Research Article

On the Study of Reverse Degree-Based Topological Properties for the Third Type of $p$th Chain Hex-Derived Network

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1. Introduction

A topological descriptor is a numerical value that represents the complete structure of a graph. In the study of topological descriptors, graph theory has shown to be a fruitful field of study. The primary elements of topological indices link the many chemical and physical characteristics of fundamental chemical substances. Vertex-edge-based topological indices are employed in the research of QSAR/QSPR for the prediction of bio-activity of different chemical compounds. With the dimension $p$, hexagonal networks create hex-derived networks, which have a wide range of implementations in engineering, computer science, and also medicine. In [1], researchers created a new form of graph known as a “third type of hex-derived networks” [2, 3] and continued this work by calculating degree-based topological descriptors for these networks, in which they computed exact values of some vertex-edge named topological indices for this network.

Researchers have used graph theory to develop a range of helpful tools, including graph labeling, topological indices, and finding numbers. The subject of graph theory has several applications and implementations in various fields of study, including chemistry, medicine, and engineering. A polynomial, a series of integers, a numeric value, or a matrix can all be used to identify a graph. A chemical compound can be represented as a graph (or a diagram) or usually denoted as a molecular graph, nodes played a role of atoms, and the bonding between atoms is usually labeled as edges in the molecular graph theory. Recently, a new topic called cheminformatics was established, which is a mix of chemistry, information science, and mathematics, in which the QSIPR/QSAR connection, bio-activity, and characterization of chemical compounds are investigated and reported in [4].

The topological descriptor is a numerical number associated with chemical compositions that maintain the relationship between chemical structures and a variety of physico-chemical characteristics, biological activity, and chemical reactivity. To describe the topology of a chemical network, it translated into a number, which is further used to create topological indices. Distance-based topological indices, degree-based topological indices, and counting-related topological indices are some of the most common...
forms of topological indices for graphs. Many academics have recently discovered topological indices for studying basic features of molecular graphs or networks. In [5–12], these networks have extremely compelling topological qualities that have been examined in distinct characteristics.

Let \( p \) and \( q \) represent the number of rows and number of triangles in each row of third-type \( p \) th chain hex-derived networks \( G_{p,q}^3 \), respectively, shown in Figure 1. Let \( G \) be a simple connected network, with a set of vertex and edges denoted by \( V \) and \( E \), respectively. \( |V| \) represents the order of \( G \) and \( |E| \) represents the size of \( G \). Let \( d_\theta \) be the degree of a vertex \( \theta \in V \) in \( G \) and \( R_\theta \) be its reverse degree that was introduced by Kulli [13] and defined as
\[
R_\theta = \frac{\sum_{\theta \in E(G)} \lambda(R_\theta, R_\theta)}{|E(G)|},
\]
then \( \lambda(R_\theta, R_\theta) = (R_\theta \times R_\theta)^{\alpha} \), then \( \Phi(G) \) represents the general reverse Randić index \( (R\;R_{\alpha}(G)) \) for \( \alpha = \frac{1}{2}, 1, -1 \),
\[
\text{If } \alpha = 1, \text{ then it is known as the second reverse Zagreb index } (R\;R_{1}(G)),
\]

\[
\text{If } \lambda(R_\theta, R_\theta) = \frac{R_\theta + R_\theta - 2}{R_\theta \times R_\theta}, \text{ then } \Phi(G) \text{ represents the reverse atom – bond connectivity index } (R\;A\;B\;C(G)),
\]

\[
\text{If } \lambda(R_\theta, R_\theta) = \frac{2\sqrt{R_\theta \times R_\theta}}{R_\theta + R_\theta}, \text{ then } \Phi(G) \text{ represents the reverse geometric – arithmetic index } (R\;G\;A\;G(G)),
\]

\[
\text{If } \lambda(R_\theta, R_\theta) = (R_\theta + R_\theta), \text{ then } \Phi(G) \text{ represents the first reverse Zagreb index } (R\;M_{1}(G)),
\]

\[
\text{If } \lambda(R_\theta, R_\theta) = (R_\theta + R_\theta)^{2}, \text{ then } \Phi(G) \text{ represents the reverse hyper Zagreb index } (R\;H\;M(G)),
\]

\[
\text{If } \lambda(R_\theta, R_\theta) = (R_\theta)^{2} + (R_\theta)^{2}, \text{ then } \Phi(G) \text{ represents the reverse forgotten index } (R\;F(G)),
\]

\[
\text{If } \lambda(R_\theta, R_\theta) = (R_\theta + R_\theta)^{\alpha} \times (R_\theta \times R_\theta)^{\beta}, \text{ then } \Phi(G) \text{ represents the first reverse redefined index } (R\;R\;Z_{1}(G)) \text{ for } \alpha = 1, \beta = -1,
\]

\[
\text{the second reverse redefined index } (R\;R\;Z_{2}(G)) \text{ for } \alpha = -1, \beta = 1,
\]

\[
\text{the third reverse redefined index } (R\;R\;Z_{3}(G)) \text{ for } \alpha = 1, \beta = 1.
\]

For latest results on topological descriptors for different chemical and computer networks and for general graphs, we refer to [14–25]. In this current research work, we determine the exact values of all the above reverse indices.

