Research Article

The Family of Multiparameter Quaternary Subdivision Schemes

Ghulam Mustafa ¹, **Muhammad Asghar** ¹, **Shafqat Ali** ¹, **Arzoo Qamar** ²,
and **Jia-Bao Liu** ³

¹Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan

²Department of Mathematics, NCBA & E Sub-Campus Bahawalpur, Bahawalpur, Pakistan

³School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China

Correspondence should be addressed to Jia-Bao Liu; liujiabaoad@163.com

Received 8 May 2021; Revised 28 September 2021; Accepted 19 October 2021; Published 14 December 2021

Academic Editor: Efthymios G. Tsionas

Copyright © 2021 Ghulam Mustafa et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In the field of subdivision, the smoothness increases as the arity of schemes increases. The family of high arity schemes gives high smoothness comparative to low arity schemes. In this paper, we propose a simple and generalized formula for a family of multiparameter quaternary subdivision schemes. The conditions for convergence of subdivision schemes are also presented. Moreover, we derive subdivision schemes after substituting the different values of parameters. We also analyzed the important properties of the proposed family of subdivision schemes. After comparison with existing schemes, we analyze that the proposed family of subdivision schemes gives better smoothness and approximation compared with the existing subdivision schemes.

1. Introduction

Subdivision schemes are the backbone of Computer Aided Geometric Design (CAGD). Subdivision schemes are used for the generation of smooth curves from the initial polygon. If the rules of subdivision schemes are four, then subdivision schemes are called quaternary subdivision schemes.

In 2009, a 4-point quaternary scheme is presented in [1]. The purposed scheme has C^3 -continuity. A family of quaternary schemes is presented in [2]. They used the Cox–De Boor recursion formula for the construction of quaternary schemes. In 2013, Ghaffar et al. [3] presented a generalized formula for the generation of 4-point subdivision schemes of binary, ternary, and quaternary subdivision schemes. In the same year, Amat and Liandrat [4] presented a 4-point scheme for the elimination of the Gibbs phenomenon.

In 2018, Pervaz [5] presented a 4-point quaternary scheme. They discuss the shape preserving properties of the subdivision scheme. Ashraf et al. [6, 7] presented and analyzed the geometrical properties of four point interpolating subdivision schemes. Hameed et al. [8] presented a 4-point subdivision scheme for regular curves and surfaces design.

Hussain et al. [9] presented a generalized formula for 5-point subdivision schemes of any arity. Khan et al. [10] presented a computational method for the generation of subdivision schemes. Conti and Romani [11] presented an algebraic technique for the generation of m -ary subdivision schemes. Romani [12] presented an algorithm for the generation of dual interpolating m -ary subdivision schemes. Romani and Viscardi [13] presented a new class of univariate stationary interpolating subdivision schemes of arity m . Recently, Mustafa et al. [14] presented a family of integer-point ternary parametric subdivision schemes.

1.1. Our Contributions. In the field of subdivision, as arity increases, the smoothness also increases. The main purpose of this work is to present a simple and generalized formula for derivation of multiparametric quaternary subdivision schemes based on Laurent polynomial. The conditions for the construction of subdivision schemes are also presented. Our schemes give better approximation and smoothness compared to the same type of existing subdivision schemes (see Figures 1 and 2).

FIGURE 1: (a–c) Comparison of limit curves for close polygons produced by 4-point schemes $\lambda_{3,3}$ and α_4^4 .

The paper is organized as follows. In Section 2, we present the general formula with different cases of a family of quaternary subdivision schemes. Analysis of the proposed family is presented in Section 3. Section 4 is for the comparison of the proposed family of subdivision schemes with existing subdivision schemes. Conclusions are drawn in Section 5.

2. General Formula for Multiparameter Family of Quaternary Subdivision Schemes

In this section, we present a general formula for the multiparameter family of quaternary approximating subdivision schemes based on Laurent polynomial. The general formula is

$$\lambda_{l,q}(z) = (1 + z + z^2 + z^3)^{l+1} (a_0 + a_1 z + a_2 z^2 + \cdots + a_q z^q). \quad (1)$$

The value of l controls the complexity and that of q controls the parameters in subdivision schemes. By using different values of l and q , we get the Laurent polynomial of family of $(l+1)$ -point quaternary $(q+1)$ parametric subdivision schemes. Here, we will discuss the different cases and conditions for family of quaternary subdivision schemes.

Case 1. By putting $l = 2, q = 1$ and $a_0 + a_1 = 1/16$, with $a_0 < 1/4, a_1 < 1/4$, in (1), we can obtain the Laurent polynomial of subdivision scheme

$$\begin{aligned} \lambda_{2,1}(z) = & a_1 z^{10} + (a_0 + 3a_1)z^9 + (3a_0 + 6a_1)z^8 + (6a_0 + 10a_1)z^7 \\ & + (10a_0 + 12a_1)z^6 + (12a_0 + 12a_1)z^5 + (12a_0 + 10a_1)z^4 + (10a_0 + 6a_1)z^3 \\ & + (6a_0 + 3a_1)z^2 + (a_1 + 3a_0)z + a_0. \end{aligned} \quad (2)$$

FIGURE 2: (a–c) Comparison of limit curves for close polygons produced by 4-point schemes $\lambda_{3,3}$ and KP .

