Research Article

A New Method of Matrix Decomposition to Get the Determinants and Inverses of r -Circulant Matrices with Fibonacci and Lucas Numbers

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We use a new method of matrix decomposition for r -circulant matrix to get the determinants of $A_n = \text{Circ}_r(F_1, F_2, \dots, F_n)$ and $B_n = \text{Circ}_r(L_1, L_2, \dots, L_n)$, where F_n is the Fibonacci numbers and L_n is the Lucas numbers. Based on these determinants and the nonsingular conditions, inverse matrices are derived. The expressions of the determinants and inverse matrices are represented by Fibonacci and Lucas Numbers. In this study, the formulas of determinants and inverse matrices are much simpler and concise for programming and reduce the computational time.

1. Introduction

The good properties and structure of the circulant matrix are widely given in the area of coding theory, image processing, signal system, and cybernetics [1–4]. For instance, Carra-squilha et al. [5] propose an innovative method for image reconstruction based on circulant matrices. This method significantly reduces the computational time than the principal component analysis (PCA), the discrete Fourier transform (DFT), and the randomized singular value decomposition (RSVD). By using the circulant matrices, Nguyen and Kim [6] get a new construction method of entanglement-assisted quantum error correction code (EAQECC). The circulant matrix also plays an important role in cryptography. Yu [7] investigates the security of a cryptosystem which is based on compressed sensing. The cryptosystem encrypts a plaintext with a secret circulant matrix and transmits the ciphertext over a wireless channel. When the EA and AE robustness and X-robustness of interval circulant matrices over max-min algebra are defined, Myšková and Plavka [8] give polynomial algorithms for checking these types of robustness and X-robustness. By comparing the coefficients in the expansion of the

permanent with those in the expansion of the determinant of a three-line circulant matrix, Poi et al. [9] solve a conjecture stated in the minimality of GT systems. Using block circulant matrices, Dong [10] considers the GPU acceleration of deep neural networks.

Recently, many scholars focus on the norms, determinants, and inverses of circulant and r -circulant matrices whose entries are special integer sequences. In particular, various algorithms for the determinants and inverses of nonsingular circulant matrices and r -circulant matrices were gained. For example, Shen and Cen [11] get the upper and lower bounds for the spectral norms of r -circulant matrices in the forms $A = \text{Circ}_r(F_0, F_1, F_2, \dots, F_{n-1})$ and $B = \text{Circ}_r(L_0, L_1, L_2, \dots, L_{n-1})$ and obtain some bounds for the spectral norms of Kronecker and Hadamard products of A and B . Shen et al. [12] prove that circulant matrices $A = \text{Circ}(F_0, F_1, F_2, \dots, F_{n-1})$ is invertible for $n > 2$ and $B = \text{Circ}(L_0, L_1, L_2, \dots, L_{n-1})$ is invertible for any positive integer n . The formulas for calculating the determinants and inverses of A and B are derived, respectively. Afterwards, Bozkurt and Tam [13] also find the explicit determinants and inverses of circulant matrices with Jacobsthal and

Jacobsthal–Lucas numbers by the same matrix decompositions. Then, Bozkurt and Tam [14] analyze the determinants and inverses of r -circulant matrices with generalized linear recursive sequence and give more accurate bounds for spectral norms of r -circulant matrix involving Fibonacci and Lucas numbers. Yazlik and Taskara [15] summarize the computing method for inverse of circulant matrices, in which the elements include all second-order sequences and polynomials. Zheng and Shon [16] consider the invertibility of generalized Lucas skew circulant matrix and give the determinant and inverse matrix by constructing the transformation matrices. Carmona et al. [17] propose the necessary and sufficient conditions for the invertibility of some circulant matrices that depend on three parameters and compute explicitly the inverse of a general symmetric, circulant, and tridiagonal matrix. Hong and Jing [18] investigate the invertibility of the Tribonacci r -circulant matrix and show the determinant and the inverse matrix based on constructing the transformation matrices. He et al. [19] and Türkmen and Gökbas [20] study the spectral norm of r -circulant matrices with Fibonacci and Lucas numbers and Pell and Pell–Lucas numbers, respectively. When certain circulant matrices having k ones and $k + 1$ zeros in the first row are nonsingular, Chen [21] proves the conditions which nonsingular matrices satisfy.

Shen et al. [12] give the techniques for calculating the determinants and inverses of circulant matrices including Fibonacci and Lucas numbers. The authors construct two matrices P and Q_1 , which are convenient to obtain the determinant. Then, simplify the circulant matrix A to a simple form after promultiplied by P and postmultiplied by Q_1 . After that, they introduce an upper triangular matrix Q_2 and reduce the product of PAQ_1Q_2 to a direct sum of two matrices and finally obtain the inverse matrices of $A = \text{Circ}(F_0, F_1, F_2, \dots, F_{n-1})$ and $B = \text{Circ}(L_0, L_1, L_2, \dots, L_{n-1})$.