Figure 1: Third-type chain hex-derived network \( G_{p,q}^3 \).
2. Structure of Third-Type Hex-Derived Networks

With the help of complete graphs of order 3 (K3), Chen et al. [26] assembled a hexagonal mesh. In terms of chemistry, these K3 graphs are also called oxide graphs. Figure 1 is obtained by joining these K3 graphs. Two-dimensional mesh graph HX(2) (see Figure 2(a)) is obtained by joining six K3 graphs and three-dimensional mesh graph HX(3) (see Figure 2(b)) is obtained by putting K3 graphs around all sides of HX(2) [27]. Furthermore, repeating the same process by putting the tK3 graph around each hexagon, we obtained the tth hexagonal mesh. We have to note that the one-dimensional hexagonal mesh graph does not exist.

The novel network, labeled the third category of hex-derived networks, was developed in [1]. In [2, 3], they defined the graphically construction algorithm for the third type of hexagonal hex-derived network HHDN3(t). Huo et al. [28] explained the graphical construction algorithm for mth chain hex-derived network of third type. In this paper, we denote it by G^3_{pq}, and different priorities of p and q the chain hex-derived networks are shown in Figure 3. In [29–33], you may find related research that utilizes this idea and that may benefit from the new research’s visions.

3. Main Results

In this section, we study the third-type p th chain hex-derived networks G^3_{pq} in the following three cases.

(i) Case 1: for p = q, (p,q) ≥ 1.

(ii) Case 2: for p < q, p is odd and q is a natural number.
    For p > q, p is odd and q is a natural number. For p < q, p and q both are even. For p > q, p and q both are even.

(iii) Case 3: for p < q, p is even and q is odd. For p > q, p is even and q is odd.

3.1. Results for Case 1. We provide a formula that would be used to calculate any reverse degree topological descriptors of Case 1 for G^3_{pq}.

Lemma 1. Let G^3_{pq} be a third-type p th chain hex-derived networks. Then,

\[
\mathcal{R}_\alpha (G^3_{pq}) = \begin{cases} 
108pq + 72(p+q) + 32, & \text{for } \alpha = 1, \\
(18 + 6\sqrt{5})q + 12(p+q) + 12 - 4\sqrt{5}, & \text{for } \alpha = \frac{1}{2}, \\
\left(\frac{18}{5} + \frac{6\sqrt{5}}{5}\right)pq - \frac{12}{5}(p+q) + \frac{12}{5} - \frac{4\sqrt{5}}{5}, & \text{for } \alpha = -\frac{1}{2}, \\
108 \sqrt{25} pq - \frac{72}{25}(p+q) + \frac{32}{25} & \text{for } \alpha = -1.
\end{cases}
\]

Proof. The graph G^3_{pq} contains 12pq edges and maximum degree in G^3_{pq} graph is 8. There are two types of reverse degree vertices in G^3_{pq} that are 1 and 5. Let us partition the edges of G^3_{pq} according to its reverse degrees according to Case 1 as

\[
E_{1,1} = \{\theta \in E(G^3_{pq}) : \mathcal{R}_\theta = 1, \mathcal{R}_\theta = 1\}, \\
E_{1,5} = \{\theta \in E(G^3_{pq}) : \mathcal{R}_\theta = 1, \mathcal{R}_\theta = 5\}, \\
E_{5,5} = \{\theta \in E(G^3_{pq}) : \mathcal{R}_\theta = 5, \mathcal{R}_\theta = 5\}.
\]

Note that \(E(G^3_{pq}) = E_{1,1} \cup E_{1,5} \cup E_{5,5}\) and \(|E_{1,1}| = 3pq - 3p - 3q + 2\), \(|E_{1,5}| = 6pq - 4\), and \(|E_{5,5}| = 3pq + 3p + 3q + 2\). Hence,

\[
\mathcal{R}_\alpha (G^3_{pq}) = \sum_{\theta \in E(G^3_{pq})} \lambda (\mathcal{R}_\theta, \mathcal{R}_\theta)
= \sum_{\theta \in E_{1,1}} \lambda (1,1) + \sum_{\theta \in E_{1,5}} \lambda (1,5) + \sum_{\theta \in E_{5,5}} \lambda (5,5)
= (3pq - 3p - 3q + 2)\lambda (1,1) + (6pq - 4)\lambda (1,5) + (3pq + 3p + 3q + 2)\lambda (5,5).
\]

After simplification, we obtain

\[
\mathcal{R}_\alpha (G^3_{pq}) = 3pq(\lambda (1,1) + 2\lambda (1,5) + \lambda (5,5))
+ 3(p + q) (\lambda (5,5) - \lambda (1,1)) + 2(\lambda (1,1)
- 2\lambda (1,5) + \lambda (5,5)).
\]

Theorem 1. The general reverse Randić index of G^3_{pq} is equal to
Proof. For $R_{ABC}(G_{3}^{p,q})$ which is the general reverse Randić index of $G_{3}^{p,q}$, from equation (2), we have $\lambda(R_{\theta}, R_{\vartheta}) = (R_{\theta} \times R_{\vartheta})^{\alpha}$; therefore, $\lambda(1,1) = 1$, $\lambda(1,5) = (5)^{\alpha}$, and $\lambda(5,5) = (25)^{\alpha}$. Thus, by Lemma 1,

$$R_{ABC}(G_{3}^{p,q}) = 3pq(1 + 2(5)^{\alpha} + (25)^{\alpha}) + 3(p + q)((25)^{\alpha} - 1) + 2(1 - 2(5)^{\alpha} + (25)^{\alpha}).$$

