.

Suppose that (iii) holds. Let $\mu\in I^Y$ with $\eta(\mu)\ge r$. Then, by (iii), $H^l(\mu)\le \mathrm{GI}^*_\tau(H^l(\mu),r)$, and by (G^*), $H^l(\mu)$ is r-generalized fuzzy ℓ-open. Thus, H is fuzzy lower generalized ℓ-continuous. Hence, the proof is complete. □

Corollary 1. *Let $H\colon (X,\tau,\ell)\multimap(Y,\eta)$ and $F\colon (Y,\eta)\multimap(Z,\gamma)$ be two fuzzy multifunctions. Then, we have the following:*

(i) If H is fuzzy lower generalized ℓ-continuous and F is fuzzy lower semicontinuous [12], then $F^\circ H$ is fuzzy lower generalized ℓ-continuous.

(ii) If H is normalized fuzzy upper generalized ℓ-continuous and F is normalized fuzzy upper semicontinuous [15], then $F^\circ H$ is fuzzy upper generalized ℓ-continuous.

5. Some Applications of r-Generalized Fuzzy ℓ-Closed Sets

In this section, some separation axioms of r-generalized fuzzy ℓ-closed sets are introduced and studied. Also, the notion of r-fuzzy G^*-connected sets is defined and studied with help of r-generalized fuzzy ℓ-closed sets.

Definition 10.

(i) (X,τ,ℓ) is called r-GF*-regular iff $x_t\overline{q}\lambda$ for each r-generalized fuzzy ℓ-closed set, $\lambda\in I^X$ implies that there exist $\mu_i\in I^X$ with $\tau(\mu_i)\ge r$ for $i\in\{1,2\}$ such that $x_t\in\mu_1$, $\lambda\le\mu_2$, and $\mu_1\overline{q}\mu_2$.

(ii) (X,τ,ℓ) is called r-GF*-normal iff $\lambda_1\overline{q}\lambda_2$ for each r-generalized fuzzy ℓ-closed sets, $\lambda_i\in I^X$ for $i\in\{1,2\}$ imply that there exist $\mu_i\in I^X$ with $\tau(\mu_i)\ge r$ such that $\lambda_i\le\mu_i$ and $\mu_1\overline{q}\mu_2$.

Lemma 5. *Every r-GF-regular (resp., r-GF-normal) space (X, τ, ℓ) [12] is r-GF*-regular (resp., r-GF*-normal) space.*

Proof. It follows from Lemma 1. □

Lemma 6. *If $\ell = \ell_0$, r-GF*-regular (resp., r-GF*-normal) space and r-GF-regular (resp., r-GF-normal) space are equivalent.*

Proof. It follows from Lemma 2. □

Theorem 7. *For any (X, τ, ℓ), the following are equivalent:*

(i) (X, τ, ℓ) is r-GF*-regular

(ii) If $x_t \in \lambda$ for each r-generalized fuzzy ℓ-open set $\lambda \in I^X$, there exists $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $x_t \in \mu \leq C_\tau(\mu, r) \leq \lambda$

(iii) If $x_t \overline{q} \lambda$ for each r-generalized fuzzy ℓ-closed set $\lambda \in I^X$, there exist $\mu_i \in I^X$ with $\tau(\mu_i) \geq r$ for $i \in \{1, 2\}$ such that $x_t \in \mu_1$, $\lambda \leq \mu_2$, and $C_\tau(\mu_1, r) \overline{q} C_\tau(\mu_2, r)$

Proof.

(i) ($\Longrightarrow$) (ii) Let $x_t \in \lambda$ for each r-generalized fuzzy ℓ-open λ. Then, $x_t \overline{q} \lambda^c$ for r-generalized fuzzy ℓ-closed λ^c. Since (X, τ) is r-GF*-regular, there exist $\mu, \nu \in I^X$ with $\tau(\mu) \geq r$ and $\tau(\nu) \geq r$ such that $x_t \in \mu$, $\lambda^c \leq \nu$, and $\mu \overline{q} \nu$. It implies that $x_t \in \mu \leq \nu^c \leq \lambda$. Since $\tau(\nu) \geq r$, $x_t \in \mu \leq C_\tau(\mu, r) \leq \lambda$.

