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Research Article

New Weighted Hermite–Hadamard Type Inequalities for Differentiable h-Convex and Quasi h-Convex Mappings

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In this paper, new weighted Hermite–Hadamard type inequalities for differentiable h-convex and quasi h-convex functions are proved. These results generalize many results proved in earlier works for these classes of functions. Applications of some of our results to $\check{\mathfrak{s}}$-divergence and to statistics are given.

1. Introduction

The theory of convex functions is based on convex functions stated as follows.

A function $p\colon \varnothing \neq \mathfrak{K} \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ is said to be convex on a convex set $\mathfrak{K}$ if the inequality given as follows:

$$p(e\xi_1 + (1-e)\xi_2) \leq e p(\xi_1) + (1-e)p(\xi_2), \tag{1}$$

holds for all $\xi_1, \xi_2 \in \mathfrak{K}$ and $e \in [0,1]$. If (1) holds in reverse direction, then p is said to be concave.

The inequality which can be considered as the necessary and sufficient condition of a function $p\colon \varnothing \neq \mathfrak{K} \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ to be convex on $[n_1, n_2]$ is given by [1]

$$p\left(\frac{n_1+n_2}{2}\right) \leq \frac{1}{n_2-n_1}\int_{n_1}^{n_2} p(\xi_1)\mathrm{d}\xi_1 \leq \frac{p(n_1)+p(n_2)}{2}, \tag{2}$$

where $n_1, n_2 \in \mathfrak{K}$ with $n_1 < n_2$.

Inequality (2) is known as Hermite–Hadamard inequality, and it holds in reversed direction if the function p is concave on $[n_1, n_2]$.

Over the past three decades, the definition of convex functions and inequality (2) has been subjected to immense research. The definition of convex functions has been modified in various forms, and hence a number of different weighted and nonweighted forms of inequality (2) have been obtained by many researchers.

Kirmachi [2] obtained the following estimate for $|p(n_1+n_2/2) - (1/n_2-n_1)\int_{n_1}^{n_2} p(\xi_1)\mathrm{d}\xi_1|$.

Theorem 1 (see [2]). *Let $p\colon \mathfrak{K}\subseteq\mathfrak{R} \longrightarrow \mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$; let $n_1, n_2 \in \mathfrak{K}^\circ$ with $n_1 < n_2$ and $p \in L([n_1, n_2])$. If $|p'|$ is convex on $[n_1, n_2]$, then*

$$\left|p\left(\frac{n_1+n_2}{2}\right) - \frac{1}{n_2-n_1}\int_{n_1}^{n_2} p(\xi_1)\mathrm{d}\xi_1\right| \leq \frac{(n_2-n_1)\left(\left|p'(n_1)\right| + \left|p'(n_2)\right|\right)}{8}. \tag{3}$$

Pearce and Pecaric [3] improved this estimate by proving the following result.

Theorem 2 (see [3]). *Let p: $\mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$, let $n_1, n_2 \in \mathfrak{K}^\circ$ with $n_1 < n_2$, and let $q \geq 1$. If $p \in L([n_1, n_2])$ and $|p'|^q$ is convex on $[n_1, n_2]$, then*

$$\left|p\left(\frac{n_1+n_2}{2}\right)-\frac{1}{n_2-n_1}\int_{n_1}^{n_2}p(\xi_1)\mathrm{d}\xi_1\right|\leq\frac{(n_2-n_1)}{4}\left[\frac{|p'(n_1)|^q+|p'(n_2)|^q}{2}\right]^{(1/q)}. \quad (4)$$

The weighted version of the results in Theorems 1 and 2 was obtained in [4].

Theorem 3 (see [4]). *Let p: $\mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$, let $n_1, n_2 \in \mathfrak{K}^\circ$ with $n_1 < n_2$, and let g: $[n_1, n_2]\longrightarrow[0,\infty)$ be a continuous positive mapping symmetric with respect to $(n_1 + n_2/2)$. If $pg \in L([n_1, n_2])$ and $|p'|$ is convex on $[n_1, n_2]$, then*

$$\left|p\left(\frac{n_1+n_2}{2}\right)\int_{n_1}^{n_2}g(\xi_1)\mathrm{d}\xi_1-\int_{n_1}^{n_2}p(\xi_1)g(\xi_1)\mathrm{d}\xi_1\right|$$
$$\leq\frac{(n_2-n_1)\left(|p'(n_1)|+|p'(n_2)|\right)}{2}\int_0^1 M(g;n_1,n_2,e)\mathrm{d}e, \quad (5)$$

where $M(g;n_1,n_2,e)=\int_{n_1}^{\xi_{n_1,n_2}(e)}g(\xi_1)d\xi_1$ *and* $\xi_{n_1,n_2}(e)=((1+e)/2)n_1+((1-e)/2)n_2$.

Theorem 4 (see [4]). *Let p: $\mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$, let $n_1, n_2 \in \mathfrak{K}^\circ$ with $n_1 < n_2$, and let g: $[n_1, n_2]\longrightarrow[0,\infty)$ be a continuous positive mapping symmetric with respect to $(n_1 + n_2/2)$. If $pg \in L([n_1, n_2])$ and $|p'|^q$ is convex on $[n_1, n_2]$ for $q \geq 1$, then*

$$\left|p\left(\frac{n_1+n_2}{2}\right)\int_{n_1}^{n_2}g(\xi_1)\mathrm{d}\xi_1-\int_{n_1}^{n_2}p(\xi_1)g(\xi_1)\mathrm{d}\xi_1\right|$$
$$\leq(n_2-n_1)\left[\frac{|p'(n_1)|^q+|p'(n_2)|^q}{2}\right]^{(1/q)}\int_0^1 M(g;n_1,n_2,e)\mathrm{d}e, \quad (6)$$

where $M(g;n_1,n_2,e)$ and $\xi_{n_1,n_2}(e)$ are as defined in Theorem 3.

Under the assumptions of Theorem 2, a bound of $|((p(n_1)+p(n_2))/2)-(1/(n_2-n_1))\int_{n_1}^{n_2}p(\xi_1)\mathrm{d}\xi_1|$ was proposed by Pearce and Pecaric in [3].

Theorem 5 (see [3]). *Let p: $\mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$, let $n_1, n_2 \in \mathfrak{K}^\circ$ with $n_1 < n_2$, and let $q \geq 1$. If $p \in L([n_1, n_2])$ and $|p'|^q$ is convex on $[n_1, n_2]$, then*

$$\left|\frac{p(n_1)+p(n_2)}{2}-\frac{1}{n_2-n_1}\int_{n_1}^{n_2}p(\xi_1)\mathrm{d}\xi_1\right|\leq\frac{(n_2-n_1)}{4}\left[\frac{|p'(n_1)|^q+|p'(n_2)|^q}{2}\right]^{(1/q)}. \quad (7)$$

The bound of the result of Theorem 5 in weighed form was given by Hwang in [5].

Theorem 6 (see [5]). *Let p: $\mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$, let $n_1, n_2 \in \mathfrak{K}^\circ$ with $n_1 < n_2$, and let g: $[n_1, n_2]\longrightarrow[0,\infty)$ be a continuous positive mapping symmetric with respect to $(n_1 + n_2/2)$. If $pg \in L([n_1, n_2])$ and $|p'|^q$ is convex on $[n_1, n_2]$ for $q \geq 1$, then*

$$\left|\frac{p(n_1)+p(n_2)}{2}\int_{n_1}^{n_2}g(\xi_1)\mathrm{d}\xi_1-\int_{n_1}^{n_2}p(\xi_1)g(\xi_1)\mathrm{d}\xi_1\right|$$
$$\leq\frac{(n_2-n_1)}{2}\left[\frac{|p'(n_1)|^q+|p'(n_2)|^q}{2}\right]^{(1/q)}\int_0^1\int_{\xi_{n_1,n_2}(e)}^{\eta_{n_1,n_2}(e)}g(\xi_1)\mathrm{d}\xi_1, \quad (8)$$

where $\xi_{n_1,n_2}(e)=((1+e)/2)n_1+((1-e)/2)n_2$ *and* $\eta_{n_1,n_2}(e)=((1-e)/2)n_1+((1+e)/2)n_2$.