The mask of the scheme corresponding to the Laurent polynomial $\lambda_{2,1}(z)$ is

$$\lambda_{2,1} = \left\{ a_1, (a_0 + 3a_1), (3a_0 + 6a_1), (6a_0 + 10a_1), (10a_0 + 12a_1), (12a_0 + 12a_1), \right. \\ \left. (12a_0 + 10a_1), (10a_0 + 6a_1), (6a_0 + 3a_1), (3a_0 + a_1), a_0 \right\}. \quad (3)$$

The scheme corresponding to mask (3) is

$$\begin{aligned} P_{4i}^{k+1} &= (10a_0 + 6a_1)P_{i-1}^k + (6a_0 + 10a_1)P_i^k, \\ P_{4i+1}^{k+1} &= (6a_0 + 3a_1)P_{i-1}^k + (10a_0 + 12a_1)P_i^k + a_1P_{i+1}^k, \\ P_{4i+2}^{k+1} &= (3a_0 + a_1)P_{i-1}^k + (12a_0 + 12a_1)P_i^k + (a_0 + 3a_1)P_{i+1}^k, \\ P_{4i+3}^{k+1} &= a_0P_{i-1}^k + (12a_0 + 10a_1)P_i^k + (3a_0 + 6a_1)P_{i+1}^k. \end{aligned} \quad (4)$$

Case 2. By setting $l = 2$, $q = 2$ and $a_0 + a_1 + a_2 = 1/16$ with $a_0 < 1/4$, $a_1 < 1/4$, $a_2 < 1/4$ in (2), we get the Laurent polynomial of 3-point scheme

$$\begin{aligned} \lambda_{2,2}(z) &= a_2z^{11} + (a_1 + 3a_2)z^{10} + (a_0 + 3a_1 + 6a_2)z^9 + (3a_0 + 6a_1 + 10a_2)z^8 \\ &\quad + (6a_0 + 10a_1 + 12a_2)z^7 + (10a_0 + 12a_1 + 12a_2)z^6 + (12a_0 + 12a_1 + 10a_2)z^5 \\ &\quad + (12a_0 + 10a_1 + 6a_2)z^4 + (10a_0 + 6a_1 + 3a_2)z^3 + (6a_0 + a_2 + 3a_1)z^2 \\ &\quad + (a_1 + 3a_0)z + a_0. \end{aligned} \quad (5)$$

The mask of the scheme corresponding to the Laurent polynomial (5) is

TABLE 1: The mask of the unified family of quaternary curve subdivision schemes corresponding to different values of l and q .

l, q	Complexity	Mask
2, 1	3-point	$\{a_1, (a_0 + 3a_1), (3a_0 + 6a_1), (6a_0 + 10a_1), (10a_0 + 12a_1),$ $(12a_0 + 12a_1), (12a_0 + 10a_1), (10a_0 + 6a_1), (6a_0 + 3a_1),$ $(3a_0 + a_1), a_0\}.$
2, 2	3-point	$\{a_2, (3a_2 + a_1), (a_0 + 3a_1 + 6a_2), (3a_0 + 6a_1 + 10a_2), (6a_0 + 10a_1 + 12a_2),$ $(10a_0 + 12a_1 + 12a_2), (12a_0 + 12a_1 + 10a_2), (12a_0 + 10a_1 + 6a_2), (10a_0 + 6a_1 + 3a_2),$ $(6a_0 + a_2 + 3a_1), (a_1 + 3a_0), a_0\}.$
3, 2	4-point	$\{a_2, (4a_2 + a_1), (a_0 + 4a_1 + 10a_2), (4a_0 + 10a_1 + 20a_2), (10a_0 + 20a_1 + 31a_2),$ $(20a_0 + 31a_1 + 40a_2), (31a_0 + 40a_1 + 44a_2), (40a_0 + 44a_1 + 40a_2),$ $(44a_0 + 40a_1 + 31a_2), (40a_0 + 20a_2 + 31a_1), (20a_1 + 31a_0 + 10a_2),$ $(20a_0 + 10a_1 + 4a_2), (10a_0 + 4a_1 + a_2), (4a_0 + a_1), a_0\}.$
3, 3	4-point	$\{a_3, (4a_3 + a_2), (a_1 + 4a_2 + 10a_3), (a_0 + 4a_1 + 10a_2 + 20a_3), (4a_0 + 10a_1 + 20a_2 + 31a_3),$ $(10a_0 + 20a_1 + 31a_2 + 40a_3), (20a_0 + 31a_1 + 40a_2 + 44a_3), (31a_0 + 40a_1 + 44a_2 + 40a_3),$ $(40a_0 + 44a_1 + 40a_2 + 31a_3), (44a_0 + 40a_1 + 31a_2 + 20a_3), (40a_0 + 31a_1 + 20a_2 + 10a_3),$ $(31a_0 + 20a_1 + 10a_2 + 4a_3), (20a_0 + 10a_1 + 4a_2 + a_3), (10a_0 + 4a_1 + a_2), (4a_0 + a_1), a_0\}.$

TABLE 2: The degree of generation of family of unified quaternary curve subdivision scheme for different cases.