(14)

Put $\alpha = 1$, and we have

$$R_{ABC}(G_{3}^{p,q}) = 108pq + 72(p + q) + 32.$$  
(15)

Put $\alpha = (1/2)$, and we have

$$R_{ABC}(G_{3}^{p,q}) = (18 + 6\sqrt{5})pq + 12(p + q) + 12 - 4\sqrt{5}.$$  
(16)

Put $\alpha = (-1/2)$, and we have

$$R_{ABC}(G_{3}^{p,q}) = \left(\frac{18}{5} + \frac{6\sqrt{5}}{5}\right)pq - \frac{12}{5}(p + q) + \frac{12}{5} - \frac{4\sqrt{5}}{5}. $$

(17)

Put $\alpha = -1$, and we have

$$R_{ABC}(G_{3}^{p,q}) = 108\frac{pq}{25} - 72\frac{pq + 32}{25}.$$  
(18)

□

Theorem 2. Let $G_{3}^{p,q}$ be a third-type $p$th chain hex-derived networks. Then, the reverse atom-bond connectivity index is

$$R_{ABC}(G_{3}^{p,q}) = \left(\frac{12\sqrt{5}}{5} + \frac{6\sqrt{2}}{5}\right)pq - \frac{8\sqrt{5}}{5} + 2(3p + 3q + 2)\sqrt{2}.$$  
(19)

The reverse geometric-arithmetic index is

$$R_{GA}(G_{3}^{p,q}) = (6 + 2\sqrt{5})pq + 4 - \frac{4\sqrt{5}}{3}.$$  
(20)

The first reverse Zagreb index is

$$R_{M_{1}}(G_{3}^{p,q}) = 72pq + 24p + 24q.$$  
(21)

The reverse hyper-Zagreb index is

$$R_{HM}(G_{3}^{p,q}) = 528pq + 288p + 288q + 64.$$  
(22)

The reverse forgotten index is

$$R_{F}(G_{3}^{p,q}) = 312pq + 144p + 144q.$$  
(23)

Proof. For $R_{ABC}(G_{3}^{p,q})$ which is the reverse atom-bond connectivity index of $G_{3}^{p,q}$ from equation (3), we have $\lambda(R_{\theta}, R_{\vartheta}) = \sqrt{(R_{\theta} + R_{\vartheta} - 2)/R_{\theta} \times R_{\vartheta}}$; therefore, $\lambda(1,1)$
For \( \mathcal{RG}_{1}(G_{pq}^{3}) \) which is the reverse geometric-arithmetic index of \( G_{pq}^{3} \) from equation (4), we have
\[
\lambda(\mathbf{R}_{p,q}, \mathbf{R}_{q}) = \left( \frac{\sqrt{5}}{2} \right)^{2} \left( \frac{\sqrt{5}}{2} \right)^{2} \left( \frac{\sqrt{5}}{2} \right)^{2} \]

therefore, \( \lambda(1,1) = 2 \), \( \lambda(1,5) = (\sqrt{5}/3) \), and \( \lambda(5,5) = 1 \). Thus, by Lemma 1 and after simplification,
\[
\mathcal{RG}_{1}(G_{pq}^{3}) = \left( 6 + (2\sqrt{5})pq + 4 - \frac{4\sqrt{5}}{3} \right). \tag{25}
\]

For \( \mathcal{RM}_{1}(G_{pq}^{3}) \) which is the first reverse Zagreb index of \( G_{pq}^{3} \) from equation (5), we have \( \lambda(\mathbf{R}_{p,q}, \mathbf{R}_{q}) = (\mathbf{R}_{p,q} + \mathbf{R}_{q})^{2} \); therefore, \( \lambda(1,1) = 5 \), \( \lambda(1,5) = 6 \), and \( \lambda(5,5) = 10 \). Thus, by Lemma 1 and after simplification,
\[
\mathcal{RM}_{1}(G_{pq}^{3}) = 72pq + 24p + 24q. \tag{26}
\]

For \( \mathcal{RM}_{3}(G_{pq}^{3}) \) which is the first reverse Zagreb index of \( G_{pq}^{3} \) from equation (6), we have \( \lambda(\mathbf{R}_{p,q}, \mathbf{R}_{q}) = (\mathbf{R}_{p,q} + \mathbf{R}_{q})^{2} \); therefore, \( \lambda(1,1) = 4 \), \( \lambda(1,5) = 26 \), and \( \lambda(5,5) = 50 \). Thus, by Lemma 1 and after simplification,
\[
\mathcal{RM}_{3}(G_{pq}^{3}) = 528pq + 288p + 288q + 64. \tag{27}
\]

For \( \mathcal{RF}(G_{pq}^{3}) \) which is the reverse forgotten index of \( G_{pq}^{3} \) from equation (7), we have \( \lambda(\mathbf{R}_{p,q}, \mathbf{R}_{q}) = (\mathbf{R}_{p,q} + \mathbf{R}_{q})^{2} \); therefore, \( \lambda(1,1) = 2 \), \( \lambda(1,5) = 26 \), and \( \lambda(5,5) = 50 \). Thus, by Lemma 1 and after simplification,
\[
\mathcal{RF}(G_{pq}^{3}) = 312pq + 144p + 144q. \tag{28}
\]

