Research Article

Two Classes of Infrasoftware Separation Axioms

Tareq M. Al-shami¹ and Jia-Bao Liu²

¹Department of Mathematics, Sana'a University, Sana'a, Yemen

²School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China

Correspondence should be addressed to Tareq M. Al-shami; tareqalshami83@gmail.com

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One of the considerable topics in the soft setting is the study of soft topology which has enticed the attention of many researchers. To contribute to this scope, we devote this work to investigate two classes of separation axioms with respect to the distinct ordinary elements through one of the generalizations of soft topology called infrasoftware topology. We first formulate the concepts of infrasoftware T_j -soft T_j using total belong and partial nonbelong relations and then introduce the concepts of infrasoftware tt -soft T_j -spaces using total belong and partial nonbelong relations. To illustrate the relationships between them, we provide some examples. We discuss their fundamental properties and study their behaviors under some special types of infrasoftware topologies. An extensive discussion is given for the transmission of these two classes between infrasoftware topology and its parametric infratopologies. In the end, we demonstrate which ones have topological and hereditary properties, and we show their behaviors under the finite product of soft spaces.

1. Introduction

One of the powerful mathematical approaches to cope with uncertain problems is the soft set. It was proposed in 1999 by Molodtsov [1] who explained its applications to different areas such as smoothness of functions, theory of measurement, game theory, Riemann integration, operations research, etc. In 2002, soft sets were successfully applied to handle decision-making problems by Maji et al. [2]. They introduced some operations and operators on soft sets such as their soft intersection and union and the complement of a soft set in [3]. Despite the weakness of some concepts and results in these early references, they form the essential start point of soft set theory. To keep some properties and results of crisp set theory, Ali et al. [4] displayed new types of these operations and operators. Novel types of them were established such as lower and upper soft equality [5], gf -soft union [6], and T -soft subset and T -soft equality [7]. Since the advent of soft sets, many authors applied successfully to address problems in some disciplines such as computer science [8], decision-making [9, 10], and medical science [11, 12]. These applications prove the adequacy of soft sets to treat and model a lot of real-life issues.

In 2011, Shabir and Naz [13] defined soft topologies as hybridization of classical topology and soft sets. Then, numerous researchers have discussed the topological concepts, properties, and results in soft topologies; see [14–17]. Soft point [18, 19] is an essential concept in soft set theory; it represents the soft version of an ordinary element. Some types of soft topologies called enriched and extended soft topologies are studied [19].

Some topological properties are kept under conditions weaker than topology's conditions; also, some phenomena are described under structures relaxing a topology. The structures of suprasoft topology [20] and infrasoftware topology [21] have been born with these goals. They have become two of the most interesting developments of soft topology in recent years. Supratopology is a class of subsets that extend the concept of topological space by dispensing with the postulate that the class is closed under finite intersections, whereas infratopology is a class of subsets that extend the concept of topological space by dispensing with the postulate that the class is closed under arbitrary unions. Generalized soft topology [22] and soft weak structure are other generalizations of soft topology. Recently, the concepts of

compactness and connectedness have been studied through the frame of infrasoft topology in [23, 24], respectively.

Another technique of generalizing soft topology was introduced by combining other structures with a soft topology. In this regard, Ittanagi [25] initiated the concept of soft bitopology.

Our contribution to this field concerns the analysis of what type of “separation axioms” are meaningful in the study of infrasoft topology. As was the case of classical topology, soft separation axioms are among the most interesting and substantial ideas in soft topology. They form a tool to establish more restricted (and wider) classes of well-behaved soft spaces. It should be noted that a large variety of separation axioms in the soft structures is attributed to two factors. One is the distinguished objects that we intend to separate: they can be either ordinary points or soft points. The other is the type of belongingness and nonbelongingness relations that we require in the definitions: they can be either partial or total. For more details about this subject, see [26], which investigated new sorts of separation axioms on suprasoft topologies, and [10, 27], which studied two kinds of soft T_i -spaces on soft topologies. In [28], the researchers presented soft separation axioms with respect to distinct soft points. Terepeta [29] discussed soft separation axioms using their counterparts in classical topological spaces. Singh and Noorie [30] conducted a comparative study among some soft separation axioms published in the literature.