The concept of quasiconvex functions generalizes the concept of convex functions.

Definition 1 (see [6]). A function p: $\varnothing\neq\mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ is said quasiconvex on $\mathfrak{K}$ if

$$p(e\xi_1+(1-e)\xi_2)\leq\max\{p(\xi_1),p(\xi_2)\}, \quad (9)$$

holds for all $\xi_1, \xi_2 \in \mathfrak{K}$ and $e \in [0,1]$.

There are quasiconvex functions which are not convex functions (see, for example, [6]).

Alomari et al. [7] obtained the bound of the result of Theorem 5 by using the quasiconvexity of the differentiable mappings.

Theorem 7 (see [7]). *Let p: $\mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$, and let $n_1, n_2 \in \mathfrak{K}^\circ$ with $n_1 < n_2$. If $p \in L([n_1, n_2])$ and $|p'|$ is quasiconvex on $[n_1, n_2]$, then*

$$\left|\frac{p(n_1)+p(n_2)}{2}-\frac{1}{n_2-n_1}\int_{n_1}^{n_2}p(\xi_1)\mathrm{d}\xi_1\right|$$
$$\leq\frac{(n_2-n_1)}{8}\left[\sup\left\{|p'(n_1)|,\left|p'\left(\frac{n_1+n_2}{2}\right)\right|\right\}+\sup\left\{\left|p'\left(\frac{n_1+n_2}{2}\right)\right|,|p'(n_2)|\right\}\right]. \tag{10}$$

A general form of the result of Theorem 7 has been proved by Hwang in [5].

Theorem 8 (see [5]). *Under the assumptions of ieorem 6, if $|p'|$ is quasiconvex on $[n_1,n_2]$, then*

$$\left|\frac{p(n_1)+p(n_2)}{2}\int_{n_1}^{n_2}g(\xi_1)\mathrm{d}\xi_1-\int_{n_1}^{n_2}p(\xi_1)g(\xi_1)\mathrm{d}\xi_1\right|$$
$$\leq\frac{(n_2-n_1)}{4}\left[\sup\left\{|p'(n_1)|,\left|p'\left(\frac{n_1+n_2}{2}\right)\right|\right\}+\sup\left\{\left|p'\left(\frac{n_1+n_2}{2}\right)\right|,|p'(n_2)|\right\}\right]\int_0^1\int_{\xi_{n_1,n_2}(e)}^{\eta_{n_1,n_2}(e)}g(\xi_1)\mathrm{d}\xi_1, \tag{11}$$

where $\xi_{n_1,n_2}(e)$ and $\eta_{n_1,n_2}(e)$ are as defined in Theorem 6.

Gavrea [8] extended inequality (10) to weighted form and generalized inequalities (5) and (6) in such a way that the weight function $g(\xi_1)$ is not necessarily symmetric with respect to the midpoint $((n_1+n_2)/2)$.

Varošanec [9] generalized the concept of convex functions by giving the concept of h-convex functions.

Definition 2. Let $\mathfrak{K}$ and J be intervals in $\mathfrak{R}$ with $(0,1)\supseteq J$ and h: $J\longrightarrow\mathfrak{R}$ be a nonnegative function, where $h\not\equiv 0$. A p: $\mathfrak{K}\longrightarrow\mathfrak{R}$ is an h-convex function or that p belongs to the class $\mathcal{SX}(h,\mathfrak{K})$ if p is nonnegative, and for all $\xi_1,\xi_2\in\mathfrak{K}$, $e\in(0,1)$, the inequality

$$p(e\xi_1+(1-e)\xi_2)\leq h(e)p(\xi_1)+h(1-e)p(\xi_2), \tag{12}$$

holds. If inequality (12) is reversed, then p is said to be h-concave or p is said to belong to the class $\mathcal{SV}(h,\mathfrak{K})$.

The class $\mathcal{SX}(h,\mathfrak{K})$ of h-convex functions contains all nonnegative convex functions, s-convex functions in the second sense [10], Godunova–Levin functions [11], s -Godunova–Levin type, **tgs**-convex, and P-functions [12] as special cases.

Inspired by the research towards this direction, the main objectives of this paper are to introduce the notion of quasi h-convex functions and to acquire new weighted Hermite–Hadamard type inequalities for h -convex and quasi h-convex mappings. The results of this paper generalize the results of Gavrea [8] and in particular contain the results for all nonnegative convex functions, s-convex functions, Godunova–Levin functions, s-Godunova–Levin functions, **tgs**-convex, quasi convex functions, and P-functions.

In Section 2, we recall some integral identities for a differentiable mapping and a symmetric function with respect to $((n_1+n_2)/2)$ defined over an interval $[n_1,n_2]$. In Section 2, an important inequality for positive linear functional on $C([n_1,n_2])$ and an h-convex function is proved to obtain some very stimulating results of this manuscript. Section 3 contains some new weighted Hermite–Hadamard type integral inequalities related with the left and right parts of Hermite–Hadamard inequalities (2). The results of Section 3 provide weighted generalization of a number of results proved so far in the field of mathematical inequalities for differentiable h-convex and quasi h-convex functions [13–22].

2. Some Auxiliary Results

The following notations and results have been used in [8].

Let r: $[n_1,n_2]\longrightarrow[0,\infty)$ be a continuous function with

$$\int_{n_1}^{n_2}r(\xi_1)\mathrm{d}\xi_1=1, \tag{13}$$

and the integral $\int_{n_1}^{n_2}\xi_1 r(\xi_1)\mathrm{d}\xi_1$ is denoted by α, that is,

$$\alpha=\int_{n_1}^{n_2}\xi_1 r(\xi_1)\mathrm{d}\xi_1. \tag{14}$$

In case, when $r(\xi_1)$: $[n_1,n_2]\longrightarrow[0,\infty)$ is symmetric with respect to $((n_1+n_2)/2)$, that is, if

$$r(n_1+n_2-\xi_1)=r(\xi_1), \tag{15}$$

then the following result holds.

Lemma 1 (see [8]). *If $r(\xi_1)$: $[n_1,n_2]\longrightarrow[0,\infty)$ is symmetric with respect to $((n_1+n_2)/2)$, then*

$$\alpha=\frac{n_1+n_2}{2}. \tag{16}$$

Now, we introduce the notion of the quasi h-convex functions as follows.

Definition 3. Let $\mathfrak{K}$ and J be intervals in $\mathfrak{R}$ with $(0,1)\supseteq J$ and h: $J\longrightarrow\mathfrak{R}$ be a nonnegative function, where $h\not\equiv 0$. A p: $\mathfrak{K}\longrightarrow\mathfrak{R}$ is an quasi h-convex function, or that p belongs to the class $\mathrm{SQ}(h,\mathfrak{K})$ if p is nonnegative, and for all $\xi_1,\xi_2\in\mathfrak{K}$, $e\in(0,1)$, the inequality

$$p(e\xi_1+(1-e)\xi_2)\leq\sup\{\lambda p(\xi_1),\mu p(\xi_2)\}, \tag{17}$$

holds, where $\lambda = \sup_{e\in(0,1)} h(e)$ and $\mu = \sup_{e\in(0,1)} h(1-e)$. If inequality (17) is reversed, then p is said to be quasi h-concave or p is said to belong to the class $SQ'(h, \mathfrak{K})$.