Cases	l, q	D_g	Values of $a_{q'}s$	τ	Parametrization
1	$l = 2, q = 1$	2	$a_0 = -(3/32), a_1 = 5/32$ $a_0 = 5/32, a_1 = -(3/32)$	7 3	Primal Primal
2	$l = 2, q = 2$	2	$a_0 = a_2 = -(15/128), a_1 = 19/64$	$\frac{11}{2}$	Dual
3	$l = 3, q = 2$	3	$a_0 = a_2 = -(5/128), a_1 = 3/32$	$\frac{7}{2}$	Primal
4	$l = 3, q = 3$	3	$a_0 = a_3 = w, a_1 = a_2 = 1/128 - w$	$\frac{15}{2}$	Dual

$$\lambda_{2,2} = \left\{ \begin{array}{c} a_2, (3a_2 + a_1), (a_0 + 3a_1 + 6a_2), (3a_0 + 6a_1 + 10a_2), (6a_0 + 10a_1 + 12a_2), \\ (10a_0 + 12a_1 + 12a_2), (12a_0 + 12a_1 + 10a_2), (12a_0 + 10a_1 + 6a_2), (10a_0 + 6a_1 + 3a_2), \\ (6a_0 + a_2 + 3a_1), (a_1 + 3a_0), a_0 \end{array} \right\}. \quad (6)$$

The scheme corresponding to mask (6) is

$$\begin{aligned} P_{4i}^{k+1} &= (10a_0 + 6a_1 + 3a_2)P_{i-1}^k + (6a_0 + 10a_1 + 12a_2)P_i^k + a_2P_{i+1}^k, \\ P_{4i+1}^{k+1} &= (6a_0 + 3a_1 + a_2)P_{i-1}^k + (10a_0 + 12a_1 + 12a_2)P_i^k + (a_1 + 3a_2)P_{i+1}^k, \\ P_{4i+2}^{k+1} &= (3a_0 + a_1)P_{i-1}^k + (12a_0 + 12a_1 + 10a_2)P_i^k + (a_0 + 3a_1 + 6a_2)P_{i+1}^k, \\ P_{4i+3}^{k+1} &= a_0P_{i-1}^k + (6a_2 + 10a_1 + 12a_2)P_i^k + (3a_0 + 6a_1 + 10a_2)P_{i+1}^k. \end{aligned} \quad (7)$$

Case 3. By setting $l = 3, q = 2$, and $a_0 + a_1 + a_2 = 1/64$, with $a_0 < 1/4, a_1 < 1/4, a_2 < 1/4$ in (2), we can obtain the Laurent polynomial of 4-point scheme

$$\begin{aligned} \lambda_{3,2}(z) &= a_2z^{14} + (a_1 + 4a_2)z^{13} + (a_0 + 4a_1 + 10a_2)z^{12} + (4a_0 + 10a_1 + 20a_2)z^{11} \\ &\quad + (10a_0 + 20a_1 + 31a_2)z^{10} + (20a_0 + 31a_1 + 40a_2)z^9 + (31a_0 + 40a_1 + 44a_2)z^8 \\ &\quad + (40a_0 + 44a_1 + 40a_2)z^7 + (44a_0 + 40a_1 + 31a_2)z^6 + (40a_0 + 20a_2 + 31a_1)z^5 \\ &\quad + (31a_0 + 20a_1 + 10a_2)z^4 + (20a_0 + 10a_1 + 4a_2)z^3 + (10a_0 + 4a_1 + a_2)z^2 \\ &\quad + (4a_0 + a_1)z + a_0. \end{aligned} \quad (8)$$

The mask of the scheme corresponding to the Laurent polynomial (8) is

$$\lambda_{3,2} = \left\{ \begin{array}{c} a_2, (4a_2 + a_1), (a_0 + 4a_1 + 10a_2), (4a_0 + 10a_1 + 20a_2), (10a_0 + 20a_1 + 31a_2), \\ (20a_0 + 31a_1 + 40a_2), (31a_0 + 40a_1 + 44a_2), (40a_0 + 44a_1 + 40a_2), (44a_0 + 40a_1 + 31a_2), \\ (40a_0 + 20a_2 + 31a_1), (20a_1 + 31a_0 + 10a_2), (20a_0 + 10a_1 + 4a_2), (10a_0 + 4a_1 + a_2), (4a_0 + a_1), a_0 \end{array} \right\}. \quad (9)$$

The scheme corresponding to mask (9) is

$$\begin{aligned}
P_{4i}^{k+1} &= (20a_0 + 10a_1 + 4a_2)P_{i-1}^k + (40a_0 + 44a_1 + 40a_2)P_i^k + (4a_0 + 10a_1 + 20a_2)P_{i+1}^k, \\
P_{4i+1}^{k+1} &= (10a_0 + 4a_1 + a_2)P_{i-1}^k + (44a_0 + 40a_1 + 31a_2)P_i^k + (10a_0 + 20a_1 + 31a_2)P_{i+1}^k + a_2P_{i+2}^k, \\
P_{4i+2}^{k+1} &= (4a_0 + a_1)P_{i-1}^k + (40a_0 + 20a_2 + 31a_1)P_i^k + (20a_0 + 31a_1 + 40a_2)P_{i+1}^k + (4a_2 + a_1)P_{i+2}^k, \\
P_{4i+3}^{k+1} &= a_0P_{i-1}^k + (20a_1 + 31a_0 + 10a_2)P_i^k + (31a_0 + 40a_1 + 44a_2)P_{i+1}^k + (a_0 + 4a_1 + 10a_2)P_{i+2}^k.
\end{aligned} \tag{10}$$