In this article, we note that many properties of soft topological spaces are still valid on infrasoft topological spaces and initiating examples that show some relationships between certain topological concepts is easier on infrasoft topological spaces. Therefore, we aim in this paper to perform an exhaustive analysis of infrasoft topological spaces.

The organization of this manuscript is as follows. Section 2 mentions some definitions and properties related to soft set theory and infrasoft topology. Section 3 initiates the concepts of infra- tp -soft T_i -spaces ($i = 0, 1, 2, 3, 4$) and studies basic properties. Section 4 defines the concepts of infra- tt -soft T_i -spaces ($i = 0, 1, 2, 3, 4$) and discusses main properties. Elucidative examples are supplied to validate the obtained relationships and results. Ultimately, Section 5 outlines the paper’s achievements and proposes some future works.

2. Preliminaries

The concepts and results that we need in this paper are recalled in this section.

2.1. Soft Set Theory. The concepts and results of soft set theory that we need in this paper are mentioned in this part.

Definition 1 (see [1]). A map E from Σ which is set of parameters to the power set $2^{\mathbb{Y}}$ of $\mathbb{Y}$ is called a soft set over $\mathbb{Y}$. It is denoted by E_{Σ} and identified as $E_{\Sigma} = \{(\sigma, E(\sigma)): \sigma \in \Sigma \text{ and } E(\sigma) \in 2^{\mathbb{Y}}\}$.

The set of all soft sets over $\mathbb{Y}$ with Σ as a set of parameters is symbolized by $S(\mathbb{Y}_{\Sigma})$.

Definition 2 (see [4]). The complement of a soft set E_{Σ} , symbolized by E_{Σ}^c , is given by $E_{\Sigma}^c = (E^c)_{\Sigma}$, where $E^c: \Sigma \longrightarrow 2^{\mathbb{Y}}$ is a map defined by $E^c(\sigma) = \mathbb{Y} \setminus E(\sigma)$ for each $\sigma \in \Sigma$.

Definition 3 (see [2]). If the image of each parameter of Σ under a map $E: \Sigma \longrightarrow \mathbb{Y}$ is the empty set, then E_{Σ} is called the null soft set over $\mathbb{Y}$. Its complement is called the absolute soft set. The notations Φ and $\mathbb{Y}$, respectively, denote the null and absolute soft sets.

Definition 4 (see [18, 19]). If the image of one parameter, say σ , under a map $P: \Sigma \longrightarrow \mathbb{Y}$ is a singleton set, say $\{\nu\}$, and the image of each parameter $\sigma' \in \Sigma, \{\sigma'\}$ is the empty set, then a soft set P_{Σ} is called a soft point over $\mathbb{Y}$. It is briefly symbolized by P_{σ}^{ν} .

Definition 5 (see [4]). The intersection of two soft sets E_{Ω} and F_{Γ} over $\mathbb{Y}$, symbolized by $E_{\Omega} \cap F_{\Gamma}$, is a soft set H_{Σ} , where $\Sigma = \Omega \cap \Gamma \neq \emptyset$, and a map $H: \Sigma \longrightarrow 2^{\mathbb{Y}}$ is given by $H(\sigma) = E(\sigma) \cap F(\sigma)$ for each $\sigma \in \Sigma$.