Example 1. Consider the function $p\colon [-2,2] \longrightarrow \mathfrak{R}$ defined as

$$p(e) = \begin{cases} 1, & e \in [-2,-1], \\ e^2, & e \in (-1,2], \end{cases}$$
$$h_{n_2}(e) = e^{n_2}, \quad n_2 > 0, 0 < e < 1. \tag{18}$$

Then, p is quasi h_{n_2}-convex but not h_{n_2}-convex on $[-2,2]$.

Now onwards, we suppose that $\kappa_{n_1,n_2}(\xi_1) = ((n_2-\xi_1)/(n_2-n_1))$ and $\eta_{n_1,n_2}(\xi_2) = ((1-\xi_2)/2)n_1 + ((1+\xi_2)/2)n_2$.

Lemma 2. *Let $p\colon \mathfrak{K}\subseteq\mathfrak{R} \longrightarrow \mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$ and $p' \in L([n_1,n_2])$, where $[n_1,n_2]\subseteq\mathfrak{K}^\circ$. Let $r\colon [n_1,n_2] \longrightarrow [0,\infty)$ be a continuous mapping and $h\colon J\supseteq(0,1) \longrightarrow \mathfrak{R}$ be a real nonnegative function, such that $h \not\equiv 0$. Then,*

$$\begin{aligned} &p(n_1)\int_{n_1}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)n_2\xi_1 + p(n_2)\int_{n_1}^{n_2} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 \\ &\quad - \int_{n_1}^{n_2}\Big[h\big(\kappa_{n_1,n_2}(\xi_1)\big) + h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)\Big]p(\xi_1)r(\xi_1)\mathrm{d}\xi_1 \\ &= (\alpha-n_1)\int_0^1 \mathcal{J}(r,n_1,\alpha;\xi_2)p'(\xi_2 n_1 + (1-\xi_2)\alpha)\mathrm{d}\xi_2 \\ &\quad + (n_2-\alpha)\int_0^1 \mathcal{J}(r,\alpha,n_2;\xi_2)p'((1-\xi_2)\alpha + \xi_2 n_2)\mathrm{d}\xi_2, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mathcal{J}(r,\alpha,\beta;\xi_2) = &\int_{n_1}^{(1-\xi_2)\alpha+\xi_2\beta} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 \\ &- \int_{(1-\xi_2)\alpha+\xi_2\beta}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1, \end{aligned} \tag{20}$$

$\alpha, \beta \in [n_1,n_2]$.

Proof. The following identities hold:

$$p(\xi_1) - p(n_1) = \int_{n_1}^{n_2} \sigma(\xi_1 - e)p'(e)\mathrm{d}e, \tag{21}$$

$$p(\xi_1) - p(n_2) = -\int_{n_1}^{n_2} \sigma(e - \xi_1)p'(e)\mathrm{d}e, \tag{22}$$

where $\sigma(\cdot)$ is the Heavyside function defined by

$$\sigma(\xi_1) = \begin{cases} 0, & \xi_1 < 0, \\ 1, & \xi_1 > 0. \end{cases} \tag{23}$$

Multiplying both sides of (21) with $h(\kappa_{n_1,n_2}(\xi_1))r(\xi_1)$ and integrating over $[n_1,n_2]$, we have

$$\begin{aligned} &\int_{n_1}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)p(\xi_1)\mathrm{d}\xi_1 - p(n_1)\int_{n_1}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 \\ &= \int_{n_1}^{n_2}\left(\int_e^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1\right)p'(e)\mathrm{d}e. \end{aligned} \tag{24}$$

Similarly, multiplying both sides of (22) with $h(1-\kappa_{n_1,n_2}(\xi_1))r(\xi_1)$ and integrating over $[n_1,n_2]$, we also have

$$\begin{aligned} &\int_{n_1}^{n_2} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)p(\xi_1)\mathrm{d}\xi_1 - p(n_2)\int_{n_1}^{n_2} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 \\ &= -\int_{n_1}^{n_2}\left(\int_{n_1}^{e} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1\right)p'(e)\mathrm{d}e. \end{aligned} \tag{25}$$

From (24) and (25), we get

$$\begin{aligned} &p(n_1)\int_{n_1}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 + p(n_2)\int_{n_1}^{n_2} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 \\ &\quad - \int_{n_1}^{n_2}\big[h\big(\kappa_{n_1,n_2}(\xi_1)\big) + h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)\big]p(\xi_1)r(\xi_1)\mathrm{d}\xi_1 \\ &= \int_{n_1}^{n_2}\left[\int_{n_1}^{e} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 - \int_{e}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1\right]p'(e)\mathrm{d}e \\ &= \int_{n_1}^{\alpha}\left[\int_{n_1}^{e} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 - \int_{e}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)n_2\xi_1\right]p'(e)\mathrm{d}e \\ &\quad + \int_{\alpha}^{n_2}\left[\int_{n_1}^{e} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\mathrm{d}\xi_1 - \int_{e}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)r(\xi_1)\right]p'(e)\mathrm{d}e. \end{aligned} \tag{26}$$

In the last identity, we set $e = (1-\xi_2)n_1 + \xi_2\alpha$ for the first integral and $e = (1-\xi_2)\alpha + \xi_2 n_2$ for the second integral, and we obtain (19). □

Remark 1. If we take $h(e) = e$ in Lemma 2, then we get the result for nonnegative convex functions similar to that of (see page 94 in Lemma 2.2. in [8]).

Corollary 1. *If we take $r(\xi_1) = (1/(n_2-n_1))$, for all $\xi_1 \in [n_1, n_2]$, then (19) reduces to*

$$\begin{aligned} &\frac{p(n_1)+p(n_2)}{n_2-n_1}\int_{n_1}^{n_2} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)\mathrm{d}\xi_1 \\ &\quad - \frac{1}{n_2-n_1}\int_{n_1}^{n_2}\big[h\big(\kappa_{n_1,n_2}(\xi_1)\big) + h\big(1-\kappa_{n_1,n_2}(\xi_1)\big)\big]p(\xi_1)\mathrm{d}\xi_1 = \left(\frac{n_2-n_1}{2}\right) \\ &\quad \times \int_0^1 \mathcal{J}_1(n_1,n_2;\xi_2)\big[p'\big(\eta_{n_1,n_2}(\xi_2)\big) - p'\big(\eta_{n_1,n_2}(n_1+n_2-\xi_2)\big)\big]\mathrm{d}\xi_2, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \mathcal{J}_1(n_1,n_2;\xi_2) = \frac{1}{n_2-n_1}\Bigg[&\int_{n_1}^{n_1+n_2-\eta_{n_1,n_2}(\xi_2)} h\big(1-\kappa_{n_1,n_2}(\xi_1)\big) \\ &\mathrm{d}\xi_1 - \int_{n_1+n_2-\eta_{n_1,n_2}(\xi_2)}^{n_2} h\big(\kappa_{n_1,n_2}(\xi_1)\big)\mathrm{d}\xi_1\Bigg]. \end{aligned} \tag{28}$$

Proof. We know that

$$\begin{aligned} \alpha &= \frac{1}{n_2-n_1}\int_{n_1}^{n_2}\xi_1\mathrm{d}\xi_1 \\ &= \frac{n_1+n_2}{2}. \end{aligned} \tag{29}$$

Hence,

$$\mathcal{J}(r,n_1,\alpha;\xi_2)=\mathcal{J}\left(\frac{1}{n_2-n_1},n_1,\frac{n_1+n_2}{2};\xi_2\right)=\frac{1}{n_2-n_1}\int_{n_1}^{n_1+n_2-\eta_{n_1,n_2}(\xi_2)}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)\mathrm{d}\xi_1$$
$$-\frac{1}{n_2-n_1}\int_{n_1+n_2-\eta_{n_1,n_2}(\xi_2)}^{n_2}h\left(\kappa_{n_1,n_2}(\xi_1)\right)\mathrm{d}\xi_1$$
$$=\frac{1}{n_2-n_1}\int_{\eta_{n_1,n_2}(\xi_2)}^{n_2}h\left(\kappa_{n_1,n_2}(\xi_1)\right)\mathrm{d}\xi_1 \quad (30)$$
$$-\frac{1}{n_2-n_1}\int_{n_1}^{\eta_{n_1,n_2}(\xi_2)}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)\mathrm{d}\xi_1$$
$$=-\mathcal{J}\left(\frac{1}{n_2-n_1},\frac{n_1+n_2}{2},n_2;\xi_2\right)=-\mathcal{J}(r,\alpha,n_2;\xi_2).$$