(ii) ($\Longrightarrow$) (iii) Let $x_t \overline{q} \lambda$ for each r-generalized fuzzy ℓ-closed λ. Then, $x_t \in \lambda^c$ for r-generalized fuzzy ℓ-open λ^c. By (ii), there exists $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $x_t \in \mu \leq C_\tau(\mu, r) \leq \lambda^c$. Since $\tau(\mu) \geq r$, μ is r-generalized fuzzy ℓ-open and $x_t \in \mu$. Again, by (ii), there exists $\mu_1 \in I^X$ with $\tau(\mu_1) \geq r$ such that

$$x_t \in \mu_1 \leq C_\tau(\mu_1, r) \leq \mu \leq C_\tau(\mu, r) \leq \lambda^c. \quad (18)$$

It implies $\lambda \leq [(C_\tau(\mu, r))^c = I_\tau(\mu^c, r)] \leq \mu^c$. Put $\mu_2 = I_\tau(\mu^c, r)$; then, $\tau(\mu_2) \geq r$. So, $C_\tau(\mu_2, r) \leq \mu^c \leq (C_\tau(\mu_1, r))^c$, that is, $C_\tau(\mu_1, r) \overline{q} C_\tau(\mu_2, r)$.

(iii) ($\Longrightarrow$) (i) It is trivial.

In a similar way, one can prove the following theorem. □

Theorem 8. *For any (X, τ, ℓ), the following are equivalent:*

(i) (X, τ, ℓ) is r-GF*-normal.

(ii) If $\nu \leq \lambda$ for each r-generalized fuzzy ℓ-closed $\nu \in I^X$ and r-generalized fuzzy ℓ-open $\lambda \in I^X$, there exists $\mu \in I^X$ with $\tau(\mu) \geq r$ such that $\nu \leq \mu \leq C_\tau(\mu, r) \leq \lambda$

(iii) If $\lambda_1 \overline{q} \lambda_2$ for each r-generalized fuzzy ℓ-closed sets $\lambda_i \in I^X$ for $i \in \{1, 2\}$, there exist $\mu_i \in I^X$ with $\tau(\mu_i) \geq r$ such that $\lambda_i \leq \mu_i$ and $C_\tau(\mu_1, r) \overline{q} C_\tau(\mu_2, r)$

Definition 11. In (X, τ, ℓ), μ and $\lambda \in I^X$ are called r-fuzzy G^*-separated iff $\mu \overline{q} \mathrm{GC}^*_\tau(\lambda, r)$ and $\lambda \overline{q} \mathrm{GC}^*_\tau(\mu, r)$. Also, any fuzzy set which cannot be expressed as the union of two r-fuzzy G^*-separated sets is called r-fuzzy G^*-connected set.

Theorem 9. *For any (X, τ, ℓ), we have the following:*

(i) If $\mu, \lambda \in I^X$ are r-fuzzy G^*-separated and $\omega, \nu \in I^X - \{\underline{0}\}$ such that $\nu \leq \mu$ and $\omega \leq \lambda$, then ω and ν are r-fuzzy G^*-separated

(ii) If $\mu \overline{q} \lambda$ and either both are r-generalized fuzzy ℓ-closed or both are r-generalized fuzzy ℓ-open, then μ and λ are r-fuzzy G^*-separated

(iii) If μ and λ are either both r-generalized fuzzy ℓ-closed or both r-generalized fuzzy ℓ-open, then $\mu \wedge \lambda^c$ and $\lambda \wedge \mu^c$ are r-fuzzy G^*-separated

Proof. (i) and (ii) are obvious.

(iii) Let μ and λ be r-generalized fuzzy ℓ-open. Since $\mu \wedge \lambda^c \leq \lambda^c$, $\mathrm{GC}^*_\tau(\mu \wedge \lambda^c, r) \leq \lambda^c$, and hence, $\mathrm{GC}^*_\tau(\mu \wedge \lambda^c, r) \overline{q} \lambda$. Then, $\mathrm{GC}^*_\tau(\mu \wedge \lambda^c, r) \overline{q} (\lambda \wedge \mu^c)$. Again, since $\lambda \wedge \mu^c \leq \mu^c$, $\mathrm{GC}^*_\tau(\lambda \wedge \mu^c, r) \leq \mu^c$, and hence, $\mathrm{GC}^*_\tau(\lambda \wedge \mu^c, r) \overline{q} \mu$. Then, $\mathrm{GC}^*_\tau(\lambda \wedge \mu^c, r) \overline{q} (\mu \wedge \lambda^c)$. Thus, $\mu \wedge \lambda^c$ and $\lambda \wedge \mu^c$ are r-fuzzy G^*-separated. The other case follows similar lines. □