Definition 6 (see [4]). The union of two soft sets E_{Ω} and F_{Γ} over $\mathbb{Y}$, symbolized by $E_{\Omega} \cup F_{\Gamma}$, is a soft set H_{Σ} , where $\Sigma = \Omega \cup \Gamma$ and a map $H: \Sigma \longrightarrow 2^{\mathbb{Y}}$ is given as follows:

$$H(\sigma) = \begin{cases} E(\sigma), & \sigma \in \Omega, \Gamma, \\ F(\sigma), & \sigma \in \Gamma, \Omega, \\ E(\sigma) \cup F(\sigma), & \sigma \in \Omega \cap \Gamma. \end{cases} \quad (1)$$

Definition 7 (see [31]). A soft set E_{Σ} is a subset of a soft set F_{Ω} , symbolized by $E_{\Sigma} \subseteq F_{\Omega}$, if $\Sigma \subseteq \Omega$ and for all $\sigma \in \Sigma$, we have $E(\sigma) \subseteq F(\sigma)$. The soft sets E_{Σ} and E_{Ω} are called soft equal if each is a subset of the other.

Definition 8 (see [13]). A soft set E_{Σ} over $\mathbb{Y}$ which is defined by $E(\sigma) = \{\nu\}$ for each $\sigma \in \Sigma$ is called a singleton soft set. It is denoted by ν_{Σ} .

Definition 9 (see [27]). If all components of a soft set are equal, then we called it a stable soft set.

Definition 10 (see [16]). The Cartesian product of E_{Σ} and H_{Ω} , symbolized by $(G \times H)_{\Sigma \times \Omega}$, is defined as $(G \times H)(\sigma, \omega) = E(\sigma) \times H(\omega)$ for each $(\sigma, \omega) \in \Sigma \times \Omega$.

The definition of soft maps given in [32] was reformulated in a way that reduces calculation burden and gives a justification (logical explanation) for some soft concepts such as why we determine that f_{η} is injective or surjective according to its two crisp maps f and η .

Definition 11 (see [33]). Let $f: \mathbb{X} \longrightarrow \mathbb{Y}$ and $\eta: \Sigma \longrightarrow \Omega$ be two crisp maps. A soft map f_{η} of $S(\mathbb{X}_{\Sigma})$ into $S(\mathbb{Y}_{\Omega})$ is a relation such that each soft point in $S(\mathbb{X}_{\Sigma})$ is related to one and only one soft point in $S(\mathbb{Y}_{\Omega})$ such that $f_{\eta}(P_{\sigma}^{\nu}) = P_{\eta(\sigma)}^{f(\nu)}$ for each $P_{\sigma}^{\nu} \in S(\mathbb{X}_{\Sigma})$.

In addition, $f_{\eta}^{-1}(P_{\omega}^{\nu}) = \sqcup_{\substack{\sigma \in \eta^{-1}(\omega) \\ \nu \in f^{-1}(\nu)}} P_{\sigma}^{\nu}$ for each $P_{\omega}^{\nu} \in S(\mathbb{Y}_{\Omega})$.

Definition 12 (see [32]). A soft map $f_\eta: S(\mathbb{Y}_\Sigma) \longrightarrow S(\mathbb{Z}_\Omega)$ is said to be injective (resp. surjective, bijective) if both f and η are injective (resp., surjective, bijective).

Definition 13 (see [13, 27]). For a soft set E_Σ over $\mathbb{Y}$ and $\nu \in \mathbb{Y}$, we say that

- (i) $\nu \in E_\Sigma$ (it reads as ν totally belongs to E_Σ) if $\nu \in E(\sigma)$ for each $\sigma \in \Sigma$;
- (ii) $\nu \notin E_\Sigma$ (it reads as ν does not partially belong to E_Σ) if $\nu \notin E(\sigma)$ for some $\sigma \in \Sigma$;
- (iii) $\nu \in E_\Sigma$ (it reads as ν partially belongs to E_Σ) if $\nu \in E(\sigma)$ for some $\sigma \in \Sigma$;
- (iv) $\nu \notin E_\Sigma$ (it reads as ν does not totally belong to E_Σ) if $\nu \notin E(\sigma)$ for each $\sigma \in \Sigma$.