□

Corollary 2. *If the function $r(\xi_1)$ is symmetric with respect to $((n_1+n_2)/2)$ on $[n_1,n_2]$, then*

$$[p(n_1)+p(n_2)]\int_{n_1}^{n_2}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1$$
$$-\int_{n_1}^{n_2}\left[h\left(\kappa_{n_1,n_2}(\xi_1)\right)+h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)\right]p(\xi_1)r(\xi_1)\mathrm{d}\xi_1=\left(\frac{n_2-n_1}{2}\right) \quad (31)$$
$$\times\left[\int_0^1\mathcal{J}_2(r,n_1,n_2;\xi_2)\left\{p'\left(\eta_{n_1,n_2}(\xi_2)\right)-p'\left(n_1+n_2-\eta_{n_1,n_2}(\xi_2)\right)\right\}\mathrm{d}\xi_2\right],$$

where

$$\mathcal{J}_2(r,n_1,n_2;\xi_2)=\int_{n_1}^{\eta_{n_1,n_2}(\xi_2)}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1$$
$$-\int_{\eta_{n_1,n_2}(\xi_2)}^{n_2}h\left(\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1. \quad (32)$$

Proof. Since the function $r(\xi_1)$ is symmetric with respect to $((n_1+n_2)/2)$ on $[n_1,n_2]$, we have

$$\int_{n_1}^{n_2}r(\xi_1)\mathrm{d}\xi_1=1,$$
$$\int_{n_1}^{n_2}\xi_1 r(\xi_1)\mathrm{d}\xi_1=\frac{n_1+n_2}{2}. \quad (33)$$

Moreover,

$$\mathcal{J}(r,n_1,\alpha;\xi_2)=\mathcal{J}\left(r,n_1,\frac{n_1+n_2}{2};\xi_2\right)$$
$$=-\mathcal{J}\left(r,\frac{n_1+n_2}{2},n_2;\xi_2\right)=-\mathcal{J}(r,\alpha,n_2;\xi_2). \quad (34)$$

Hence, from (19), we get the required identity (31). Now, we will discuss some cases for Lemma 2.

(1) If $h(e)=1$, then we have the result for P-functions. □

Corollary 3. *Under the assumptions of Corollary 1, if p is P-function on $[n_1,n_2]$, then*

$$p(n_1)+p(n_2)-\frac{2}{n_2-n_1}\int_{n_1}^{n_2}p(\xi_1)\mathrm{d}\xi_1$$
$$=\left(\frac{n_2-n_1}{2}\right)\int_0^1\left[p'\left(\eta_{n_1,n_2}(\xi_2)\right)-p'\left(n_1+n_2-\eta_{n_1,n_2}(\xi_2)\right)\right]\mathrm{d}\xi_2. \quad (35)$$

(2) If $h(e)=e^s$, then we obtain the following result for s-convex functions.

Corollary 4. *Suppose that the conditions of Corollary 1 are fulfilled and if p is s-convex function on $[n_1,n_2]$, then*

$$\frac{p(n_1)+p(n_2)}{s+1}-\frac{1}{n_2-n_1}\int_{n_1}^{n_2}\left[\left(\kappa_{n_1,n_2}(\xi_1)\right)^s+\left(1-\kappa_{n_1,n_2}(\xi_1)\right)^s\right]p(\xi_1)\mathrm{d}\xi_1$$
$$=\left(\frac{n_2-n_1}{2}\right)\int_0^1\mu(\xi_2,s)\left[p'\left(\eta_{n_1,n_2}(\xi_2)\right)-p'\left(n_1+n_2-\eta_{n_1,n_2}(\xi_2)\right)\right]\mathrm{d}\xi_2, \tag{36}$$

where

$$\mu(\xi_2,s)=\frac{(1-\xi_2)^{s+1}-(1+\xi_2)^{s+1}}{2^{s+1}(s+1)}. \tag{37}$$

(3) If $h(e)=e^{-s}$, $e\in(0,1)$, and $s\in[0,1]$, then we obtain the result for function of s-Godunova–Levin type.

Corollary 5. *Under the assumptions of Corollary 1, if p is function of s-Godunova–Levin type on $[n_1,n_2]$, then*

$$\frac{p(n_1)+p(n_2)}{1-s}-\frac{1}{n_2-n_1}\int_{n_1}^{n_2}\left[\left(\kappa_{n_1,n_2}(\xi_1)\right)^{-s}+\left(1-\kappa_{n_1,n_2}(\xi_1)\right)^{-s}\right]p(\xi_1)\mathrm{d}\xi_1$$
$$=\left(\frac{n_2-n_1}{2}\right)\int_0^1\mu(\xi_2,-s)\left[p'\left(\eta_{n_1,n_2}(\xi_2)\right)-p'\left(n_1+n_2-\eta_{n_1,n_2}(\xi_2)\right)\right]\mathrm{d}\xi_2, \tag{38}$$

where

$$\mu(\xi_2,-s)=\frac{(1-\xi_2)^{1-s}-(1+\xi_2)^{1-s}}{2^{1-s}(1-s)},\quad \xi_2\in(0,1), s\in[0,1). \tag{39}$$

(4) If $h(e)=e(1-e)$, $e\in[0,1]$, then we obtain the result for egs-convex functions.

Corollary 6. *Under the assumptions of Corollary 1, if p is result for egs-convex functions on $[n_1,n_2]$, then*

$$\frac{p(n_1)+p(n_2)}{6}-\frac{2}{n_2-n_1}\int_{n_1}^{n_2}\left(1-\kappa_{n_1,n_2}(\xi_1)\right)\left(\kappa_{n_1,n_2}(\xi_1)\right)p(\xi_1)\mathrm{d}\xi_1$$
$$=\left(\frac{n_2-n_1}{24}\right)\int_0^1\xi_2\left(\xi_2^2-3\right)\left[p'\left(\eta_{n_1,n_2}(\xi_2)\right)-p'\left(n_1+n_2-\eta_{n_1,n_2}(\xi_2)\right)\right]\mathrm{d}\xi_2. \tag{40}$$

Lemma 3. *Let $A\colon C([n_1,n_2])\longrightarrow\mathfrak{R}$ be a positive linear functional on $C([n_1,n_2])$, and let e_i be monomials $e_i(\xi_1)=\xi_1^i$, $\xi_1\in[n_1,n_2]$, $i\in\mathbb{N}$. Let g be a h-convex function on $[n_1,n_2]$, then*

$$A(g(e_1))\le A\left(h\left(\kappa_{n_1,n_2}(e_1)\right)\right)g(n_1)+A\left(h\left(1-\kappa_{n_1,n_2}(e_1)\right)\right)g(n_2). \tag{41}$$

Proof. By using the h-convexity of g on $[n_1,n_2]$ and the given equality

$$e_1=\kappa_{n_1,n_2}(e_1)n_1+\left(1-\kappa_{n_1,n_2}(e_1)\right)n_2, \tag{42}$$

we get

$$g(e_1)=g\left(\left(\kappa_{n_1,n_2}(e_1)\right)n_1+\left(1-\kappa_{n_1,n_2}(e_1)\right)n_2\right)$$
$$\le h\left(\kappa_{n_1,n_2}(e_1)\right)g(n_1)+h\left(1-\kappa_{n_1,n_2}(e_1)\right)g(n_2). \tag{43}$$

Since A is a positive linear functional, we get inequality (41) by applying A on both sides of (43). □

3. Main Results

The following theorem generalizes the result given by Gavrea in [8].