Case 4. By putting $l = 3, q = 3$, and $a_0 + a_1 + a_2 + a_3 = 1/64$, with $a_0 < 1/4, a_1 < 1/4, a_2 < 1/4, a_3 < 1/4$ in (2), we can obtain the Laurent polynomial of 4-point scheme

$$\begin{aligned}
\lambda_{3,3}(z) &= a_3z^{15} + (a_2 + 4a_3)z^{14} + (a_1 + 4a_2 + 10a_3)z^{13} + (a_0 + 4a_1 + 10a_2 + 20a_3)z^{12} \\
&\quad + (4a_0 + 10a_1 + 20a_2 + 31a_3)z^{11} + (10a_0 + 20a_1 + 31a_2 + 40a_3)z^{10} \\
&\quad + (20a_0 + 40a_2 + 44a_3 + 31a_1)z^9 + (31a_0 + 40a_1 + 44a_2 + 40a_3)z^8 \\
&\quad + (40a_0 + 44a_1 + 40a_2 + 31a_3)z^7 + (44a_0 + 31a_2 + 40a_1 + 20a_3)z^6 \\
&\quad + (31a_1 + 20a_2 + 10a_3 + 40a_0)z^5 + (20a_1 + 10a_2 + 4a_3 + 31a_0)z^4 \\
&\quad + (20a_0 + 10a_1 + 4a_2 + a_3)z^3 + (4a_1 + 10a_0 + a_2z)^2 + (4a_0 + a_1)z + a_0.
\end{aligned} \tag{11}$$

The mask of the scheme corresponding to the Laurent polynomial (11) is

$$\lambda_{3,3} = \left\{ \begin{array}{l} a_3, (4a_3 + a_2), (a_1 + 4a_2 + 10a_3), (a_0 + 4a_1 + 10a_2 + 20a_3), (4a_0 + 10a_1 + 20a_2 + 31a_3), \\ (10a_0 + 20a_1 + 31a_2 + 40a_3), (20a_0 + 31a_1 + 40a_2 + 44a_3), (31a_0 + 40a_1 + 44a_2 + 40a_3), \\ (40a_0 + 44a_1 + 40a_2 + 31a_3), (44a_0 + 40a_1 + 31a_2 + 20a_3), (40a_0 + 31a_1 + 20a_2 + 10a_3), \\ (31a_0 + 20a_1 + 10a_2 + 4a_3), (20a_0 + 10a_1 + 4a_2 + a_3), (10a_0 + 4a_1 + a_2), (4a_0 + a_1), a_0 \end{array} \right\}. \tag{12}$$

The scheme corresponding to the mask is

$$\begin{aligned}
P_{4i}^{k+1} &= (20a_0 + 10a_1 + 4a_2 + a_3)P_{i-1}^k + (40a_0 + 44a_1 + 40a_2 + 31a_3)P_i^k \\
&\quad + (4a_0 + 10a_1 + 20a_2 + 31a_3)P_{i+1}^k + a_3P_{i+2}^k, \\
P_{4i+1}^{k+1} &= (10a_0 + 4a_1 + a_2)P_{i-1}^k + (44a_0 + 40a_1 + 31a_2 + 20a_3)P_i^k + (10a_0 + 20a_1 + 31a_2 + 40a_3)P_{i+1}^k \\
&\quad + (4a_3 + a_2)P_{i+2}^k, \\
P_{4i+2}^{k+1} &= (4a_0 + a_1)P_{i-1}^k + (40a_0 + 31a_1 + 20a_2 + 10a_3)P_i^k + (20a_0 + 31a_1 + 40a_2 + 40a_3 + 44a_3)P_{i+1}^k \\
&\quad + (4a_2 + a_1 + 10a_3)P_{i+2}^k, \\
P_{4i+3}^{k+1} &= a_0P_{i-1}^k + (31a_0 + 20a_1 + 10a_2 + 4a_3)P_i^k + (31a_0 + 40a_1 + 44a_2 + 40a_3)P_{i+1}^k \\
&\quad + (a_0 + 4a_1 + 10a_2 + 20a_3)P_{i+2}^k.
\end{aligned} \tag{13}$$

Scheme (13) is the general 4-point quaternary scheme with 4 parameters.

Similarly for different values of l and q , we get the $l+1$ -point quaternary approximating subdivision schemes

having $(q + 1)$ parameters. In Table 1, we present the mask of family members of quaternary schemes for different values of l and q .

3. Analysis of the Unified Family of Quaternary Curve Subdivision Schemes

This section contains the analysis of important properties of the proposed subdivision schemes. For this, we consider the

4-point scheme. After substituting the values of $a_0 = w$, $a_1 = 1/128 - w$, $a_2 = 1/128 - w$, and $a_3 = w$ in (13), we get a 4-point parametric scheme

$$\begin{aligned} P_{4i}^{k+1} &= \left(7w + \frac{7}{64}\right)P_{i-1}^k + \left(\frac{21}{32} - 13w\right)P_i^k + \left(5w + \frac{15}{64}\right)P_{i+1}^k + wP_{i+2}^k, \\ P_{4i+1}^{k+1} &= \left(5w + \frac{5}{128}\right)P_{i-1}^k + \left(\frac{71}{128} - 7w\right)P_i^k + \left(\frac{51}{128} - w\right)P_{i+1}^k + \left(3w + \frac{1}{128}\right)P_{i+2}^k, \\ P_{4i+2}^{k+1} &= \left(3w + \frac{1}{128}\right)P_{i-1}^k + \left(\frac{51}{128} - w\right)P_i^k + \left(\frac{71}{128} - 7w\right)P_{i+1}^k + \left(5w + \frac{5}{128}\right)P_{i+2}^k, \\ P_{4i+3}^{k+1} &= wP_{i-1}^k + \left(5w + \frac{15}{64}\right)P_i^k + \left(\frac{21}{32} - 13w\right)P_{i+1}^k + \left(7w + \frac{7}{64}\right)P_{i+2}^k. \end{aligned} \quad (14)$$

The Laurent polynomial corresponding to scheme (14) is

$$\lambda(z) = \frac{1}{128}(1 + z + z^2 + z^3)^4(1 + z)(128w + (1 - 256w)z + 128wz^2), \quad (15)$$

The Laurent polynomial method [15] is used to compute the degree of generation, degree of reproduction, and continuity analysis. Moreover, Rioul's method [16] is used to compute lower and upper bounds on Hölder regularity of scheme (14). The analysis of other schemes is similar.