From the above papers, we find that there are no simple ways to compute the determinant and inverse of r -circulant matrices. The r -circulant matrix A_n can be written as a polynomial about basic r -circulant matrix, $\pi = \text{Circ}_r(0, 1, 0, \dots, 0)$. By simplifying the subtraction of A_n and $A_n\pi$, we obtain a special factorization $(E - \alpha\pi)(E - \beta\pi)$. On the properties of Fibonacci and Lucas numbers, we use this new method of matrix decomposition to express the determinants and inverses of r -circulant matrices where the elements are Fibonacci and Lucas numbers. The nonsingular condition of the r -circulant matrices is obtained. The expression of the inverse is determined by the first row of the inverse matrix. These computing formulas are simpler than the related studies. Therefore, they are more useful for practical application.

The arrangement of this paper is as follows. In Section 2, we give the preliminary knowledge and lemmas. In Section 3, the determinants and inverses of r -circulant matrix with Fibonacci numbers are obtained. In Section 4, the determinants and inverses of r -circulant matrix with Lucas numbers are obtained. Section 5 presents numerical examples.

2. Preliminaries

For $n \in N_0$, the Fibonacci and Lucas numbers are defined by $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$ under the initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, respectively. Let $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ be the roots of the characteristic equation $x^2 - x - 1 = 0$; then, the Binet formulas of the sequences $\{F_n\}$ and $\{L_n\}$ have the form

$$\begin{aligned} F_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\ L_n &= \alpha^n + \beta^n. \end{aligned} \quad (1)$$

Definition 1 (see [8]). For any given $c_0, c_1, \dots, c_{n-1} \in C$, the $n \times n$ r -circulant matrix $C_n = \text{Circ}_r(c_0, c_1, \dots, c_{n-1})$ is defined by

$$C_n = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_1 & c_0 \end{pmatrix}, \quad (2)$$

where $r = 1$ and C_n is a classical circulant matrix, and reduced to $C_n = \text{Circ}(c_0, c_1, \dots, c_{n-1})$.

Let $\pi = \text{Circ}_r(0, 1, 0, \dots, 0)$; we have $\pi^i = \text{Circ}(0, \dots, 0, 1, 0, \dots, 0)$ and $\pi^n = rE$. If $\pi^0 = E$, we obtain

$$C_n = c_0E + c_1\pi + c_2\pi^2 + \cdots + c_{n-1}\pi^{n-1}. \quad (3)$$

Lemma 1. $\det(aE - b\pi) = a^n - rb^n$.

Lemma 2. Inverse matrix of the r -circulant matrix is a r -circulant matrix.

Lemma 3. Product of r -circulant matrices is a r -circulant matrix.

Lemma 4 (see [12]). Let $A_n = \text{Circ}(F_1, F_2, \dots, F_n)$ be a circulant matrix, and we have

$$\det A_n = (1 - F_{n+1})^{n-1} + F_n^{n-2} \sum_{k=1}^{n-1} F_k \left(\frac{1 - F_{n+1}}{F_n} \right)^{k-1}, \quad (4)$$

where F_n is the n th Fibonacci number.

Lemma 5 (see [12]). Let $A_n = \text{Circ}(F_1, F_2, \dots, F_n)$ ($n > 2$) be a circulant matrix; we have

$$A_n^{-1} = \frac{1}{f_n} \text{Circ} \left(1 + \sum_{i=1}^{n-2} \frac{F_{n-i} F_n^{i-1}}{(F_1 - F_{n+1})^i}, -1 + \sum_{i=1}^{n-2} \frac{F_{n-1-i} F_n^{i-1}}{(F_1 - F_{n+1})^i}, -\frac{1}{F_1 - F_{n+1}}, \right. \\ \left. -\frac{F_n}{(F_1 - F_{n+1})^2}, -\frac{F_n^2}{(F_1 - F_{n+1})^3}, \dots, -\frac{F_n^{n-3}}{(F_1 - F_{n+1})^{n-2}} \right), \quad (5)$$

where $f_n = F_1 - F_n + \sum_{k=1}^{n-2} F_k (F_n/F_1 - F_{n+1})^{n-(k+1)}$.

Lemma 6 (see [12]). Let $B_n = \text{Circ}(L_1, L_2, \dots, L_n)$ be a circulant matrix; the determinant of B_n is

$$\det B_n = (1 - L_{n+1})^{n-1} + (L_n - 2)^{n-2} \sum_{k=1}^{n-1} (L_{k+2} - 3L_{k+1}) \left(\frac{1 - L_{n+1}}{L_n - 2} \right)^{k-1}, \quad (6)$$

where L_n is the n th Lucas number.