Theorem 9. *Let $p\colon\mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$ and $p'\in L([n_1,n_2])$, where $[n_1,n_2]\subseteq\mathfrak{K}^\circ$. If $r\colon[n_1,n_2]\longrightarrow[0,\infty)$ is a continuous mapping and $|p'|$ is h-convex on $[n_1,n_2]$, then the following inequality holds:*

$$\left|\int_{n_1}^{n_2} p(\xi_1)r(\xi_1)n_2\xi_1 - p(\alpha)\right| \leq \mathscr{Z}(n_1,\alpha,n_2)\left|p'(n_1)\right| + \mathscr{G}(n_1,\alpha,n_2)\left|p'(n_2)\right|, \tag{44}$$

where

$$\begin{aligned}\mathscr{Z}(n_1,\alpha,n_2) &= \int_{n_1}^{\alpha}\left(\int_{n_1}^{e} r(\xi_1)n_2\xi_1\right)h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e \\ &\quad + \int_{\alpha}^{n_2}\left(\int_{e}^{n_2} r(\xi_1)n_2\xi_1\right)h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e, \\ \mathscr{G}(n_1,\alpha,n_2) &= \int_{n_1}^{\alpha}\left(\int_{n_1}^{e} r(\xi_1)n_2\xi_1\right)h\left(1-\kappa_{n_1,n_2}(e)\right)\mathrm{d}e \\ &\quad + \int_{\alpha}^{n_2}\left(\int_{e}^{n_2} r(\xi_1)n_2\xi_1\right)h\left(1-\kappa_{n_1,n_2}(e)\right)\mathrm{d}e.\end{aligned} \tag{45}$$

Proof. We can write

$$p(\xi_1) - p(\alpha) = \int_{n_1}^{n_2}\left[\sigma(\xi_1 - e) - \sigma(\alpha - e)\right]p'(e)\mathrm{d}e. \tag{46}$$

From (46), we obtain

$$\int_{n_1}^{n_2} p(\xi_1)r(\xi_1)\mathrm{d}\xi_1 - p(\alpha) = \int_{n_1}^{n_2}\left(\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha - e)\right)p'(e)\mathrm{d}e. \tag{47}$$

Taking absolute value on both sides of (47) and applying Lemma 3, we have

$$\begin{aligned}&\left|\int_{n_1}^{n_2} p(\xi_1)r(\xi_1)\mathrm{d}\xi_1 - p(\alpha)\right| \\ &\leq \int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha - e)\right|\left|p'(e)\right|\mathrm{d}e \\ &\leq \left|p'(n_1)\right|\int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha - e)\right|h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e \\ &+ \left|p'(n_2)\right|\int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha - e)\right|h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e.\end{aligned} \tag{48}$$

We notice that

$$\begin{aligned}&\int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha - e)\right|h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e \\ &= \int_{n_1}^{\alpha}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \int_{n_1}^{n_2} r(\xi_1)\mathrm{d}\xi_1\right|h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e \\ &\quad + \int_{\alpha}^{n_2}\left(\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1\right)h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e = \int_{n_1}^{\alpha}\left(\int_{n_1}^{e} r(\xi_1)\mathrm{d}\xi_1\right)h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e \\ &\quad + \int_{\alpha}^{n_2}\left(\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1\right)h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{d}e = \mathscr{Z}(n_1,\alpha,n_2).\end{aligned} \tag{49}$$

In a similar way,

$$\begin{aligned}&\int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha - e)\right|h\left(1-\kappa_{n_1,n_2}(e)\right)\mathrm{d}e \\ &= \int_{n_1}^{\alpha}\left(\int_{n_1}^{e} r(\xi_1)\mathrm{d}\xi_1\right)h\left(1-\kappa_{n_1,n_2}(e)\right)\mathrm{d}e \\ &\quad + \int_{\alpha}^{n_2}\left(\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1\right)h\left(1-\kappa_{n_1,n_2}(e)\right)\mathrm{d}e = \mathscr{G}(n_1,\alpha,n_2).\end{aligned} \tag{50}$$

We get the result from (49) and (50). □

Corollary 7. *Suppose that the assumptions of Theorem 9 are satisfied and that $r(\xi_1)$ is symmetric with respect to $((n_1 + n_2)/2)$ on $[n_1, n_2]$, then*

$$\left|\int_{n_1}^{n_2} p(\xi_1)r(\xi_1)\mathrm{d}\xi_1 - p\left(\frac{n_1+n_2}{2}\right)\right| \leq \left[\left|p'(n_1)\right| + \left|p'(n_2)\right|\right]\mathscr{Z}_1(n_1,n_2), \tag{51}$$

where

$$\begin{aligned}\mathscr{Z}_1(n_1,n_2) &= \int_{n_1}^{((n_1+n_2)/2)}\left(\int_{n_1}^{e} r(\xi_1)\mathrm{d}\xi_1\right)h\left(\kappa_{n_1,n_2}(\xi_1)\right)\mathrm{d}e \\ &\quad + \int_{((n_1+n_2)/2)}^{n_2}\left(\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1\right)h\left(\kappa_{n_1,n_2}(\xi_1)\right)\mathrm{d}e.\end{aligned} \tag{52}$$

Proof. Since the function $r(\xi_1)$ is symmetric with respect to $((n_1 + n_2)/2)$ on $[n_1, n_2]$ so $\alpha = ((n_1 + n_2)/2)$ and the function $r(\xi_1)$ is symmetric with respect $((n_1 + n_2)/2)$ on $[n_1, n_2]$, this fact gives

$$\begin{aligned}\mathscr{Z}(n_1,\alpha,n_2) &= \mathscr{Z}\left(n_1,\frac{n_1+n_2}{2},n_2\right) \\ &= \mathscr{G}\left(n_1,\frac{n_1+n_2}{2},n_2\right) \\ &= \mathscr{G}(n_1,\alpha,n_2).\end{aligned} \tag{53}$$

Thus,

$$\mathscr{L}\left(n_1, \frac{n_1+n_2}{2}, n_2\right) = \mathscr{G}\left(n_1, \frac{n_1+n_2}{2}, n_2\right)$$
$$= \int_{n_1}^{((n_1+n_2)/2)} \left(\int_{n_1}^{e} r(\xi_1)\mathrm{d}\xi_1\right) h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{de}$$
$$+ \int_{((n_1+n_2)/2)}^{n_2} \left(\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1\right) h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{de} = \mathscr{L}_1(n_1,n_2). \tag{54}$$

□

Corollary 8. *If we take $r(\xi_1) = (g(\xi_1)/\int_{n_1}^{n_2} g(\xi_1)d\xi_1)$ in (44) and $g(\xi_1)$ is symmetric with respect to $((n_1+n_2)/2)$. Then, the following inequality holds:*

$$\left|\int_{n_1}^{n_2} p(\xi_1)g(\xi_1)\mathrm{d}\xi_1 - p\left(\frac{n_1+n_2}{2}\right)\int_{n_1}^{n_2} g(\xi_1)\mathrm{d}\xi_1\right|$$
$$\leq \left[\left|p'(n_1)\right| + \left|p'(n_2)\right|\right]\mathscr{L}_2(n_1,n_2),$$
$$\mathscr{L}_2(n_1,n_2) = \int_{n_1}^{((n_1+n_2)/2)} \left(\int_{n_1}^{e} g(\xi_1)n_2\xi_1\right) h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{de}$$
$$+ \int_{((n_1+n_2)/2)}^{n_2} \left(\int_{e}^{n_2} g(\xi_1)\mathrm{d}\xi_1\right) h\left(\kappa_{n_1,n_2}(e)\right)\mathrm{de}. \tag{55}$$