Theorem 1. A 4-point quaternary subdivision scheme (14) has cubic reproduction with respect to the dual parametrization for $w = -(21/1024)$.

Proof. By taking the derivative of (15) with respect to z , we get

$$\begin{aligned} \lambda'(z) &= 4(1 + z + z^2 + z^3)^3(1 + 2z + 3z^2)\left(w + \left(\frac{1}{128} - w\right)z + \left(\frac{1}{128} - w\right)z^2 + wz^3\right) \\ &\quad + (1 + z + z^2 + z^3)^4\left(\left(\frac{1}{128} - w\right) + 2\left(\frac{1}{128} - w\right)z + 3wz^2\right), \end{aligned} \quad (16)$$

After substituting $z = 1$ in (15) and (16), we get $\lambda(1) = 4$ and $\lambda'(1) = 30$. The value of shift parameter $\tau = \lambda'(1)/4 = 15/2$. Hence, by [15], the subdivision scheme (14) has dual parametrization. Further, we can easily verify that

$$\lambda^k(1) = 4 \prod_{j=0}^{k-1} \left(\frac{15}{2} - j\right) \quad \text{for } k = 0, 1, 2, 3 \text{ with } w = -\frac{21}{1024}, \quad (17)$$

Hence, by [15], the scheme corresponding to $\lambda(z)$ has cubic reproduction with respect to the dual parametrization. $\square$

Table 2 summarizes the results of degree of generation, values of parameters, shift parameter, and parametrization of a

proposed family of quaternary subdivision schemes. Here, l , q , D_g , values of $a_{q's}$, τ , and parametrization denote the positive integer, degree of generation, parameter values, shift parameter, and parametrization of the scheme, respectively.

Theorem 2. A 4-point quaternary subdivision scheme (14) has C^3 continuity for $w \in (-1/128)(2/128)$.

Proof. Consider the Laurent polynomial

$$\lambda^1(z) = \left(\frac{4z^3}{1 + z + z^2 + z^3}\right)^4 \lambda(z), \quad (18)$$

where $\lambda(z)$ is defined in (15). This implies

$$\lambda^1(z) = \left(\frac{4z^3}{1+z+z^2+z^3} \right)^4 (1+z+z^2+z^3)^4 \left(w + \left(\frac{1}{128} - w \right) z + \left(\frac{1}{128} - w \right) z^2 + wz^3 \right), \quad (19)$$

After simplification, we get

$$\lambda^1(z) = 256z^{12} \left[w + \left(\frac{1}{128} - w \right) z + \left(\frac{1}{128} - w \right) z^2 + wz^3 \right]. \quad (20)$$

Let λ^1 be the mask of the scheme S^1 corresponding to $\lambda^1(z)$, then we have

$$\lambda^1 = 256 \left[w, \frac{1}{128} - w, \frac{1}{128} - w, w \right], \quad (21)$$

The scheme corresponding to $\lambda(z)$ is C^3 continuous if $\|1/4S^1\|_\infty < 1$; for this, we have to check that

$$\left\| \frac{1}{4}S^1 \right\|_\infty = \max \frac{1}{4} \{ |256w|, |2-256w|, |2-256w|, |256w| \}. \quad (22)$$

If $w \in (-1/128, 2/128)$, then $\|1/4S^1\|_\infty < 1$. Then, by [15], the scheme corresponding to $\lambda(z)$ has C^3 continuity, which completes the proof. $\square$

Theorem 3. The Hölder regularity of a 4-point scheme (14) is $r = 4 - \log_4(\mu)$, where μ is defined as

$$\begin{cases} \mu = 2 - 256w, & \text{if } -\frac{1}{128} < w \leq \frac{1}{256}, \\ \mu = 256w, & \text{if } \frac{1}{256} < w < \frac{2}{128}. \end{cases} \quad (23)$$

Proof. The Laurent polynomial (15) can be written as

$$\lambda(z) = \left(\frac{1+z+z^2+z^3}{4} \right)^4 b(z), \quad (24)$$

where

$$b(z) = [256w + (2-256w)z + (2-256w)z^2 + 256wz^3]. \quad (25)$$

From (25), the coefficients of z in $b(z)$ are $b_0 = 256w$, $b_1 = 2-256w$, $b_2 = 2-256w$, and $b_3 = 256w$. The number of factors in $\lambda(z)$ is $k = 4$. The matrices B_n has order 3×3 where $n = 0, 1, 2$ and 3 . The elements of the matrices B_0, B_1, B_2 , and B_3 can be derived by $(B_n)_{ij} = b_{(3+n)+i-4j}$, for $i, j = 1, 2$, and 3 ; then, we have

$$B_0 = \begin{pmatrix} 256w & 0 & 0 \\ 2-256w & 0 & 0 \\ 2-256w & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} B_1 &= \begin{pmatrix} 2-256w & 0 & 0 \\ 2-256w & 0 & 0 \\ 256w & 0 & 0 \end{pmatrix}, \\ B_2 &= \begin{pmatrix} 2-256w & 0 & 0 \\ 256w & 0 & 0 \\ 0 & 256w & 0 \end{pmatrix}, \\ B_3 &= \begin{pmatrix} 256w & 0 & 0 \\ 0 & 256w & 0 \\ 0 & 2-256w & 0 \end{pmatrix}. \end{aligned} \quad (26)$$