Lemma 7 (see [12]). Let $B_n = \text{Circ}(L_1, L_2, \dots, L_n)$ be a circulant matrix; the determinant of B_n is

$$B_n^{-1} = \frac{1}{l_n} \text{Circ} \left(1 + \sum_{i=1}^{n-2} \frac{(L_{n+2-i} - 3L_{n+1-i})(L_n - 2)^{i-1}}{(L_1 - L_{n+1})^i}, -3 \right. \\ \left. + \sum_{i=1}^{n-2} \frac{(L_{n+1-i} - 3L_{n-i})(L_n - 2)^{i-1}}{(L_1 - L_{n+1})^i}, -\frac{5}{L_1 - L_{n+1}}, -\frac{5(L_n - 2)}{(L_1 - L_{n+1})^2}, -\frac{5(L_n - 2)^2}{(L_1 - L_{n+1})^3}, \dots, -\frac{5(L_n - 2)^{n-3}}{(L_1 - L_{n+1})^{n-2}} \right), \quad (7)$$

where $l_n = L_1 - 3L_n + \sum_{k=1}^{n-2} (L_{k+2} - 3L_{k+1})(L_n - 2/L_1 - L_{n+1})^{n-(k+1)}$.

Lemma 8 (see [22]). Let $C = \text{circ}_k\{c_0, c_1, c_2, \dots, c_{n-1}\}$ be an invertible matrix with complex entries. Then, $C^{-1} = \text{circ}_k\{c'_0, c'_1, c'_2, \dots, c'_{n-1}\}$, where $(c'_0, c'_1, c'_2, \dots, c'_{n-1})$ is the unique solution of the following system of linear equations:

$$C \begin{pmatrix} x_0 \\ kx_{n-1} \\ \vdots \\ kx_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (8)$$

Lemma 9 (see [23]). Let $C = \text{circ}_n\{c_0, c_1, c_2, \dots, c_{n-1}\}$ be a nonsingular matrix with complex entries. Then, $C^{-1} = \text{circ}_n\{c'_0, c'_1, c'_2, \dots, c'_{n-1}\}$, where $(c'_0, c'_1, c'_2, \dots, c'_{n-1})$ is the unique solution of the following system of linear equations:

$$C \begin{pmatrix} x_{j-1} \\ \vdots \\ x_1 \\ x_0 \\ kx_{n-1} \\ \vdots \\ kx_j \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j, j = \overline{1, n}. \quad (9)$$

3. Determinant and Inverse of r -Circulant Matrix with Fibonacci Numbers

In this section, $A_n = \text{Circ}_r(F_1, F_2, \dots, F_n)$ is a r -circulant matrix. A determinant formula for the r -circulant matrix A_n is obtained. We prove that A_n is an invertible matrix and obtain the inverse of the r -circulant matrix A_n .

Theorem 1. When $r \neq \alpha^{-n}$ and $r \neq \beta^{-n}$, the determinant of A_n is

$$\det A_n = \frac{(1 - rF_{n+1})^n - r^{n+1}F_n^n}{1 + (-1)^n r^2 - rL_n}, \quad (10)$$

where F_n and L_n are the n th Fibonacci number and Lucas number, respectively.

Proof. From equation (3), A_n is represented by

$$A_n = F_1 E + F_2 \pi + F_3 \pi^2 + \cdots + F_n \pi^{n-1}. \quad (11)$$

Right multiplication matrix π on both sides of equation (7) is

$$A_n \pi = F_1 \pi + F_2 \pi^2 + F_3 \pi^3 + \cdots + F_n \pi^n. \quad (12)$$

From (11) and (12), we obtain

$$\begin{aligned} A_n - A_n \pi &= (F_1 - rF_n)E + (F_2 - F_1)\pi \\ &\quad + (F_3 - F_2)\pi^2 + \cdots + (F_n - F_{n-1})\pi^{n-1}. \end{aligned} \quad (13)$$

Because $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, equation (11) and equation (13) become

$$\begin{aligned} A_n - A_n \pi &= (F_1 - rF_n)E + F_0 \pi + F_1 \pi^2 + \cdots + F_{n-2} \pi^{n-1} \\ &= (1 - rF_n - rF_{n-1})E - rF_n \pi \\ &\quad + (F_1 E + F_2 \pi + F_3 \pi^2 + \cdots + F_n \pi^{n-1})\pi^2 \\ &= (1 - rF_{n+1})E - rF_n \pi + A_n \pi^2. \end{aligned} \quad (14)$$