Theorem 10. *Let $p\colon \mathfrak{K}\subseteq\mathfrak{R} \longrightarrow \mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$ and $p' \in L([n_1,n_2])$, where $[n_1,n_2]\subseteq\mathfrak{K}^\circ$. If $r\colon [n_1,n_2] \longrightarrow [0,\infty)$ is a continuous mapping and $|p'|^q$ is h-convex on $[n_1,n_2]$ for $q \geq 1$, then*

$$\left|\int_{n_1}^{n_2} p(\xi_1)r(\xi_1)\mathrm{d}\xi_1 - p(\alpha)\right| \leq \left(2\int_{\alpha}^{n_2}(\xi_1-\alpha)r(\xi_1)\mathrm{d}\xi_1\right)^{1-(1/q)}$$
$$\times\left(\mathscr{L}(n_1,\alpha,n_2)\left|p'(n_1)\right|^q + \mathscr{G}(n_1,\alpha,n_2)\left|p'(n_2)\right|^q\right)^{(1/q)}, \tag{56}$$

where $\mathscr{L}(n_1,\alpha,n_2)$ and $\mathscr{G}(n_1,\alpha,n_2)$ are given in Theorem 9.

Proof. Application of Hölder inequality in (47) yields that

$$\left|\int_{n_1}^{n_2} p(\xi_1)r(\xi_1)\mathrm{d}\xi_1 - p(\alpha)\right| \leq \int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha-e)\right|\left|p'(e)\right|^q\mathrm{de}$$
$$\leq \left(\int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha-e)\right|\mathrm{de}\right)^{1-(1/q)}$$
$$\times\left(\int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha-e)\right|\left|p'(e)\right|^q\mathrm{de}\right)^{(1/q)}. \tag{57}$$

Applying Lemma 3, we have

$$\int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha-e)\right|\left|p'(e)\right|^q\mathrm{de}$$
$$\leq \mathscr{L}(n_1,\alpha,n_2)\left|p'(n_1)\right|^q + \mathscr{G}(n_1,\alpha,n_2)\left|p'(n_2)\right|^q. \tag{58}$$

On the other hand, we have

$$\int_{n_1}^{n_2}\left|\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 - \sigma(\alpha-e)\right|\mathrm{de}$$
$$= \int_{n_1}^{\alpha}\left(\int_{n_1}^{e} r(\xi_1)\mathrm{d}\xi_1\right)\mathrm{de} + \int_{\alpha}^{n_2}\left(\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1\right)n_2 e \tag{59}$$
$$= 2\int_{\alpha}^{n_2}(\xi_1-\alpha)r(\xi_1)\mathrm{d}\xi_1.$$

A combination of (57)–(59) gives (56). □

Corollary 9. *If $r(\xi_1)$ is symmetric with respect to $((n_1+n_2)/2)$ on $[n_1,n_2]$, then from (56), we obtain*

$$\left|\int_{n_1}^{n_2} p(\xi_1)r(\xi_1)\mathrm{d}\xi_1 - p\left(\frac{n_1+n_2}{2}\right)\right| \le \mathscr{Z}_1^{(1/q)}(n_1,n_2)\left(2\int_{\alpha}^{n_2}(\xi_1-\alpha)r(\xi_1)\mathrm{d}\xi_1\right)^{1-(1/q)}\left(\left|p'(n_1)\right|^q + \left|p'(n_2)\right|^q\right)^{(1/q)}, \tag{60}$$

where $\mathscr{Z}_1(n_1,n_2)$ is as defined in Corollary 7.

Corollary 10. *If $r(\xi_1) = (g(\xi_1)/\int_{n_1}^{n_2} g(\xi_1)d\xi_1)$ and $g(\xi_1)$ is symmetric with respect to $((n_1+n_2)/2)$ on $[n_1,n_2]$, then the following inequality holds:*

$$\left|\int_{n_1}^{n_2} p(\xi_1)g(\xi_1)\mathrm{d}\xi_1 - p\left(\frac{n_1+n_2}{2}\right)\int_{n_1}^{n_2} g(\xi_1)\mathrm{d}\xi_1\right| \le \mathscr{Z}_2^{(1/q)}(n_1,n_2)\left(2\int_{n_1}^{((n_1+n_2)/2)}\left(\frac{n_1+n_2}{2}-\xi_1\right)r(\xi_1)\mathrm{d}\xi_1\right)^{1-(1/q)}\left(\left|p'(n_1)\right|^q + \left|p'(n_2)\right|^q\right)^{(1/q)}, \tag{61}$$

where $\mathscr{Z}_2(n_1,n_2)$ is as defined in Corollary 8.

For our next results, we use the following notations.

$$\begin{aligned}\varphi(r,p) &:= p(n_1)\int_{n_1}^{n_2} h\left(\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1 \\ &\quad + p(n_2)\int_{n_1}^{n_2} h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1 \\ &\quad - \int_{n_1}^{n_2}\left[h\left(\kappa_{n_1,n_2}(\xi_1)\right) + h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)\right]p(\xi_1)r(\xi_1)\mathrm{d}\xi_1.\end{aligned} \tag{62}$$

It is clear from (62) that

$$\varphi\left(\frac{1}{n_2-n_1},p\right) := \frac{p(n_1)\int_{n_1}^{n_2} h\left(\kappa_{n_1,n_2}(\xi_1)\right)\mathrm{d}\xi_1 + p(n_2)\int_{n_1}^{n_2} h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)\mathrm{d}\xi_1}{n_2-n_1} - \frac{1}{n_2-n_1}\int_{n_1}^{n_2}\left[h\left(\kappa_{n_1,n_2}(\xi_1)\right) + h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)\right]p(\xi_1)\mathrm{d}\xi_1. \tag{63}$$

The next result gives upper bound of $|\varphi(r,p)|$ when the function $p(\xi_1)$ is quasi h-convex.

Theorem 11. *Let $p: \mathfrak{K}\subseteq\mathfrak{R} \longrightarrow \mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$ and $p' \in L([n_1,n_2])$, where $[n_1,n_2]\subseteq\mathfrak{K}^\circ$. If $r: [n_1,n_2] \longrightarrow [0,\infty)$ be a continuous mapping and $|p'|$ is quasi h-convex on $[n_1,n_2]$, then the following inequality holds:*

$$\begin{aligned}|\varphi(r,p)| &\le (\alpha-n_1)\left(\sup\left\{\lambda\left|p'(n_1)\right|, \mu\left|p'(\alpha)\right|\right\}\right)\int_0^1\left|\mathscr{J}(r,n_1,\alpha;\xi_2)\right|\mathrm{d}\xi_2 \\ &\quad + (n_2-\alpha)\left(\sup\left\{\mu\left|p'(\alpha)\right|, \lambda\left|p'(n_2)\right|\right\}\right)\int_0^1\left|\mathscr{J}(r,\alpha,n_2;\xi_2)\right|\mathrm{d}\xi_2,\end{aligned} \tag{64}$$

where $\mathscr{J}(r,\alpha,\beta;\xi_2)$ *is defined as in Lemma 2,* $\lambda=\sup_{e\in(0,1)}h(e)$*, and* $\mu=\sup_{e\in(0,1)}h(1-e)$.