Theorem 3. Let \( G_{pq}^{3} \) be a third-type \( p \)th chain hex-derived networks. Then, the first reverse redefined index is
\[
\mathcal{RR}_{1}(G_{pq}^{3}) = \frac{72pq}{5} - \frac{24p}{5} - \frac{24q}{5}. \tag{29}
\]

The second reverse redefined index is
\[
\mathcal{RR}_{2}(G_{pq}^{3}) = 14pq + 6p + 6q + \frac{8}{3}. \tag{30}
\]

The third reverse redefined index is
\[
\mathcal{RR}_{3}(G_{pq}^{3}) = 936pq + 744p + 744q + 384. \tag{31}
\]

Proof. For \( \mathcal{RR}_{1}(G_{pq}^{3}) \) which is the first reverse redefined index of \( G_{pq}^{3} \) from equation (8), we have \( \lambda(\mathbf{R}_{p,q}, \mathbf{R}_{q}) = (\mathbf{R}_{p,q} + \mathbf{R}_{q})^{4} (\mathbf{R}_{p,q} \times \mathbf{R}_{q})^{4} \); therefore, \( \lambda(1,1) = 2 \), \( \lambda(1,5) = (\sqrt{5}/3) \), and \( \lambda(5,5) = (2/5) \). Thus, by Lemma 1 and after simplification,
\[
\mathcal{RR}_{1}(G_{pq}^{3}) = \frac{72pq}{5} - \frac{24p}{5} - \frac{24q}{5}. \tag{32}
\]

For \( \mathcal{RR}_{3}(G_{pq}^{3}) \) which is the second reverse redefined index of \( G_{pq}^{3} \) from equation (8), we have \( \lambda(\mathbf{R}_{p,q}, \mathbf{R}_{q}) = (\mathbf{R}_{p,q} + \mathbf{R}_{q})^{4} (\mathbf{R}_{p,q} \times \mathbf{R}_{q})^{4} \); therefore, \( \lambda(1,1) = (1/2) \), \( \lambda(1,5) = (5/6) \), and \( \lambda(5,5) = (5/2) \). Thus, by Lemma 1 and after simplification,
\[
\mathcal{RR}_{2}(G_{pq}^{3}) = 14pq + 6p + 6q + \frac{8}{3}. \tag{33}
\]

For \( \mathcal{RR}_{3}(G_{pq}^{3}) \) which is the third reverse redefined index of \( G_{pq}^{3} \) from equation (8), we have \( \lambda(\mathbf{R}_{p,q}, \mathbf{R}_{q}) = (\mathbf{R}_{p,q} + \mathbf{R}_{q})^{4} (\mathbf{R}_{p,q} \times \mathbf{R}_{q})^{4} \); therefore, \( \lambda(1,1) = 2 \), \( \lambda(1,5) = 30 \), and \( \lambda(5,5) = 250 \). Thus, by Lemma 1 and after simplification,
\[
\mathcal{RR}_{3}(G_{pq}^{3}) = 936pq + 744p + 744q + 384. \tag{34}
\]

3.2. Results for Case 2. We provide a formula that would be used to calculate any reverse degree topological descriptors of Case 2 for \( G_{pq}^{3} \).

Lemma 2. Let \( G_{pq}^{3} \) be a third-type \( p \)th chain hex-derived networks. Then,
\[
\nabla_{2}(G_{pq}^{3}) = 3pq(\lambda(1,1) + 2\lambda(1,5) + \lambda(5,5)) + 2(2p + q)(\lambda(5,5) - \lambda(1,1)) + 2(\lambda(1,1) - 2\lambda(1,5) + \lambda(5,5)). \tag{35}
\]

Proof. The graph \( G_{pq}^{3} \) contains 12pq edges and maximum degree in \( G_{pq}^{3} \) graph is 8. There are two types of reverse degree vertices in \( G_{pq}^{3} \) that are 1 and 5. Let us partition the edges of \( G_{pq}^{3} \) according to its reverse degrees according to Case 2 as
\[
E_{1,1} = \{ \theta \in E(G_{pq}^{3}) : \mathbf{R}_{p,q} = 1, \mathbf{R}_{q} = 1 \}, \tag{36}
\]
\[
E_{1,5} = \{ \theta \in E(G_{pq}^{3}) : \mathbf{R}_{p,q} = 1, \mathbf{R}_{q} = 5 \}, \tag{36}
\]
\[
E_{5,5} = \{ \theta \in E(G_{pq}^{3}) : \mathbf{R}_{p,q} = 5, \mathbf{R}_{q} = 5 \}. \tag{36}
\]

Note that \( E(G_{pq}^{3}) = E_{1,1} \cup E_{1,5} \cup E_{5,5} \) and \( |E_{1,1}| = 3pq - 4p - 2q + 2 \), \( |E_{1,5}| = 6pq - 4 \), and \( |E_{5,5}| = 3pq + 4p + 2q + 2 \). Hence,
\[
\nabla_{2}(G_{pq}^{3}) = \sum_{\theta \in E(G_{pq}^{3})} \lambda(\mathbf{R}_{p,q}, \mathbf{R}_{q}) \]
\[
= \sum_{\theta \in E_{1,1}} \lambda(1,1) + \sum_{\theta \in E_{1,5}} \lambda(1,5) + \sum_{\theta \in E_{5,5}} \lambda(5,5) \]
\[
= (3pq - 4p - 2q + 2)\lambda(1,1) + (6pq - 4)\lambda(1,5) + (3pq + 4p + 2q + 2)\lambda(5,5). \tag{37}
\]