Thus,

$$A_n(E - \pi - \pi^2) = (1 - rF_{n+1})E - rF_n \pi. \quad (15)$$

When $r \neq \alpha^{-n}$, $r \neq \beta^{-n}$, and $L_n = \alpha^n + \beta^n$, we obtain

$$\begin{aligned} \det(E - \pi - \pi^2) &= \det(E - \alpha\pi)(E - \beta\pi) \\ &= (1 - r\alpha^n)(1 - r\beta^n) \\ &= 1 + (-1)^n r^2 - rL_n \neq 0. \end{aligned} \quad (16)$$

From Lemma 1, the determinant of equation (15) is

$$\begin{aligned} \det A_n \det(E - \pi - \pi^2) &= \det A_n (E - \pi - \pi^2) \\ &= \det((1 - rF_{n+1})E - rF_n \pi) \\ &= (1 - rF_{n+1})^n - r^{n+1}F_n^n. \end{aligned} \quad (17)$$

Hence, the determinant of A_n is

$$\det A_n = \frac{(1 - rF_{n+1})^n - r^{n+1}F_n^n}{1 + (-1)^n r^2 - rL_n}. \quad (18)$$

□

□

Corollary 1. Under the assumption of Theorem 1, when $r = 1$, the determinant of A_n is

$$\det A_n = \frac{(1 - F_{n+1})^n - F_n^n}{1 + (-1)^n - L_n}, \quad (19)$$

where F_n and L_n are the n th Fibonacci number and Lucas number, respectively.

Proof. From the proof of Theorem 1, we have

$$A_n(E - \pi - \pi^2) = (1 - F_{n+1})E - F_n \pi. \quad (20)$$

When $r = 1$, we get $1 \neq \alpha^{-n}$ and $1 \neq \beta^{-n}$; thus,

$$\begin{aligned} \det(E - \pi - \pi^2) &= \det(E - \alpha\pi)(E - \beta\pi) \\ &= (1 - \alpha^n)(1 - \beta^n) \\ &= 1 + (-1)^n - L_n \neq 0 \end{aligned} \quad (21)$$

$$\det((1 - F_{n+1})E - F_n \pi) = (1 - F_{n+1})^n - F_n^n.$$

Hence, we obtain

$$\det A_n = \frac{(1 - F_{n+1})^n - F_n^n}{1 + (-1)^n - L_n}. \quad (22)$$

Obviously, the computation formula of Corollary 1 is simpler than Lemma 4. □

Corollary 2. Under the assumption of Theorem 1, when $(1 - rF_{n+1}/rF_n)^n \neq r$, A_n is an invertible matrix.

Proof. Since $(1 - rF_{n+1}/rF_n)^n \neq r$, we have

$$\det((1 - rF_{n+1})E - rF_n \pi) = (1 - rF_{n+1})^n - r^{n+1}F_n^n \neq 0. \quad (23)$$

Then,

$$\det A_n = \frac{(1 - rF_{n+1})^n - r^{n+1}F_n^n}{1 + (-1)^n r^2 - rL_n} \neq 0. \quad (24)$$

Therefore, A_n is an invertible matrix. □

Theorem 2. Under the assumption of Theorem 1, if $(1 - rF_{n+1}/rF_n)^n \neq r$, the inverse of A_n is

$$A_n^{-1} = \frac{x^{n-1}}{x^n + (-1)^{n-1} r y^n} \text{Circ}_r(y_1, y_2, y_3, \dots, y_n), \quad (25)$$

where $x = 1 - rF_{n+1}$, $y = -rF_n$, $xy \neq 0$,
 $y_1 = 1 - r(-y/x)^{n-2} - r(-y/x)^{n-1}$,
 $y_2 = -1 - (-y/x) - r(-y/x)^{n-1}$, and
 $y_i = (-y/x)^{i-1} - (-y/x)^{i-2} - (-y/x)^{i-3}$ ($i = 3, 4, \dots, n$).

Proof. From the proof of Theorem 1, we have

$$A_n(E - \pi - \pi^2) = (1 - rF_{n+1})E - rF_n \pi. \quad (26)$$

When $(1 - rF_{n+1}/rF_n)^n \neq r$, let $x = 1 - rF_{n+1}$ and $y = -rF_n$, $xy \neq 0$. From Lemma 2, the r -circulant matrix $x E + y \pi$ is an invertible r -circulant matrix. From Lemmas 8 and 9, $(xE + y\pi)^{-1} = \text{Circ}_r(x_1, x_2, x_3, \dots, x_n)$, where $x_1, x_2, x_3, \dots, x_n$ are the unique solutions of the following linear system:

$$\begin{pmatrix} x & y & 0 & \cdots & 0 \\ 0 & x & y & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y \\ ry & 0 & 0 & \cdots & x \end{pmatrix} \begin{pmatrix} x_1 \\ rx_n \\ \vdots \\ rx_3 \\ rx_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \quad (27)$$