Proposition 1 (see [27]). Consider a soft map $f_\eta: S(\mathbb{Y}_\Sigma) \longrightarrow S(\mathbb{Z}_\Omega)$ and let E_Σ and H_Ω be soft sets in $S(\mathbb{Y}_\Sigma)$ and $S(\mathbb{Z}_\Omega)$, respectively. Then the next results hold true:

- (i) If η is surjective and $\nu \in E_\Sigma$, then $f(\nu) \in f_\eta(E_\Sigma)$.
- (ii) If $\nu \notin E_\Sigma$, then $f(\nu) \notin f_\eta(E_\Sigma)$ provided that f is injective.
- (iii) If f_η is injective and $\nu \notin E_\Sigma$, then $f(\nu) \notin f_\eta(E_\Sigma)$.
- (iv) If $\mu \in H_\Omega$, then $\nu \in f_\eta^{-1}(H_\Omega)$ for each $\nu \in f^{-1}(\mu)$.
- (v) If $\mu \notin H_\Omega$, then $\nu \notin f_\eta^{-1}(H_\Omega)$ for each $\nu \in f^{-1}(\mu)$.
- (vi) If η is surjective and $\mu \notin H_\Omega$, then $\nu \notin f_\eta^{-1}(H_\Omega)$ for each $\nu \in f^{-1}(\mu)$.

2.2. Infrasoftware Topological Spaces. The concepts and results of infrasoftware topology that we need in this paper are mentioned in this part.

Definition 14 (see [21]). The collection δ of soft sets over $\mathbb{Y}$ under a parameters set Σ is said to be an infrasoftware topology on $\mathbb{Y}$ if it is closed under finite soft intersection and $\Phi \in \delta$.

The triple $(\mathbb{Y}, \delta, \Sigma)$ is called an infrasoftware topological space. Every member of δ is called an infrasoftware open set and its relative complement is called an infrasoftware closed set. We called $(\mathbb{Y}, \delta, \Sigma)$ stable if all its infrasoftware open sets are stable.

Definition 15 (see [21]). We define the infrainterior points and infraclosure points of a soft subset E_Σ of $(\mathbb{Y}, \delta, \Sigma)$ which are, respectively, denoted by $\text{Int}_\delta(E_\Sigma)$ and $\text{Cl}_\delta(E_\Sigma)$ as follows:

- (i) $\text{Int}_\delta(E_\Sigma)$ is the union of all infrasoftware open sets contained in E_Σ .
- (ii) $\text{Cl}_\delta(E_\Sigma)$ is the intersection of all infrasoftware closed sets containing E_Σ .

Theorem 1 (see [21]). Let E_Σ and F_Σ be two soft subsets of $(\mathbb{Y}, \delta, \Sigma)$. Then,

- (i) If $E_\Sigma \subseteq F_\Sigma$, then $\text{Cl}_\delta(E_\Sigma) \subseteq \text{Cl}_\delta(F_\Sigma)$;

- (ii) $P_\sigma^\nu \in \text{Cl}_\delta(E_\Sigma)$ if and only if $E_\Sigma \cap H_\Sigma \neq \Phi$ for each infrasoftware open set F_Σ containing P_σ^ν .

Proposition 2 (see [21]). Let $(\mathbb{Y}, \delta, \Sigma)$ be an infrasoftware topological space. Then the collection $\delta_\sigma = \{E(\sigma): E_\Sigma \in \delta\}$ forms an infratopology on $\mathbb{Y}$ for each $\sigma \in \Sigma$.

We called δ_σ a parametric infratopology.

Proposition 3 (see [21]). Suppose that $\{\delta_\sigma\}_{\sigma \in \Sigma}$ is a family of crisp infratopologies on $\mathbb{Y}$. Then the family

$$\{(\sigma, E(\sigma)): \sigma \in \Sigma\} \in S(\mathbb{Y}_\Sigma) \text{ such that } E(\sigma) \in \delta_\sigma \text{ for each } \sigma \in \Sigma, \quad (2)$$

defines an infrasoftware topology on $\mathbb{Y}$.