After simplification, we obtain
\[ \mathcal{T}_2(G^3_{p,q}) = 3pq(\lambda(1,1) + 2\lambda(1,5) + \lambda(5,5)) + 2(2p + q)(\lambda(5,5) - \lambda(1,1)) + 2(\lambda(1,1) - 2\lambda(1,5) + \lambda(5,5)). \] (38)

**Theorem 4.** The general reverse Randić index of \(G^3_{p,q}\) is equal to

\[
\mathfrak{RR}_\alpha(G^3_{p,q}) = \begin{cases} 
108pq + 96p + 48q + 32, & \text{for } \alpha = 1, \\
(18 + 6\sqrt{5})pq + 16p + 8q + 12 - 4\sqrt{5}, & \text{for } \alpha = \frac{1}{2}, \\
\left(\frac{18}{5} + \frac{6\sqrt{5}}{5}\right)pq - \frac{16}{5} - \frac{8}{5}q + \frac{12}{5} - \frac{4\sqrt{5}}{5}, & \text{for } \alpha = -\frac{1}{2}, \\
108 \frac{pq}{25} - \frac{96}{25}p - \frac{48}{25}q + \frac{32}{25}, & \text{for } \alpha = -1.
\end{cases}
\] (39)

Proof. For \(\mathfrak{RR}_\alpha(G^3_{p,q})\) which is the general reverse Randić index of \(G^3_{p,q}\) from equation (2), we have \(\lambda(\mathfrak{R}_p, \mathfrak{R}_q) = (\mathfrak{R}_p \times \mathfrak{R}_q)^a\); therefore, \(\lambda(1,1) = 1\), \(\lambda(1,5) = (5)^a\), and \(\lambda(5,5) = (25)^a\). Thus, by Lemma 2,

\[ \mathfrak{RR}_\alpha(G^3_{p,q}) = 3pq(1 + 2(5)^a + (25)^a) + 2(2p + q)((25)^a - 1) + 2(1 - 2(5)^a + (25)^a). \] (40)

Put \(\alpha = 1\), and we have

\[ \mathfrak{RR}_1(G^3_{p,q}) = 108pq + 96p + 48q + 32. \] (41)

Put \(\alpha = (1/2)\), and we have

\[ \mathfrak{RR}_{1/2}(G^3_{p,q}) = (18 + 6\sqrt{5})pq + 16p + 8q + 12 - 4\sqrt{5}. \] (42)

Put \(\alpha = (-1/2)\), and we have

\[ \mathfrak{RR}_{-1/2}(G^3_{p,q}) = \left(\frac{18}{5} + \frac{6\sqrt{5}}{5}\right)pq - \frac{16}{5} - \frac{8}{5}q + \frac{12}{5} - \frac{4\sqrt{5}}{5}. \] (43)

Put \(\alpha = -1\), and we have

\[ \mathfrak{RR}_{-1}(G^3_{p,q}) = \frac{108}{25}pq - \frac{96}{25}p - \frac{48}{25}q + \frac{32}{25}. \] (44)

**Theorem 5.** Let \(G^3_{p,q}\) be a third-type \(p\)th chain hex-derived networks. Then, the reverse atom-bond connectivity index is

\[ \mathfrak{RABC}(G^3_{p,q}) = \left(\frac{12\sqrt{5}}{5} + \frac{6\sqrt{2}}{5}\right)pq - \frac{8\sqrt{5}}{5} + \frac{2(4p + 2q + 2)\sqrt{5}}{5}. \] (45)

The reverse geometric-arithmetic index is

\[ \mathfrak{RGA}(G^3_{p,q}) = (6 + 2\sqrt{5})pq + 4 - \frac{4\sqrt{5}}{3}. \] (46)

The first reverse Zagreb index is

\[ \mathfrak{RM}_2(G^3_{p,q}) = 72pq + 32p + 16q. \] (47)

The reverse hyper-Zagreb index is

\[ \mathfrak{RM}_h(G^3_{p,q}) = 528pq + 384p + 192q + 64. \] (48)

The reverse forgotten index is

\[ \mathfrak{RF}(G^3_{p,q}) = 312pq + 192p + 96q. \] (49)

Proof. For \(\mathfrak{RABC}(G^3_{p,q})\) which is the reverse atom-bond connectivity index of \(G^3_{p,q}\) from equation (3), we have \(\lambda(\mathfrak{R}_p, \mathfrak{R}_q) = \sqrt{(\mathfrak{R}_p \times \mathfrak{R}_q - 2)} / (\mathfrak{R}_p + \mathfrak{R}_q)\); therefore, \(\lambda(1,1) = 0\), \(\lambda(1,5) = \sqrt{4/5}\), and \(\lambda(5,5) = (\sqrt{8}/5)\). Thus, by Lemma 2 and after simplification,

\[ \mathfrak{RABC}(G^3_{p,q}) = \left(\frac{12\sqrt{5}}{5} + \frac{6\sqrt{2}}{5}\right)pq - \frac{8\sqrt{5}}{5} + \frac{2(4p + 2q + 2)\sqrt{5}}{5}. \] (50)