Then, the solution of equation (27) is

$$\begin{aligned} x_1 &= \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n}, \\ x_2 &= \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \left(-\frac{y}{x} \right), \\ x_3 &= \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \left(-\frac{y}{x} \right)^2, \\ &\vdots \\ x_n &= \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \left(-\frac{y}{x} \right)^{n-1}. \end{aligned} \quad (28)$$

Let

$$(E - \pi - \pi^2) \text{Circ}_r \left(1, -\frac{y}{x}, \left(-\frac{y}{x} \right)^2, \dots, \left(-\frac{y}{x} \right)^{n-1} \right) \quad (29)$$

$$= \text{Circ}_r(y_1, y_2, \dots, y_n).$$

We obtain

$$\begin{aligned} y_1 &= 1 - r \left(-\frac{y}{x} \right)^{n-2} - r \left(-\frac{y}{x} \right)^{n-1}, \\ y_2 &= -1 - \frac{y}{x} - r \left(-\frac{y}{x} \right)^{n-1}, \\ y_i &= \left(-\frac{y}{x} \right)^{i-1} - \left(-\frac{y}{x} \right)^{i-2} - \left(-\frac{y}{x} \right)^{i-3}, \quad (i = 3, 4, \dots, n). \end{aligned} \quad (30)$$

Hence, the inverse of A_n is

$$A_n^{-1} = \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \text{Circ}_r(y_1, y_2, y_3, \dots, y_n). \quad (31)$$

□

Corollary 3. Under the assumption of Theorem 2, when $(1 - F_{n+1}/F_n)^n \neq 1$ and $r = 1$, the inverse of A_n is

$$\begin{aligned} A_n^{-1} &= \frac{x^{n-1}}{x^n + (-1)^{n-1}y^n} \text{Circ}_r \left(1 - r \left(-\frac{y}{x} \right)^{n-2} - r \left(-\frac{y}{x} \right)^{n-1}, -1 \right. \\ &\quad \left. - \frac{y}{x} - r \left(-\frac{y}{x} \right)^{n-1}, \left(-\frac{y}{x} \right)^2 - \left(-\frac{y}{x} \right) - 1, \dots, \left(-\frac{y}{x} \right)^{i-1} \right. \\ &\quad \left. - \left(-\frac{y}{x} \right)^{i-2} - \left(-\frac{y}{x} \right)^{i-3} \right), \end{aligned} \quad (32)$$

where $x = 1 - F_{n+1}$ and $y = -F_n, xy \neq 0$.

Proof. Under the condition of Corollary 2, when $(1 - F_{n+1}/F_n)^n \neq 1$ and $r = 1$, then A_n is an invertible circulant matrix. From Theorem 2, we obtain

$$\begin{aligned} A_n^{-1} &= \frac{x^{n-1}}{x^n + (-1)^{n-1}y^n} \text{Circ}_r \left(1 - r \left(-\frac{y}{x} \right)^{n-2} - r \left(-\frac{y}{x} \right)^{n-1}, \right. \\ &\quad \left. -1 - \frac{y}{x} - r \left(-\frac{y}{x} \right)^{n-1}, \left(-\frac{y}{x} \right)^2 - \left(-\frac{y}{x} \right) \right. \\ &\quad \left. -1, \dots, \left(-\frac{y}{x} \right)^{i-1} - \left(-\frac{y}{x} \right)^{i-2} - \left(-\frac{y}{x} \right)^{i-3} \right), \end{aligned} \quad (33)$$

where $x = 1 - F_{n+1}$ and $y = -F_n, xy \neq 0$.

Obviously, the formula of Corollary 3 is simpler than Lemma 5. □

4. Determinant and Inverse of r -Circulant Matrix with Lucas Numbers

Let $B_n = \text{Circ}_r(L_1, L_2, \dots, L_n)$ be a r -circulant matrix, where L_n is the Lucas numbers. We discuss the nonsingular conditions for B_n and obtain the determinant formula and inverse of the r -circulant matrix B_n .

Theorem 3. When $r \neq \alpha^{-n}$ and $r \neq \beta^{-n}$, the determinant of B_n is

$$\det B_n = \frac{(1 - rL_{n+1})^n - r(rL_n - 2)^n}{1 + (-1)^n r^2 - rL_n}, \quad (34)$$

where L_n is the Lucas numbers.