The eigenvalues of B_0, B_1, B_2 , and B_3 are $\{0, 0, 256w\}$, $\{0, 0, 2-256w\}$, $\{0, 0, 2-256w\}$, and $\{0, 256w, 256w\}$, respectively. For bounds on Hölder regularity, we calculate $\max\{\rho(B_0), \rho(B_1), \rho(B_2), \rho(B_3)\} \leq \mu \leq \max\{\|B_0\|, \|B_1\|, \|B_2\|, \|B_3\|\}$, with $\|\cdot\|$ denoting the infinity norm, since μ is bounded from below by the spectral radii and from above by the infinity norm of the matrices B_0, B_1, B_2, B_3 . So $\max\{\rho(B_3), \rho(B_1), \rho(B_2), \rho(B_3)\} = \max(|2-256w|, |256w|)$ and $\max(\|B_0\|, \|B_1\|, \|B_2\|, \|B_3\|) = \max(|2-256w|, |256w|)$. Then by [16], we have $\mu = \max(|2-256w|, |256w|)$. So Hölder regularity of the scheme S is computed by $r = 4 - \log_4(\mu)$, where μ is defined as

$$\begin{cases} \mu = 2 - 256w & \text{if } -\frac{1}{128} < w \leq \frac{1}{256}, \\ \mu = 256w & \text{if } \frac{1}{256} < w < \frac{2}{128}, \end{cases} \quad (27)$$

which completes the proof. $\square$

Corollary 1. The 4-point scheme (14) is C^3 continuous if and only if $1 \leq \mu < 4$, i.e., if and only if $-1/128 < w < 2/128$.

Theorem 4. The limit stencils providing the evaluations of the basic limit function of the 4-point scheme (14) at integers and half integers are $\{(128w^2/15 + 7w/12 + 1/5120), (31w/5 - 512w^2/15 + 223/1280), (256w^2/5 - 407w/30 + 1667/2560), (31w/5 - 512w^2/15 + 223/1280), (128w^2/15 + 7w/12 + 1/5120)\}$ and $\{(64w/15 + 1/40), (19/40 - 64w/15), (19/40 - 64w/15), (64w/15 + 1/40)\}$, respectively.

Proof. The local subdivision matrices for limit stencils of 4-point scheme (14) at integers and half integers are $P_I^{k+1} = S_I P_I^k$ and $P_{I/2}^{k+1} = S_{I/2} P_{I/2}^k$, respectively, with

$$S_I = \begin{bmatrix} \frac{1}{128} + 3w & \frac{51}{128} - w & \frac{71}{128} - 7w & \frac{5}{128} + 5w & 0 \\ w & \frac{15}{64} + 5w & \frac{21}{32} - 13w & \frac{7}{64} + 7w & 0 \\ 0 & \frac{7}{64} + 7w & \frac{21}{32} - 13w & \frac{15}{64} + 5w & w \\ 0 & \frac{5}{128} + 5w & \frac{71}{128} - 7w & \frac{51}{128} - w & \frac{1}{128} + 3w \end{bmatrix}, \quad (28)$$

$$P_I^{k+1} = \begin{bmatrix} P_{-2}^{k+1} \\ P_{-1}^{k+1} \\ P_0^{k+1} \\ P_1^{k+1} \end{bmatrix}, \quad P_I^k = \begin{bmatrix} P_{-2}^k \\ P_{-1}^k \\ P_0^k \\ P_1^k \\ P_2^k \end{bmatrix},$$

$$S_{I/2} = \begin{bmatrix} 7w + \frac{7}{64} & \frac{21}{32} - 13w & 5w + \frac{15}{64} & w \\ 5w + \frac{5}{128} & \frac{71}{128} - 7w & \frac{51}{128} - w & 3w + \frac{1}{128} \\ 3w + \frac{1}{128} & \frac{51}{128} - w & \frac{71}{128} - 7w & 5w + \frac{5}{128} \\ w & 5w + \frac{15}{64} & \frac{21}{32} - 13w & 7w + \frac{7}{64} \end{bmatrix}, \quad (29)$$

$$P_{I/2}^{k+1} = \begin{bmatrix} P_{-(3/2)}^{k+1} \\ P_{-(1/2)}^{k+1} \\ P_{1/2}^{k+1} \\ P_{3/2}^{k+1} \end{bmatrix} \text{ and } P_{I/2}^k = \begin{bmatrix} P_{-(3/2)}^k \\ P_{-(1/2)}^k \\ P_{1/2}^k \\ P_{3/2}^k \end{bmatrix}.$$

The eigenvalues of the matrix $S_{I/2}$ are $\{1/16, 1, 1/64, 1/4\}$. The eigenvectors of local subdivision matrix $S_{I/2}$ corresponding to eigenvalues are

$$Q_{I/2} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ \frac{512w+3}{512w-57} & 1 & -\frac{128w+2}{384w-9} & -\frac{1}{3} \\ \frac{512w+3}{512w-57} & 1 & \frac{128w+2}{384w-9} & \frac{1}{3} \\ 1 & 1 & 1 & 1 \end{bmatrix}. \quad (30)$$