For \(\mathfrak{RGA}(G^3_{p,q})\) which is the reverse geometric-arithmetic index of \(G^3_{p,q}\) from equation (4), we have \(\lambda(\mathfrak{R}_p, \mathfrak{R}_q) = (2\sqrt{\mathfrak{R}_p \times \mathfrak{R}_q}) / (\mathfrak{R}_p + \mathfrak{R}_q)\); therefore, \(\lambda(1,1) = 1\), \(\lambda(1,5) = (\sqrt{5}/3)\), and \(\lambda(5,5) = 1\). Thus, by Lemma 2 and after simplification,

\[ \mathfrak{RGA}(G^3_{p,q}) = (6 + 2\sqrt{5})pq + 4 - \frac{4\sqrt{5}}{3}. \] (51)

For \(\mathfrak{RM}_2(G^3_{p,q})\) which is the first reverse Zagreb index of \(G^3_{p,q}\) from equation (5), we have \(\lambda(\mathfrak{R}_p, \mathfrak{R}_q) = (\mathfrak{R}_p + \mathfrak{R}_q)\); therefore, \(\lambda(1,1) = 2\), \(\lambda(1,5) = 6\), and \(\lambda(5,5) = 10\). Thus, by Lemma 2 and after simplification,
For $\mathcal{RH}^M(G_{p,q}^3)$ which is the first reverse hyper-Zagreb index of $G_{p,q}^3$, from equation (6), we have 
\[ \lambda(\mathcal{R}_G, \mathcal{R}_G) = (\mathcal{R}_G)^2; \] 
therefore, $\lambda(1, 1) = 4$, $\lambda(1, 5) = 36$, and $\lambda(5, 5) = 100$. Thus, by Lemma 2 and after simplification,
\[ \mathcal{RH}^M(G_{p,q}^3) = 528pq + 384p + 192q + 64. \] (53)

For $\mathcal{RF}(G_{p,q}^3)$ which is the reverse forgotten index of $G_{p,q}^3$, from equation (7), we have 
\[ \lambda(\mathcal{R}_G, \mathcal{R}_G) = ((\mathcal{R}_G)^2 + (\mathcal{R}_G)^2); \] 
therefore, $\lambda(1, 1) = 2$, $\lambda(1, 5) = 26$, and $\lambda(5, 5) = 50$. Thus, by Lemma 2 and after simplification,
\[ \mathcal{RF}(G_{p,q}^3) = 312pq + 192p + 96q. \] (54)

**Theorem 6.** Let $G_{p,q}^3$ be a third-type $p$th chain hex-derived networks. Then, the first reverse redefined index is
\[ \mathcal{RRZ}_1(G_{p,q}^3) = \frac{72pq}{5} - \frac{32p}{5} - \frac{16q}{5}. \] (55)

The second reverse redefined index is
\[ \mathcal{RRZ}_2(G_{p,q}^3) = 14pq + 8p + 4q + \frac{8}{3}. \] (56)

The third reverse redefined index is
\[ \mathcal{RRZ}_3(G_{p,q}^3) = 936pq + 992p + 496q + 384. \] (57)

**Proof.** For $\mathcal{RRZ}_1(G_{p,q}^3)$ which is the first reverse redefined index of $G_{p,q}^3$, from equation (8), we have 
\[ \lambda(\mathcal{R}_G, \mathcal{R}_G) = (\mathcal{R}_G)^2 \frac{1}{1} (\mathcal{R}_G \times \mathcal{R}_G)^{-1}; \] 
therefore, $\lambda(1, 1) = 2$, $\lambda(1, 5) = (6/5)$, and $\lambda(5, 5) = (2/5)$. Thus, by Lemma 2 and after simplification,
\[ \mathcal{RRZ}_1(G_{p,q}^3) = \frac{72pq}{5} - \frac{32p}{5} - \frac{16q}{5}. \] (58)

For $\mathcal{RRZ}_2(G_{p,q}^3)$ which is the second reverse redefined index of $G_{p,q}^3$, from equation (8), we have 
\[ \lambda(\mathcal{R}_G, \mathcal{R}_G) = (\mathcal{R}_G)^2 \frac{1}{1} (\mathcal{R}_G \times \mathcal{R}_G)^{-1}; \] 
therefore, $\lambda(1, 1) = (1/2)$, $\lambda(1, 5) = (5/6)$, and $\lambda(5, 5) = (5/2)$. Thus, by Lemma 2 and after simplification,
\[ \mathcal{RRZ}_2(G_{p,q}^3) = 14pq + 8p + 4q + \frac{8}{3}. \] (59)

For $\mathcal{RRZ}_3(G_{p,q}^3)$ which is the third reverse redefined index of $G_{p,q}^3$, from equation (8), we have 
\[ \lambda(\mathcal{R}_G, \mathcal{R}_G) = (\mathcal{R}_G)^2 + (\mathcal{R}_G)^2; \] 
therefore, $\lambda(1, 1) = 2$, $\lambda(1, 5) = 30$, and $\lambda(5, 5) = 250$. Thus, by Lemma 2 and after simplification,
\[ \mathcal{RRZ}_3(G_{p,q}^3) = 936pq + 992p + 496q + 384. \] (60)

3.3 Results for Case 3. We provide a formula that would be used to calculate any reverse degree topological descriptors of Case 3 for $G_{p,q}^3$.