Proof. By the proof of Theorem 1, we have

$$B_n(E - \pi - \pi^2) = (1 - rL_{n+1})E + (2 - rL_n)\pi. \quad (35)$$

When $r \neq \alpha^{-n}$, $r \neq \beta^{-n}$, and $L_n = \alpha^n + \beta^n$, we obtain

$$\det(E - \pi - \pi^2) = 1 + (-1)^n r^2 - rL_n \neq 0. \quad (36)$$

From Lemma 1, we obtain

$$\det((1 - rL_{n+1})E + (2 - rL_n)\pi) = (1 - rL_{n+1})^n - r(rL_n - 2)^n. \quad (37)$$

Hence, the determinant of B_n is

$$\det B_n = \frac{(1 - rL_{n+1})^n - r(rL_n - 2)^n}{1 + (-1)^n r^2 - rL_n}. \quad (38)$$

□

Corollary 4. Under the assumption of Theorem 3, when $r = 1$, the determinant of B_n is

$$\det B_n = \frac{(1 - L_{n+1})^n - (L_n - 2)^n}{1 + (-1)^n - L_n}, \quad (39)$$

where L_n is the n th Lucas number.

Obviously, the computation formula of Corollary 4 is simpler than Lemma 6.

Corollary 5. Under the assumption of Theorem 3, when $(1 - rL_{n+1}/rL_n - 2)^n \neq r$, B_n is an invertible matrix.

Proof. Since $(1 - rL_{n+1}/rL_n - 2)^n \neq r$, we obtain

$$\det((1 - rL_{n+1})E + (2 - rL_n)\pi) = (1 - rL_{n+1})^n - r(rL_n - 2)^n \neq 0. \quad (40)$$

Because

$$\det B_n \neq 0, \quad (41)$$

B_n is an invertible matrix. $\square$

Theorem 4. Under the assumption of Theorem 3, if $(1 - rL_{n+1}/rL_n - 2)^n \neq r$, the inverse of B_n is

$$B_n^{-1} = \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \text{Circ}_r(y_1, y_2, y_3, \dots, y_n), \quad (42)$$

where $x = 1 - rL_{n+1}$, $y = 2 - rL_n$, $xy \neq 0$, $y_1 = 1 - r(-y/x)^{n-2} - r(-y/x)^{n-1}$, $y_2 = -1(-y/x) - r(-y/x)^{n-1}$, and $y_i = (-y/x)^{i-1} - (-y/x)^{i-2} - (-y/x)^{i-3}$ ($i = 3, 4, \dots, n$).

Proof. From the proof of Theorem 3, we have

$$B_n(E - \pi - \pi^2) = (1 - rL_{n+1})E + (2 - rL_n)\pi. \quad (43)$$

When $(1 - rL_{n+1}/rL_n - 2)^n \neq r$, let $x = 1 - rL_{n+1}$ and $y = 2 - rL_n$, $xy \neq 0$; from Lemma 2, the r -circulant matrix $xE + y\pi$ is an invertible r -circulant matrix. From Lemmas 8 and 9, $(xE + y\pi)^{-1} = \text{Circ}_r(x_1, x_2, x_3, \dots, x_n)$, where $x_1, x_2, x_3, \dots, x_n$ are the unique solution of the following linear system:

$$\begin{pmatrix} x & y & 0 & \cdots & 0 \\ 0 & x & y & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y \\ ry & 0 & 0 & \cdots & x \end{pmatrix} \begin{pmatrix} x_1 \\ rx_n \\ \vdots \\ rx_3 \\ rx_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \quad (44)$$

Then, the solution of equation (44) is

$$\begin{aligned} x_1 &= \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n}, \\ x_2 &= \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \left(-\frac{y}{x} \right), \\ x_3 &= \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \left(-\frac{y}{x} \right)^2, \\ &\vdots \\ x_n &= \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \left(-\frac{y}{x} \right)^{n-1}. \end{aligned} \quad (45)$$

Since

$$(E - \pi - \pi^2) \text{Circ}_r \left(1, -\frac{y}{x}, \left(-\frac{y}{x} \right)^2, \dots, \left(-\frac{y}{x} \right)^{n-1} \right) = \text{Circ}_r(y_1, y_2, \dots, y_n). \quad (46)$$

We obtain

$$\begin{aligned} y_1 &= 1 - r \left(-\frac{y}{x} \right)^{n-2} - r \left(-\frac{y}{x} \right)^{n-1}, \\ y_2 &= -1 - \frac{y}{x} - r \left(-\frac{y}{x} \right)^{n-1}, \\ y_i &= \left(-\frac{y}{x} \right)^{i-1} - \left(-\frac{y}{x} \right)^{i-2} - \left(-\frac{y}{x} \right)^{i-3} \quad (i = 3, 4, \dots, n). \end{aligned} \quad (47)$$