The inverse of $Q_{I/2}$ is

$$Q_{I/2}^{-1} = \begin{bmatrix} \frac{19}{40} - \frac{64w}{15} & -\frac{19}{40} + \frac{64w}{15} & -\frac{19}{40} + \frac{64w}{15} & \frac{19}{40} - \frac{64w}{15} \\ \frac{1}{40} + \frac{64w}{15} & \frac{19}{40} - \frac{64w}{15} & \frac{19}{40} - \frac{64w}{15} & \frac{1}{40} - \frac{64w}{15} \\ -\frac{3}{10} + \frac{64w}{5} & \frac{9}{10} - \frac{192w}{5} & -\frac{9}{10} + \frac{192w}{5} & \frac{3}{10} - \frac{64w}{5} \\ -\frac{1}{5} - \frac{64w}{5} & -\frac{9}{10} + \frac{192w}{5} & \frac{9}{10} - \frac{192w}{5} & \frac{1}{5} + \frac{64w}{5} \end{bmatrix}. \quad (31)$$

For the decomposition of matrix $S_{I/2}$, we need $\Delta_{I/2}$, where $\Delta_{I/2}$ is the scalar matrix in which eigenvalues are arranged diagonally; therefore, we now compute $\lim_{k \rightarrow \infty} \Delta_{I/2}^k$

$$\Delta_{I/2}^k = \begin{bmatrix} \left(\frac{1}{16}\right)^k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \left(\frac{1}{64}\right)^k & 0 \\ 0 & 0 & 0 & \left(\frac{1}{4}\right)^k \end{bmatrix} \quad \text{and} \quad \lim_{k \rightarrow \infty} \Delta_{I/2}^k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (32)$$

$P_{I/2}^{k+1} = S_{I/2} P_{I/2}^k$; therefore, $P_{I/2}^{k+1} = S_{I/2}^k P_{I/2}^0$. This implies

$$\begin{bmatrix} P_{-(3/2)}^\infty \\ P_{-(1/2)}^\infty \\ P_{1/2}^\infty \\ P_{3/2}^\infty \end{bmatrix} = \begin{bmatrix} \frac{1}{40} + \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{1}{40} + \frac{64}{15}w \\ \frac{1}{40} + \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{1}{40} + \frac{64}{15}w \\ \frac{1}{40} + \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{1}{40} + \frac{64}{15}w \\ \frac{1}{40} + \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{1}{40} + \frac{64}{15}w \end{bmatrix} \begin{bmatrix} P_{-(3/2)}^0 \\ P_{-(1/2)}^0 \\ P_{1/2}^0 \\ P_{3/2}^0 \end{bmatrix}. \quad (33)$$

Since the subdivision scheme is dual, after computing the limit stencil at half integers by the local matrix $S_{I/2}$, the limit stencil at integers must be computed as

$$\text{limit stencil integers} = \text{limit stencil half integers} \times S_I. \quad (34)$$

with

$$S_I^\infty = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \\ l_1 & l_2 & l_3 & l_4 & l_5 \\ l_1 & l_2 & l_3 & l_4 & l_5 \\ l_1 & l_2 & l_3 & l_4 & l_5 \end{bmatrix}, \quad (36)$$

The matrix of limit stencils at half integers is

$$S_{I/2}^\infty = \begin{bmatrix} \frac{1}{40} + \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{1}{40} + \frac{64}{15}w \\ \frac{1}{40} + \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{1}{40} + \frac{64}{15}w \\ \frac{1}{40} + \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{1}{40} + \frac{64}{15}w \\ \frac{1}{40} + \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{19}{40} - \frac{64}{15}w & \frac{1}{40} + \frac{64}{15}w \end{bmatrix}. \quad (35)$$

After multiplying the matrix of limit stencil at half integers $S_{I/2}^\infty$ with local subdivision matrix S_I , we get

$$\begin{aligned} l_1 &= \frac{128w^2}{15} + \frac{7w}{12} + \frac{1}{5120}, \\ l_2 &= \frac{31w}{5} - \frac{512w^2}{15} + \frac{223}{1280}, \\ l_3 &= \frac{256w^2}{5} - \frac{407w}{30} + \frac{1667}{2560}, \\ l_4 &= \frac{31w}{5} - \frac{512w^2}{15} + \frac{223}{1280}, \\ l_5 &= \frac{128w^2}{15} + \frac{7w}{12} + \frac{1}{5120}. \end{aligned} \quad (37)$$

Hence, the limit stencils providing the evaluations of the basic limit functions of the 4-point scheme (14) at integers and half integers are

FIGURE 3: Basic limit functions produced by the 4-point scheme corresponding to (14) for different values of parameter w . The red lines denote the initial polyline, and the blue, black, and green lines represent the basic limit functions produced with $w = -1/256, 0, 1/256$ respectively. The circles denote the evaluations of the basic limit function at integers, and the asterisks denote the evaluations of the basic limit function at half integers.

$$\left\{ \begin{array}{l} (128w^2/15 + 7w/12 + 1/5120), (31w/5 - 512w^2/15 + 223/1280), (256w^2/5 - 407w/30 + 1667/2560), \\ (31w/5 - 512w^2/15 + 223/1280), (128w^2/15 + 7w/12 + 1/5120) \end{array} \right\}, \quad (38)$$

$$\{(64w/15 + 1/40), (19/40 - 64w/15), (19/40 - 64w/15), (64w/15 + 1/40)\}$$

respectively, which completes the proof. $\square$

In Figure 3, we present the basic limit functions of the proposed 4-point quaternary approximating subdivision scheme for different values of w and show its evaluations at integers and half integers which coincide with the limit stencils computed in Theorem 4.