The infrasoftware topological space given in the above proposition is called an extended infrasoftware topology.

Definition 16 (see [21]). Let $(\mathbb{Y}, \delta, \Sigma)$ be an infrasoftware topological space and $\mathbb{Z}$ be a nonempty subset of $\mathbb{Y}$. Then $\delta_\mathbb{Z} = \{\mathbb{Z} \cap E_\Sigma: E_\Sigma \in \delta\}$ is called an infrasoftware relative topology on $\mathbb{Z}$ and $(\mathbb{Z}, \delta_\mathbb{Z}, \Sigma)$ is called an infrasoftware subspace of $(\mathbb{Y}, \delta, \Sigma)$.

Theorem 2 (see [21]). Let $(\mathbb{Z}, \delta_\mathbb{Z}, \Sigma)$ be an infrasoftware subspace of $(\mathbb{Y}, \delta, \Sigma)$. Then H_Σ is an infrasoftware closed subset of $(\mathbb{Z}, \delta_\mathbb{Z}, \Sigma)$ iff there exists an infrasoftware closed subset F_Σ of $(\mathbb{Y}, \delta, \Sigma)$ such that $H_\Sigma = \mathbb{Z} \cap F_\Sigma$.

Definition 17 (see [33]). A soft mapping $f_\phi: (\mathbb{Y}, \delta, \Sigma) \longrightarrow (\mathbb{Z}, \theta, \Omega)$ is said to be

- (i) Infrasoftware continuous if the preimage of any infrasoftware open set is an infrasoftware open set;
- (ii) Infrasoftware open (resp., infrasoftware closed) if the image of any infrasoftware open (resp., infrasoftware closed) set is an infrasoftware open (resp., infrasoftware closed) set;
- (iii) An infrasoftware homeomorphism if it is bijective, infrasoftware continuous, and infrasoftware open.

3. Infra-tp-Soft T_i -Spaces ($i = 0, 1, 2, 3, 4$)

In this section, we display new family of separation axioms, namely, infra-tp-soft T_i -spaces ($i = 0, 1, 2, 3, 4$), where t and p are the first letters of total belong and partial nonbelong relations that are used to define these spaces. We explore main properties and supply various examples to elucidate the relationships between them.

Definition 18. $(\mathbb{Y}, \delta, \Sigma)$ is said to be

- (i) An infra-tp-soft T_0 -space if for every $\nu \neq \mu \in \mathbb{Y}$, there exists an infrasoftware open set E_Σ such that $\nu \in E_\Sigma$ and $\mu \notin E_\Sigma$, or $\mu \in E_\Sigma$ and $\nu \notin E_\Sigma$;
- (ii) An infra-tp-soft T_1 -space if for every $\nu \neq \mu \in \mathbb{Y}$, there exist infrasoftware open sets E_Σ and F_Σ such that $\nu \in E_\Sigma, \mu \notin E_\Sigma$ and $\mu \in F_\Sigma, \nu \notin F_\Sigma$;
- (iii) An infra-tp-soft T_2 -space (or an infra-tp-soft Hausdorff space) if for every $\nu \neq \mu \in \mathbb{Y}$, there exist

disjoint infrasoft open sets E_Σ and F_Σ such that $\nu \in E_\Sigma, \mu \notin E_\Sigma$ and $\mu \in F_\Sigma, \nu \notin F_\Sigma$;

- (iv) An infra- tp -soft regular space if for every infrasoft closed set H_Σ such that $\nu \notin H_\Sigma$, there exist disjoint infrasoft open sets E_Σ and F_Σ such that $H_\Sigma \subseteq E_\Sigma$ and $\nu \in F_\Sigma$;
- (v) An infrasoft normal space if disjoint infrasoft closed sets are separated by disjoint infrasoft open sets;
- (vi) An infra- tp -soft T_3 (resp., infra- tp -soft T_4) -space if it is infra- tp -soft regular (resp., infrasoft normal) and infra- tp -soft T_1 .