Hence, the inverse of B_n is

$$A^{-1} = \frac{x^{n-1}}{x^n + (-1)^{n-1}ry^n} \text{Circ}_r(y_1, y_2, y_3, \dots, y_n). \quad (48)$$

$\square$

Corollary 6. Under the assumption of Theorem 4, when $(1 - L_{n+1}/L_n - 2)^n \neq 1$ and $r = 1$, the inverse of B_n is

$$\begin{aligned} B_n^{-1} &= \frac{x^{n-1}}{x^n + (-1)^{n-1}y^n} \text{Circ}_r \left(1 - r \left(-\frac{y}{x} \right)^{n-2} - r \left(-\frac{y}{x} \right)^{n-1}, -1 - \frac{y}{x} \right. \\ &\quad \left. - r \left(-\frac{y}{x} \right)^{n-1}, \left(-\frac{y}{x} \right)^2 - \left(-\frac{y}{x} \right) - 1, \dots, \left(-\frac{y}{x} \right)^{i-1} - \left(-\frac{y}{x} \right)^{i-2} - \left(-\frac{y}{x} \right)^{i-3} \right), \end{aligned} \quad (49)$$

	1	2	3	4	5	6	7	8	9	10	11	12
1	-1.4219e-2...	2.3006e-209	0	0	0	0	0	0	0	0	0	0
2	0	-1.4219e-2...	2.3006e-209	0	0	0	0	0	0	0	0	0
3	0	0	-1.4219e-2...	2.3006e-209	0	0	0	0	0	0	0	0
4	0	0	0	-1.4219e-2...	2.3006e-209	0	0	0	0	0	0	0
5	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0	0	0	0	0
6	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0	0	0	0
7	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0	0	0
8	0	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0	0
9	0	0	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0
10	0	0	0	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0
11	0	0	0	0	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209
12	0	0	0	0	0	0	0	0	0	0	0	-1.4219e-2...

FIGURE 1: The result of Lemma 5.

	1	2	3	4	5	6	7	8	9	10	11	12
1	-1.4219e-2...	2.3006e-209	0	0	-7.8929e-2...	7.8929e-226	0	1.9732e-226	-9.8661e-2...	9.8661e-227	-4.9331e-2...	2.4665e-227
2	0	-1.4219e-2...	2.3006e-209	0	0	-7.8929e-2...	7.8929e-226	0	1.9732e-226	-9.8661e-2...	9.8661e-227	-4.9331e-2...
3	0	0	-1.4219e-2...	2.3006e-209	0	0	-7.8929e-2...	7.8929e-226	0	1.9732e-226	-9.8661e-2...	9.8661e-227
4	0	0	0	-1.4219e-2...	2.3006e-209	0	0	-7.8929e-2...	7.8929e-226	0	1.9732e-226	-9.8661e-2...
5	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0	-7.8929e-2...	7.8929e-226	0	1.9732e-226
6	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0	-7.8929e-2...	7.8929e-226	0
7	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0	-7.8929e-2...	7.8929e-226
8	0	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0	-7.8929e-2...
9	0	0	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0	0
10	0	0	0	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209	0
11	0	0	0	0	0	0	0	0	0	0	-1.4219e-2...	2.3006e-209
12	0	0	0	0	0	0	0	0	0	0	0	-1.4219e-2...

FIGURE 2: The result of Corollary 3.

where $x = 1 - L_{n+1}$ and $y = 2 - L_n$, $xy \neq 0$.

Proof. Under the condition of Corollary 5, when $(1 - L_{n+1}/L_n - 2)^n \neq 1$ and $r = 1$, then B_n is an invertible circulant matrix. From proof of Theorem 4, we obtain

$$B_n^{-1} = \frac{x^{n-1}}{x^n + (-1)^{n-1}y^n} \text{Circ}_r \left(1 - r \left(-\frac{y}{x} \right)^{n-2} - r \left(-\frac{y}{x} \right)^{n-1}, \right. \\ \left. -1 - \frac{y}{x} - r \left(-\frac{y}{x} \right)^{n-1}, \left(-\frac{y}{x} \right)^2 - \left(-\frac{y}{x} \right) - 1, \dots, \left(-\frac{y}{x} \right)^{i-1} - \left(-\frac{y}{x} \right)^{i-2} - \left(-\frac{y}{x} \right)^{i-3} \right), \quad (50)$$

where $x = 1 - L_{n+1}$ and $y = 2 - L_n$, $xy \neq 0$. Obviously, the formula of Corollary 6 is simpler than Lemma 7. $\square$

5. Numerical Examples

Next, we further discuss our conclusion through numerical simulation.