Proof. Since $|p'|$ is quasi h-convex on $[n_1,n_2]$, we have

$$\begin{aligned}&\left|p'(\xi_2 n_1+(1-\xi_2)\alpha)\right|\le\sup\left\{\lambda\left|p'(n_1)\right|,\mu\left|p'(\alpha)\right|\right\},\\&\left|p'((1-\xi_2)\alpha+\xi_2 n_2)\right|\le\sup\left\{\mu\left|p'(\alpha)\right|,\lambda\left|p'(n_2)\right|\right\},\end{aligned}\tag{65}$$

for all $\xi_2\in[0,1]$. Hence, inequality (64) follows from (19). □

Theorem 12. *Let* $p\colon \mathfrak{K}\subseteq\mathfrak{R}\longrightarrow\mathfrak{R}$ *be a differentiable mapping on* $\mathfrak{K}^\circ$ *and* $p'\in L([n_1,n_2])$*, where* $[n_1,n_2]\subseteq\mathfrak{K}^\circ$*. If* $r\colon[n_1,n_2]\longrightarrow[0,\infty)$ *be a continuous mapping and* $|p'|$ *is quasi h-convex on* $[n_1,n_2]$*, then*

$$|\varphi(r,p)|\le\mathscr{Z}_3(n_1,n_2)\left[\sup\left\{\lambda\left|p'(n_1)\right|,\mu\left|p'\left(\frac{n_1+n_2}{2}\right)\right|\right\}+\sup\left\{\lambda\left|p'\left(\frac{n_1+n_2}{2}\right)\right|,\mu\left|p'(n_2)\right|\right\}\right],\tag{66}$$

where

$$\mathscr{Z}_3(n_1,n_2)=\int_{n_1}^{((n_1+n_2)/2)}\left(\int_{n_1}^{e}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1-\int_{e}^{n_2}h\left(\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1\right)\mathrm{d}e.\tag{67}$$

λ and μ are defined as in Theorem 11.

Proof. The symmetry of $r(\xi_1)$ with respect to $((n_1+n_2)/2)$ on $[n_1,n_2]$ gives

$$\begin{aligned}\varphi(r,p)=&\left[p(n_1)+p(n_2)\right]\int_{n_1}^{n_2}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1\\&-\int_{n_1}^{n_2}\left[h\left(\kappa_{n_1,n_2}(\xi_1)\right)+h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)\right]p(\xi_1)r(\xi_1)\mathrm{d}\xi_1.\end{aligned}\tag{68}$$

We also observe that

$$\begin{aligned}(\alpha-n_1)\int_0^1\left|\mathscr{J}(r,n_1,\alpha;\xi_2)\right|\mathrm{d}\xi_2&=\int_{n_1}^{\alpha}\left|\int_{n_1}^{e}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1-\int_{e}^{n_2}h\left(\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1\right|\mathrm{d}e,\\(n_2-\alpha)\int_0^1\left|\mathscr{J}(r,\alpha,n_2;\xi_2)\right|\mathrm{d}\xi_2&=\int_{\alpha}^{n_2}\left|\int_{n_1}^{e}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1-\int_{e}^{n_2}h\left(\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1\right|\mathrm{d}e.\end{aligned}\tag{69}$$

Consider the function $p\colon[n_1,n_2]\longrightarrow\mathfrak{R}$ defined by

$$p(e)=\int_{n_1}^{e}h\left(1-\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1-\int_{e}^{n_2}h\left(\kappa_{n_1,n_2}(\xi_1)\right)r(\xi_1)\mathrm{d}\xi_1.\tag{70}$$

Then,

$$p'(e)=\left[h\left(1-\kappa_{n_1,n_2}(e)\right)+h\left(\kappa_{n_1,n_2}(e)\right)\right]r(e)>0,\quad e\in[n_1,n_2].\tag{71}$$

This shows that $p(e)$ is an increasing function on $[n_1,n_2]$ and

$$p\left(\frac{n_1+n_2}{2}\right)=0.\tag{72}$$

Now, it is easy to see that

$$\begin{aligned}
&(\alpha - n_1)\int_0^1 \left|\mathcal{J}(r, n_1, \alpha; \xi_2)\right| \mathrm{d}\xi_2 \\
&\quad = \int_{n_1}^{((n_1+n_2)/2)} \left(\int_{n_1}^{e} h\left(1-\kappa_{n_1,n_2}(\xi_1)\right) r(\xi_1)\mathrm{d}\xi_1 - \int_e^{n_2} h\left(\kappa_{n_1,n_2}(\xi_1)\right) r(\xi_1)\mathrm{d}\xi_1 \right)\mathrm{d}e, \\
&(n_2 - \alpha)\int_0^1 \left|\mathcal{J}(r, \alpha, n_2; \xi_2)\right| \mathrm{d}\xi_2 \\
&\quad = \int_{((n_1+n_2)/2)}^{n_2} \left(\int_{n_1}^{e} h\left(1-\kappa_{n_1,n_2}(\xi_1)\right) r(\xi_1)\mathrm{d}\xi_1 - \int_e^{n_2} h\left(\kappa_{n_1,n_2}(\xi_1)\right) r(\xi_1)\mathrm{d}\xi_1 \right)\mathrm{d}e \\
&\quad = \int_{n_1}^{((n_1+n_2)/2)} \left(\int_{n_1}^{e} h\left(1-\kappa_{n_1,n_2}(\xi_1)\right) r(\xi_1)\mathrm{d}\xi_1 - \int_e^{n_2} h\left(\kappa_{n_1,n_2}(\xi_1)\right) r(\xi_1)\mathrm{d}\xi_1 \right)\mathrm{d}e.
\end{aligned} \tag{73}$$

Hence, inequality (66) follows from the inequality. □

Remark 2. If we choose $h(e) = e$ in Theorem 9, Corollary 7 and 8, Theorem 10, Corollaries 9 and 10, and Theorems 11 and 12, we get the results for nonnegative convex functions and quasi-convex functions (see Theorem 3.1, Corollary 3.1, Theorem 3.2, Remark 3.2, Theorem 3.3 and Theorem 3.4 in [8]). Moreover, one can obtain inequalities for s-convex functions, Godunova–Levin functions, s-Godunova–Levin functions, egs-convex, P-functions, and quasiconvex functions from the result of Theorem 9, Corollaries 7 and 8, Theorem 10, Corollaries 9 and 10, and Theorems 11 and 12 by choosing $h(e) = e^s$, e^{-1}, e^{-s}, $e(1-e)$ and 1, respectively.

4. Applications

4.1. $\breve{\mathbf{s}}$-Divergence Measures. Here, we provide some applications on $\breve{\mathbf{s}}$-divergence measure and probability density function by using the results proved in Section 3. Let the set ϕ and the σ finite measure μ be given, and let the set of all probability densities on μ to be defined on $\Omega \coloneqq \{\xi_2 | \xi_2\colon \phi \longrightarrow \mathfrak{R}, \xi_2(\lambda) > 0, \int_\phi \xi_2(\lambda)\mathrm{d}\mu(\lambda) = 1\}$. Let $\breve{\mathbf{s}}\colon (0,\infty) \longrightarrow \mathfrak{R}$ be given mapping and consider $\mathfrak{D}_{\mathbf{s}}(\xi_2, \xi_1)$ defined by

$$\mathfrak{D}_{\mathbf{s}}(\xi_2, \xi_1) \coloneqq \int_\phi \xi_2(\lambda)\breve{\mathbf{s}}\left[\frac{\xi_1(\lambda)}{\xi_2(\lambda)}\right]\mathrm{d}\mu(\lambda), \quad \xi_2, \xi_1 \in \Omega. \tag{74}$$

If $\breve{\mathbf{s}}$ is convex, then (74) is called as the Csisźar $\breve{\mathbf{s}}$-divergence. Consider the following Hermite–Hadamard $(\mathcal{H}\mathcal{H})$ divergence:

$$\mathfrak{D}^{\breve{\mathbf{s}}}_{\mathcal{H}\mathcal{H}}(\xi_2, \xi_1) \coloneqq \int_\phi \xi_2(\lambda) \frac{\int_1^{(\xi_1(\lambda)/\xi_2(\lambda))} \breve{\mathbf{s}}(\xi_1)\mathrm{d}\xi_1}{(\xi_1(\lambda)/\xi_2(\lambda)) - 1}\mathrm{d}\mu(\lambda), \quad \xi_2, \xi_1 \in \Omega, \tag{75}$$

where $\breve{\mathbf{s}}$ is convex on $(0,\infty)$ with $\breve{\mathbf{s}}(1) = 0$. Note that $\mathfrak{D}^{\breve{\mathbf{s}}}_{\mathcal{H}\mathcal{H}}(\xi_2, \xi_1) \geq 0$ with the equality holds if and only if $\xi_2 = \xi_1$.