Before we investigate the relationships between the above infrasoft axioms, we need the following auxiliary result.

Lemma 1. *Every infrasoft open (infrasoft closed) subset of an infra- tp -soft regular space is stable; i.e., every infra- tp -soft regular space is stable.*

Proof. Let E_Σ be an infrasoft open subset of an infra- tp -soft regular space $(\mathbb{Y}, \delta, \Sigma)$. Suppose that E_Σ is unstable; i.e., there exist $\nu \in \mathbb{Y}$ and $\sigma, \sigma' \in \Sigma$ such that $\nu \in E(\sigma)$ and $\nu \notin E(\sigma')$. Now, E_Σ^c is an infrasoft closed set such that $\nu \notin E^c(\sigma)$ and $\nu \in E^c(\sigma')$. It is clear that for any infrasoft open set F_Σ containing E_Σ^c we obtain $\nu \in F_\Sigma$; this implies we cannot find an infrasoft open set whose intersection with F_Σ is the null soft set such that ν totally belongs to it. Hence, E_Σ must be stable.

One can similarly prove the lemma for an infrasoft closed set. $\square$

Corollary 1. *The number of infrasoft open subsets of a finite infra- tp -soft regular space $(\mathbb{Y}, \delta, \Sigma)$ is greater than or equal to $2^{|\mathbb{Y}|}$.*

Proposition 4. *Every infra- tp -soft T_i -space is an infra- tp -soft T_{i-1} -space for $i = 1, 2, 3$.*

Proof. The proof is clear for the cases $i = 1, 2$. To prove the case of $i = 3$, let ν, μ be two distinct elements in an infra- tp -soft T_3 -space. Then there exists an infrasoft open set E_Σ such that $\nu \in E_\Sigma$ and $\mu \notin E_\Sigma$. Now, E_Σ^c is infrasoft closed such that $\nu \notin E_\Sigma^c$. By hypothesis, there exist disjoint infrasoft open sets U_Σ and V_Σ such that $E_\Sigma^c \subseteq U_\Sigma$ and $\nu \in V_\Sigma$. It follows from the above lemma that $\mu \notin E_\Sigma$. Then $\mu \in U_\Sigma$; hence, we obtain the desired result.

Examples below explain that we cannot reverse the above proposition in general. $\square$

Example 1. The collections $\delta_1 = \{\Phi, \tilde{\mathbb{Y}}, E_{1\Sigma}, E_{3\Sigma}\}$, $\delta_2 = \{\Phi, \tilde{\mathbb{Y}}, E_{4\Sigma}, E_{5\Sigma}, E_{6\Sigma}\}$, and $\delta_3 = \{\Phi, \tilde{\mathbb{Y}}, E_{1\Sigma}, E_{2\Sigma}, E_{3\Sigma}\}$ are three infrasoft topologies on the universal set $\mathbb{Y} = \{\nu_1, \nu_2\}$ with a set of parameters $\Sigma = \{\sigma_1, \sigma_2\}$, where

$$\begin{aligned} E_{1\Sigma} &= \{(\sigma_1, \{\nu_1\}), (\sigma_2, \{\nu_1\})\}; \\ E_{2\Sigma} &= \{(\sigma_1, \{\nu_2\}), (\sigma_2, \{\nu_2\})\}; \\ E_{3\Sigma} &= \{(\sigma_1, \{\nu_2\}), (\sigma_2, \emptyset)\}; \\ E_{4\Sigma} &= \{(\sigma_1, \{\nu_1\}), (\sigma_2, \mathbb{Y})\}; \\ E_{5\Sigma} &= \{(\sigma_1, \{\nu_2\}), (\sigma_2, \mathbb{Y})\}; \\ E_{6\Sigma} &= \{(\sigma_1, \emptyset), (\sigma_2, \mathbb{Y})\}; \end{aligned} \quad (3)$$