Example 1. When $r = 1$ and $n = 1000$, we obtain the inverse matrix of A_n by Lemma 5 and Corollary 3.

Example 2. When $r = 1$ and $n = 1000$, we obtain the inverse matrix of B_n by Lemma 7 and Corollary 6.

Examples 1 and 2 calculate the inverse of a matrix for 1000 orders. Due to the limited space, we only cut 12 orders. Through Figures 1–4, we can find that our conclusion is more accurate than Lemmas 5 and 7 in the process of numerical simulation. At the same time, in the process of

practical application, our conclusion will further reduce the error.

Example 3. When $r = 1$, we compare the running time of Lemma 4 and Corollary 1 in milliseconds (ms).

Example 4. When $r = 1$, we compare the running time of Lemma 6 and Corollary 4 in milliseconds (ms).

Through calculating, it can be concluded that the time complexity of these formulas in Lemmas 4–7 are $O(n^2)$. And the time complexity of formulas in Corollaries 1–6 are $O(n)$. This new method of matrix decomposition reduces computation by an order of magnitude. We also find that the computation time is greatly reduced through programming with C language. From Tables 1 and 2, the same conclusions are easily found. We calculate the inverse matrices by MATLAB; when $n \geq 1000$, the expression for the inverse matrices are more accurate than Lemmas 5 and 7.

	1	2	3	4	5	6	7	8	9	10	11	12
1	-6.3588e-2...	1.0289e-209	0	0	0	0	0	0	0	0	0	0
2	0	-6.3588e-2...	1.0289e-209	0	0	0	0	0	0	0	0	0
3	0	0	-6.3588e-2...	1.0289e-209	0	0	0	0	0	0	0	0
4	0	0	0	-6.3588e-2...	1.0289e-209	0	0	0	0	0	0	0
5	0	0	0	0	-6.3588e-2...	1.0289e-209	0	0	0	0	0	0
6	0	0	0	0	0	-6.3588e-2...	1.0289e-209	0	0	0	0	0
7	0	0	0	0	0	0	-6.3588e-2...	1.0289e-209	0	0	0	0
8	0	0	0	0	0	0	0	-6.3588e-2...	1.0289e-209	0	0	0
9	0	0	0	0	0	0	0	0	-6.3588e-2...	1.0289e-209	0	0
10	0	0	0	0	0	0	0	0	0	-6.3588e-2...	1.0289e-209	0
11	0	0	0	0	0	0	0	0	0	0	-6.3588e-2...	1.0289e-209
12	0	0	0	0	0	0	0	0	0	0	0	-6.3588e-2...

FIGURE 3: The result of Lemma 7.

	1	2	3	4	5	6	7	8	9	10	11	12
1	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...	3.5298e-226	-1.7649e-2...	0	-1.7649e-2...	4.4123e-227	-4.4123e-2...	0	-1.1031e-2...
2	0	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...	3.5298e-226	-1.7649e-2...	0	-1.7649e-2...	4.4123e-227	-4.4123e-2...	0
3	0	0	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...	3.5298e-226	-1.7649e-2...	0	-1.7649e-2...	4.4123e-227	-4.4123e-2...
4	0	0	0	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...	3.5298e-226	-1.7649e-2...	0	-1.7649e-2...	4.4123e-227
5	0	0	0	0	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...	3.5298e-226	-1.7649e-2...	0	-1.7649e-2...
6	0	0	0	0	0	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...	3.5298e-226	-1.7649e-2...	0
7	0	0	0	0	0	0	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...	3.5298e-226	-1.7649e-2...
8	0	0	0	0	0	0	0	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...	3.5298e-226
9	0	0	0	0	0	0	0	0	-6.3588e-2...	1.0289e-209	7.0596e-226	-7.0596e-2...

FIGURE 4: The result of Corollary 6.

TABLE 1: Running time comparison of Lemma 4 and Corollary 1.

n	Lemma 4	Corollary 1
20	0	0
25	31	16
30	172	109
35	2278	1201
36	2855	1996
37	4758	3276
38	7550	5164

TABLE 2: Running time comparison of Lemma 6 and Corollary 4.

n	Lemma 6	Corollary 4
20	0	0
25	47	15
30	281	109
35	3073	1186
36	4976	1903
37	8112	3026
38	12979	4992

6. Conclusion

In this paper, we use a new matrix decomposition method to solve the determinant and inverse m of the r -circulant matrix with Fibonacci sequence and Lucas sequence. Some simpler forms are obtained and the corresponding proofs are given. Through numerical examples, we find that the new formulas not only reduce the computation time of

determinant but also increase the accuracy of solving inverse matrix and are more conducive to the application of real life.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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