**Lemma 3.** Let $G_{p,q}^3$ be a third-type $p$th chain hex-derived networks. Then,
\[ \mathcal{F}_3(G_{p,q}^3) = 3pq(\lambda(1, 1) + 2\lambda(5, 5) + \lambda(5, 5)) \]
\[ + 2(\lambda(5, 5) - \lambda(1, 1))(2p + q) \]
\[ + 2(\lambda(1, 1) - 4\lambda(1, 5) + 3\lambda(5, 5)). \] (61)

**Proof.** The graph $G_{p,q}^3$ contains 12pq edges, and maximum degree in $G_{p,q}^3$ graph is 8. There are two types of reverse degree vertices in $G_{p,q}^3$ that are 1 and 5. Let us partition the edges of $G_{p,q}^3$ according to its reverse degrees according to Case 3 as
\[ E_{1,1} = \{ \theta \in E(G_{p,q}^3) : \mathcal{R}_G = 1, \mathcal{R}_G = 1 \} \]
\[ E_{1,5} = \{ \theta \in E(G_{p,q}^3) : \mathcal{R}_G = 1, \mathcal{R}_G = 5 \} \]
\[ E_{5,5} = \{ \theta \in E(G_{p,q}^3) : \mathcal{R}_G = 5, \mathcal{R}_G = 5 \} \]
\[ E_{1,1} = \{ \theta \in E(G_{p,q}^3) : \mathcal{R}_G = 1, \mathcal{R}_G = 1 \} \]
\[ \|E_{1,1}\| = 3pq - 4p - 2q + 2 \]
\[ \|E_{1,5}\| = 6pq - 8 \]
\[ \|E_{5,5}\| = 3pq + 4p + 4q + 6 \]

Hence,
\[ \mathcal{F}_3(G_{p,q}^3) = \sum_{\theta \in E(G_{p,q}^3)} \lambda(\mathcal{R}_G, \mathcal{R}_G) \]
\[ = \sum_{\theta \in E_{1,1}} \lambda(1, 1) + \sum_{\theta \in E_{1,5}} \lambda(1, 5) + \sum_{\theta \in E_{5,5}} \lambda(5, 5) \]
\[ = (3pq - 4p - 2q + 2)\lambda(1, 1) + (6pq - 8)\lambda(1, 5) \]
\[ + (3pq + 4p + 4q + 6)\lambda(5, 5) \] (63)

After simplification, we obtain
\[ \mathcal{F}_3(G_{p,q}^3) = 3pq(\lambda(1, 1) + 2\lambda(5, 5) + \lambda(5, 5)) \]
\[ + 2(\lambda(5, 5) - \lambda(1, 1))(2p + q) + 2(\lambda(1, 1) - 4\lambda(1, 5) + 3\lambda(5, 5)). \] (64)

**Theorem 7.** The general reverse Randić index of $G_{p,q}^3$ is equal to
Proof. For $\mathfrak{R}_{R_1}(G^3_{pq})$ which is the general reverse Randić index of $G^3_{pq}$, from equation (2), we have

$$\mathfrak{R}_{R_1}(G^3_{pq}) = \begin{cases} 
108pq + 96p + 48q + 112, & \text{for } \alpha = 1, \\
(18 + 6\sqrt{5})pq + 16p + 8q + 32 - 8\sqrt{5}, & \text{for } \alpha = \frac{1}{2}, \\
(\frac{18}{5} + \frac{6\sqrt{5}}{5})pq - \frac{16p}{5} - \frac{8q}{5} + \frac{16}{5} + \frac{8\sqrt{5}}{5}, & \text{for } \alpha = -\frac{1}{2}, \\
108pq - \frac{96p}{25} - \frac{48q}{25} + 16 + \frac{16}{25} & \text{for } \alpha = -1.
\end{cases}$$

(65)

Put $\alpha = 1$, and we have

$$\mathfrak{R}_{R_1}(G^3_{pq}) = 108pq + 96p + 48q + 112. \quad (67)$$

Put $\alpha = (1/2)$, and we have

$$\mathfrak{R}_{R_{(1/2)}}(G^3_{pq}) = (18 + 6\sqrt{5})pq + 16p + 8q + 32 - 8\sqrt{5}. \quad (68)$$

Put $\alpha = (-1/2)$, and we have

$$\mathfrak{R}_{R_{(-1/2)}}(G^3_{pq}) = \left(\frac{18}{5} + \frac{6\sqrt{5}}{5}\right)pq - \frac{16p}{5} - \frac{8q}{5} + \frac{16}{5} + \frac{8\sqrt{5}}{5}. \quad (69)$$

The reverse geometric-arithmetic index is

$$\mathfrak{R}_{GA}(G^3_{pq}) = (6 + 2\sqrt{5})pq + 8 - \frac{8\sqrt{5}}{3}. \quad (72)$$

The first reverse Zagreb index is

$$\mathfrak{R}_{M_1}(G^3_{pq}) = 72pq + 32p + 16q + 16. \quad (73)$$

The reverse hyper-Zagreb index is

$$\mathfrak{R}_{HM}(G^3_{pq}) = 528pq + 384p + 192q + 320. \quad (74)$$

The reverse forgotten index is

$$\mathfrak{R}_{F}(G^3_{pq}) = 312pq + 192p + 96q + 96. \quad (75)$$

Proof. For $\mathfrak{R}_{ABC}(G^3_{pq})$ which is the reverse atom-bond connectivity index of $G^3_{pq}$, from equation (3), we have

$$\mathfrak{R}_{ABC}(G^3_{pq}) = \frac{(12\sqrt{5} + 6\sqrt{2})pq - 16\sqrt{5} + 2(4p + 2q + 6)\sqrt{2}}{5}. \quad (71)$$