Proposition 1. *Let $\breve{\mathbf{s}}\colon \mathfrak{K} \subseteq \mathfrak{R} \longrightarrow \mathfrak{R}$ be a differentiable mapping on $\mathfrak{K}^\circ$, where $[n_1, n_2] \subseteq \mathfrak{K}^\circ$. If $r(\xi_1) = (1/(n_2 - n_1))$, $\xi_1 \in [n_1, n_2]$, $|\breve{\mathbf{s}}'|^q$ is h-convex on $[n_1, n_2]$ for $q \geq 1$ with $h = 1$ | No Image for this Article and $\breve{\mathbf{s}}(1) = 0$, then*

$$\begin{aligned}
&\left|\mathfrak{D}^{\breve{\mathbf{s}}}_{\mathcal{H}\mathcal{H}}(\xi_2, \xi_1) - \int_\phi \xi_2(\lambda)\breve{\mathbf{s}}\left(\frac{\xi_1(\lambda) + \xi_2(\lambda)}{2\xi_2(\lambda)}\right)\mathrm{d}\mu(\lambda)\right| \\
&\quad \leq \int_\phi \xi_2(\lambda)\frac{|\xi_1(\lambda) - \xi_2(\lambda)|}{4\xi_2(\lambda)}\left(\left|\breve{\mathbf{s}}'(1)\right|^q + \left|\breve{\mathbf{s}}'\left(\frac{\xi_1(\lambda)}{\xi_2(\lambda)}\right)\right|^q\right)^{(1/q)}\mathrm{d}\mu(\lambda).
\end{aligned} \tag{76}$$

Proof. Let $\vartheta_1 \coloneqq \{\lambda \in \phi\colon \xi_1(\lambda) > \xi_2(\lambda)\}$, $\vartheta_2 \coloneqq \{\lambda \in \phi\colon \xi_1(\lambda) < \xi_2(\lambda)\}$, and $\vartheta_3 \coloneqq \{\lambda \in \phi\colon \xi_1(\lambda) = \xi_2(\lambda)\}$. Obviously, if $\lambda \in \vartheta_3$, then equality holds in (76). Now, if $\lambda \in \vartheta_1$, then for $n_1 = 1$ and $n_2 = (\xi_1(\lambda)/\xi_2(\lambda))$ in Theorem 10, multiplying both sides to the obtained result by $\xi_2(\lambda)$ and integrating over ϑ_1, we have

$$\left|\int_{\vartheta_1} \frac{\xi_2(\lambda)\left(\int_1^{(\xi_1(\lambda)/\xi_2(\lambda))} \breve{\mathbf{s}}(\xi_1)\mathrm{d}\xi_1\right)}{(\xi_1(\lambda)/\xi_2(\lambda))-1}\mathrm{d}\mu(\lambda) - \int_{\vartheta_1} \xi_2(\lambda)\breve{\mathbf{s}}\left(\frac{\xi_1(\lambda)+\xi_2(\lambda)}{2\xi_2(\lambda)}\right)\mathrm{d}\mu(\lambda)\right|$$
$$\leq \int_{\vartheta_1} \xi_2(\lambda)\left(\frac{\xi_1(\lambda)-\xi_2(\lambda)}{4\xi_2(\lambda)}\right)\left(\left|\breve{\mathbf{s}}'(1)\right|^q + \left|\breve{\mathbf{s}}'\left(\frac{\xi_1(\lambda)}{\xi_2(\lambda)}\right)\right|^q\right)^{(1/q)}\mathrm{d}\mu(\lambda). \tag{77}$$

Similarly, if $\lambda \in \vartheta_2$, then for $n_2 = 1$ and $n_1 = (\xi_1(\lambda)/\xi_2(\lambda))$ in Theorem 9, multiplying both sides to the obtained result by $\xi_2(\lambda)$ and integrating over ϑ_2, we have

$$\left|\int_{\vartheta_2} \frac{\xi_2(\lambda)\left(\int_1^{(\xi_1(\lambda)/\xi_2(\lambda))} \breve{\mathbf{s}}(\xi_1)\mathrm{d}\xi_1\right)}{(\xi_1(\lambda)/\xi_2(\lambda))-1}\mathrm{d}\mu(\lambda) - \int_{\vartheta_2} \xi_2(\lambda)\breve{\mathbf{s}}\left(\frac{\xi_1(\lambda)+\xi_2(\lambda)}{2\xi_2(\lambda)}\right)\mathrm{d}\mu(\lambda)\right|$$
$$\leq \int_{\vartheta_2} \left(\frac{\xi_2(\lambda)-\xi_1(\lambda)}{4\xi_2(\lambda)}\right)\left(\left|\breve{\mathbf{s}}'\left(\frac{\xi_1(\lambda)}{\xi_2(\lambda)}\right)\right|^q + \left|\breve{\mathbf{s}}'(1)\right|^q\right)^{(1/q)}\mathrm{d}\mu(\lambda). \tag{78}$$

Adding inequalities (77) and (78) and utilizing triangle inequality, we get the desired result (76). □

4.2. Applications to Statistics. Let $r\colon [n_1, n_2] \longrightarrow [0, \infty)$ be the probability density function of a continuous random variable X symmetric to $((n_1 + n_2)/2)$ with $0 < n_1 < n_2$. The rth moment of X is defined as

$$E_r(X) = \int_{n_1}^{n_2} \xi_1^r r(\xi_1)\mathrm{d}\xi_1, \tag{79}$$

which is assumed to be finite.

Theorem 13. *Suppose that $0 < n_1 < n_2$ and $r \geq 2$, then the following inequality holds:*

$$\left|E_r(X) - \left(\frac{n_1+n_2}{2}\right)^r\right| \leq r(n_2 - n_1)\left(\frac{n_1^{r-1} + n_2^{r-1}}{2}\right). \tag{80}$$

Proof. Let $p(\xi_1) = \xi_1^r$ on $[n_1, n_2]$ for $r \geq 2$, and we have $|p'(\xi_1)| = r\xi_1^{r-1}$ is convex on $[n_1, n_2]$. Since the functions $r\colon [n_1, n_2] \longrightarrow [0, \infty)$ are symmetric with respect to $((n_1 + n_2)/2)$,

$$\int_{n_1}^{e} r(\xi_1)\mathrm{d}\xi_1 \leq \int_{n_1}^{n_2} r(\xi_1)\mathrm{d}\xi_1 = 1,$$
$$\int_{e}^{n_2} r(\xi_1)\mathrm{d}\xi_1 \leq \int_{n_1}^{n_2} r(\xi_1)\mathrm{d}\xi_1 = 1, \tag{81}$$
$$\alpha = \frac{n_1+n_2}{2}.$$

Therefore, from inequality (44), we obtain inequality (80). □

5. Conclusion

In this study, we propose new definition, namely, the definition of quasi h-convex functions and provide an example of such type of functions. We prove some new weighted Hermite–-Hadamard type inequalities for differentiable for h-convex and quasi h-convex functions when the weight function is not necessarily symmetric about the midpoint of the interval. These results generalize many results proved in earlier works for these classes of functions. Applications of some of our results to $\breve{\mathbf{s}}$-divergence and to statistics are given. We believe that the results of the current study may be a motivation to explore more new results relevant to this field of research for people working in the rich field of mathematical inequalities.

Data Availability

Data sharing is not applicable to this paper as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors' Contributions

This study was carried out in collaboration of all authors. The author read and approved the final manuscript.

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