Theorem 10. *For any (X, τ, ℓ), μ and $\lambda \in I^X - \{\underline{0}\}$ are r-fuzzy G^*-separated iff there exist two r-generalized fuzzy ℓ-open sets ω and ν such that $\lambda \leq \omega$, $\mu \leq \nu$, $\lambda \overline{q} \nu$, and $\mu \overline{q} \omega$.*

Proof. ($\Longrightarrow$) Let μ and $\lambda \in I^X - \{\underline{0}\}$ be r-fuzzy G^*-separated, $\lambda \leq (\mathrm{GC}^*_\tau(\mu, r))^c = \omega$, and $\mu \leq (\mathrm{GC}^*_\tau(\lambda, r))^c = \nu$, where ν and ω are r-generalized fuzzy ℓ-open; then, $\nu \overline{q} \mathrm{GC}^*_\tau(\lambda, r)$ and $\omega \overline{q} \mathrm{GC}^*_\tau(\mu, r)$. Thus, $\mu \overline{q}\, \omega$ and $\lambda \overline{q}\, \nu$. Hence, we obtain the desired result.

($\Longleftarrow$) Let ω and ν be r-generalized fuzzy ℓ-open sets such that $\mu \leq \nu$, $\lambda \leq \omega$, $\mu \overline{q}\, \omega$, and $\lambda \overline{q}\, \nu$. Then, $\mu \leq \omega^c$ and $\lambda \leq \nu^c$. Hence, $\mathrm{GC}^*_\tau(\mu, r) \leq \omega^c$ and $\mathrm{GC}^*_\tau(\lambda, r) \leq \nu^c$. Then, $\mathrm{GC}^*_\tau(\mu, r) \overline{q} \lambda$ and $\mathrm{GC}^*_\tau(\lambda, r) \overline{q} \mu$. Thus, μ and λ are r-fuzzy G^*-separated. Hence, we obtain the desired result. □

Theorem 11. *For any (X, τ, ℓ), if $\mu \in I^X - \{\underline{0}\}$ is a r-fuzzy G^*-connected set such that $\mu \leq \lambda \leq GC^*_\tau(\mu, r)$, then λ is r-fuzzy G^*-connected.*

Proof. Let λ be not r-fuzzy G^*-connected. Then, there exist r-fuzzy G^*-separated sets ν_1 and $\nu_2 \in I^X$ such that $\lambda = \nu_1 \vee \nu_2$. Let $\omega = \mu \wedge \nu_2$, $\nu = \mu \wedge \nu_1$, and $\mu = \nu \vee \omega$. Since $\omega \leq \nu_2$ and $\nu \leq \nu_1$, by Theorem 9 (i), ω and ν are r-fuzzy G^*-separated, and it is a contradiction. Thus, λ is r-fuzzy G^*-connected, as required. □

6. Conclusion

In this article, we have continued to study the fuzzy sets on a fuzzy ideal topological space (X, τ, ℓ) in Šostak sense. First, we defined the class of r-generalized fuzzy ℓ-closed sets, and some basic properties are given. Moreover, we show that r-generalized fuzzy closed set $\Longrightarrow$ r-generalized fuzzy

ℓ-closed set, but the converse need not be true. Also, if we take $\ell = \ell_0$, r-generalized fuzzy ℓ-closed set and r-generalized fuzzy closed set are equivalent. After that, we define fuzzy upper (lower) generalized ℓ-continuous multifunctions, and some properties of these multifunctions along with their mutual relationships are discussed with the help of examples. In the end, some separation axioms of r-generalized fuzzy ℓ-closed sets are introduced and studied. Also, the notion of r-fuzzy G^*-connected sets is defined and studied with help of r-generalized fuzzy ℓ-closed sets. We hope that the concepts initiated herein will find their applications in many fields soon.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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