Theorem 8. Let $G^3_{pq}$ be a third-type $p$th chain hex-derived networks. Then, the reverse atom-bond connectivity index is

$$\mathfrak{R}_{ABC}(G^3_{pq}) = \left(\frac{12\sqrt{5} + 6\sqrt{2}}{5}\right)pq - \frac{16\sqrt{5}}{5} + \frac{2(4p + 2q + 6)\sqrt{2}}{5}. \quad (76)$$

For $\mathfrak{R}_{GA}(G^3_{pq})$ which is the reverse geometric-arithmetic index of $G^3_{pq}$, from equation (4), we have

$$\mathfrak{R}_{GA}(G^3_{pq}) = \sqrt{\mathfrak{R}_{g} \mathfrak{R}_{g} - \frac{2}{\mathfrak{R}_{g}}} \mathfrak{R}_{g}; \text{ therefore, } \lambda(1,1) = 0, \lambda(1,5) = (\sqrt{4}/5), \text{ and } \lambda(5,5) = (\sqrt{8}/5). \text{ Thus, by Lemma 3 and after simplification,}
\mathfrak{R}_{ABC}(G^3_{pq}) = \left(\frac{12\sqrt{5} + 6\sqrt{2}}{5}\right)pq - \frac{16\sqrt{5}}{5} + \frac{2(4p + 2q + 6)\sqrt{2}}{5}. \quad (76)$$

(For $\mathfrak{R}_{GA}(G^3_{pq})$ which is the reverse geometric-arithmetic index of $G^3_{pq}$, from equation (4), we have

$$\mathfrak{R}_{GA}(G^3_{pq}) = \left(\frac{12\sqrt{5} + 6\sqrt{2}}{5}\right)pq - \frac{16\sqrt{5}}{5} + \frac{2(4p + 2q + 6)\sqrt{2}}{5}. \quad (76)$$

(77)
Lemma 3 and after simplification, therefore, \( \lambda \) \((G)\), \( \lambda \) \((R)\), and \( \lambda \) \((\theta)\) are all equal to 10. Thus, by Lemma 3 and after simplification, the first reverse redefined index is
\[
\mathcal{RF}(G^3_{pq}) = 312pq + 192p + 96q + 96.
\] (80)

\[ \square \]

**Theorem 9.** Let \( G_{pq}^3 \) be a third-type \( p \)th chain hex-derived networks. Then, the first reverse redefined index is
\[
\mathcal{RF}(G^3_{pq}) = \frac{72pq}{5} - \frac{32p}{5} + \frac{16q}{5} + \frac{16}{5}.
\] (81)

The second reverse redefined index is
\[
\mathcal{RF}(G^3_{pq}) = 14pq + 8p + 4q + \frac{28}{3}.
\] (82)

The third reverse redefined index is
Proof. For $\mathcal{RZ}_1(G_{p,q}^3)$ which is the first reverse redefined index of $G_{p,q}^3$, from equation (8), we have

$$\lambda(\mathcal{R}_p, \mathcal{R}_q) = (\mathcal{R}_p + \mathcal{R}_q)^{-1}(\mathcal{R}_p \times \mathcal{R}_q)^{-1};$$

therefore, $\lambda(1,1) = 2, \lambda(1,5) = (6/5)$, and $\lambda(5,5) = (2/5)$. Thus, by Lemma 3 and after simplification,

$$\mathcal{RZ}_1(G_{p,q}^3) = \frac{72pq}{5} - \frac{32p}{5} - \frac{16q}{5} - \frac{16}{5}. \quad (84)$$

For $\mathcal{RZ}_2(G_{p,q}^3)$ which is the second reverse redefined index of $G_{p,q}^3$, from equation (8), we have

$$\lambda(\mathcal{R}_p, \mathcal{R}_q) = (\mathcal{R}_p + \mathcal{R}_q)^{-1}(\mathcal{R}_p \times \mathcal{R}_q)^{-1};$$

therefore, $\lambda(1,1) = (1/2), \lambda(1,5) = (5/6)$, and $\lambda(5,5) = (5/2)$. Thus, by Lemma 3 and after simplification,

$$\mathcal{RZ}_2(G_{p,q}^3) = 14pq + 8p + 4q + \frac{28}{3}. \quad (85)$$

For $\mathcal{RZ}_3(G_{p,q}^3)$ which is the third reverse redefined index of $G_{p,q}^3$, from equation (8), we have

$$\lambda(\mathcal{R}_p, \mathcal{R}_q) = (\mathcal{R}_p + \mathcal{R}_q)^{-1}(\mathcal{R}_p \times \mathcal{R}_q)^{-1};$$

therefore, $\lambda(1,1) = 2, \lambda(1,5) = 30$, and $\lambda(5,5) = 250$. Thus, by Lemma 3 and after simplification,

$$\mathcal{RZ}_3(G_{p,q}^3) = 936pq + 992p + 496q + 1264. \quad (86)$$
4. Numerical and Graphical Representation

In this section, we determine the numerical values of $R_{ABC}$, $R_{GA}$, $R_{M1}$, $R_{HM}$, and $R_{F}$ in Tables 1–3, for Case 1, Case 2, and Case 3, respectively. We represent these results graphically in Figures 4–6.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