4. Comparison with Existing Schemes

Here we will present the comparison of our proposed family of quaternary subdivision schemes with existing quaternary subdivision schemes in visual performance. In Figure 1, we present the comparison of proposed 4-point scheme $\lambda_{3,3}$ with 4-point scheme α_4^4 presented in [1] ((a), (b)&(c)) and 4-point scheme presented in [4] ((d), (e)&(f)), respectively. Here, black dotted lines show the initial polygon, red solid lines are the limit curves of 4-point scheme $\lambda_{3,3}$, and blue solid lines are the limit curves of 4-point scheme α_4^4 presented in [1] and 4-point scheme presented in [4]. We see that, our proposed schemes $\lambda_{3,3}$ give maximum smoothness and best approximation compared with the schemes presented in [1, 4].

In Figure 2, we present the comparison of proposed 4-point scheme $\lambda_{3,3}$ with 4-point scheme KP presented in [5]. Here, black dotted lines show the initial polygon, red solid lines are the limit curve of 4-point scheme $\lambda_{3,3}$, and blue solid lines are the limit curve of 4-point scheme KP

presented in [5]. We see that the approximating scheme KP presented in [5] gives interpolating behavior, but our proposed schemes $\lambda_{3,3}$ give maximum smoothness and best approximation compared with the schemes presented in [5].

5. Conclusions

In this paper, we have presented a general formula for the derivation of multiparametric family of quaternary subdivision schemes. We present the complete analysis of the proposed family of the multiparametric quaternary subdivision schemes. We also present the comparison with existing quaternary subdivision schemes. The comparison shows that our proposed family gives maximum smoothness compared with existing quaternary subdivision schemes.

Data Availability

The data used to support the findings of the study are available within this paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this study.

References

- [1] G. Mustafa and F. Khan, "A new 4-point C^3 quaternary approximating subdivision scheme," *Abstract and Applied Analysis*, vol. 2009, Article ID 301967, 14 pages, 2009.
- [2] S. S. Siddiqi and M. Younis, "The m-point quaternary approximating subdivision schemes," *American Journal of Computational Mathematics*, vol. 3, pp. 6–10, 2013.
- [3] A. Ghaffar, G. Mustafa, and K. Qin, "The 4-point a-ary approximating subdivision scheme," *Open Journal of Applied Sciences*, vol. 3, pp. 106–111, 2013.
- [4] S. Amat and J. Liandrat, "On a nonlinear 4-point quaternary approximating subdivision scheme eliminating the Gibbs phenomenon," *SeMA Journal*, vol. 62, pp. 15–25, 2013.
- [5] K. Pervez, "Shape preservation of the stationary 4-point quaternary subdivision schemes," *Communications in Mathematics and Applications*, vol. 9, no. 3, pp. 249–264, 2018.
- [6] P. Ashraf, B. Nawaz, D. Baleanu et al., "Analysis of geometric properties of ternary four-point rational interpolating subdivision scheme," *Mathematics*, vol. 8, pp. 1–19, 2020.
- [7] P. Ashraf, M. Sabir, A. Ghaffar, K. S. Nisar, and I. Khan, "Shape-preservation of the four-point ternary interpolating non-stationary subdivision scheme," *Frontiers in Physics*, vol. 2020, pp. 1–10, 2020.
- [8] R. Hameed, G. Mustafa, A. Liaqat et al., "A new approach to increase the flexibility of curves and regular surfaces produced by 4-point ternary subdivision scheme," *Mathematical Problems in Engineering*, vol. 2020, Article ID 6096545, 17 pages, 2020.
- [9] S. M. Hussain, A. Rehman, D. Baleanu, K. S. Nisar, A. Ghaffar, and S. A. A. Karim, "Generalized 5-point approximating subdivision scheme of varying arity," *Mathematics*, vol. 8, 2020.
- [10] F. Khan, G. Mustafa, A. Shahzad, D. Baleanu, and M. M. Al-Qurashi, "A computational method for subdivision depth of ternary schemes," *Mathematics*, vol. 8, no. 5, 2020.
- [11] C. Conti and L. Romani, "Dual univariate m-ary subdivision schemes of de Rham-type," *Journal of Mathematical Analysis and Applications*, vol. 407, pp. 443–456, 2013.
- [12] L. Romani, "Interpolating m-refinable functions with compact support: The second generation class," *Applied Mathematics and Computation*, vol. 361, pp. 735–746, 2019.
- [13] L. Romani and A. Viscardi, "Dual univariate interpolatory subdivision of every arity: Algebraic characterization and construction," *Journal of Mathematical Analysis and Applications*, vol. 484, Article ID 123713, 2020.
- [14] G. Mustafa, M. Asghar, A. Afzal, and J.-B. Liu, "A family of integer-point ternary parametric subdivision schemes," *Journal of Mathematics*, vol. 2021, Article ID 9281006, 10 pages, 2021.
- [15] N. Dyn and D. Levin, "Subdivision schemes in geometric modelling," *Acta Numerica*, vol. 11, pp. 73–144, 2002.
- [16] O. Rioul, "Simple regularity criteria for subdivision schemes," *SIAM Journal on Mathematical Analysis*, vol. 23, no. 6, pp. 1544–1576, 1992.