Obviously, $(\mathbb{Y}, \delta_1, \Sigma)$, $(\mathbb{Y}, \delta_2, \Sigma)$, and $(\mathbb{Y}, \delta_3, \Sigma)$ are, respectively, infra- tp -soft T_0 , infra- tp -soft T_1 , and infra- tp -soft T_2 . On the other hand, $(\mathbb{Y}, \delta_1, \Sigma)$ is not infra- tp -soft T_1 , $(\mathbb{Y}, \delta_2, \Sigma)$ is not infra- tp -soft T_2 , and $(\mathbb{Y}, \delta_3, \Sigma)$ is not infra- tp -soft T_3 .

Remark 1. In the above example, $(\mathbb{Y}, \delta_3, \Sigma)$ is an infra- tp -soft T_4 -space. As we know, the soft topology and general topology (infrasoft topology and general infratopology) are identical if a set of parameters is a singleton. Then we can say that there is an infrasoft topology which is infra- tp -soft T_3 , but not infra- tp -soft T_4 . Hence, the concepts of infra- tp -soft T_3 and infra- tp -soft T_4 -spaces are independent of each other.

Now, we explore and discuss some properties of infra- tp -soft T_i -spaces.

Proposition 5. *If P_σ^ν is an infrasoft closed subset of $(\mathbb{Y}, \delta, \Sigma)$ for each $\nu \in \mathbb{Y}$, then $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tp -soft T_1 -space.*

Proof. Let $\nu \neq \mu \in \mathbb{Y}$. By assumption, $(P_\sigma^\nu)^c$ and $(P_\sigma^\mu)^c$ are infrasoft open sets. Obviously, $\mu \in (P_\sigma^\nu)^c, \nu \notin (P_\sigma^\nu)^c$, and $\nu \in (P_\sigma^\mu)^c, \mu \notin (P_\sigma^\mu)^c$. Hence, $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_1 .

In Example 1, $(\mathbb{Y}, \delta_2, \Sigma)$ is infra- tp -soft T_1 . However, $P_{\sigma_2}^{\nu_1}$ and $P_{\sigma_2}^{\nu_2}$ are not infrasoft closed sets. So, the converse of the above proposition need not be true in general. $\square$

Proposition 6. *$(\mathbb{Y}, \delta, \Sigma)$ is an infra- tp -soft T_2 -space iff for each $\nu \neq \mu$, there exists an infrasoft open set E_Σ such that $\nu \in E_\Sigma$ and $\mu \notin Cl_\delta(E_\Sigma)$, where $\mu \notin Cl_\delta(E_\Sigma)$ if there is an infrasoft open set G_Σ such that $\mu \in G_\Sigma$ and $E_\Sigma \cap G_\Sigma = \Phi$.*

Proof. Let $\nu \neq \mu$ in an infra- tp -soft T_2 -space. Then there exist disjoint infrasoft open sets E_Σ and F_Σ such that $\nu \in E_\Sigma$ and $\mu \in F_\Sigma$. Now, $E_\Sigma \subseteq F_\Sigma^c$ which means $Cl_\delta(E_\Sigma) \subseteq F_\Sigma^c$. Since $\mu \notin F^c(\sigma)$ for each $\sigma \in \Sigma$, $\mu \notin Cl_\delta(E_\Sigma)$. Conversely, let $\nu \neq \mu \in \mathbb{Y}$ and E_Σ be an infrasoft open set such that $\nu \in E_\Sigma$ and $\mu \notin Cl_\delta(E_\Sigma)$. Then there is an infrasoft open set G_Σ such that $\mu \in G_\Sigma$ and $E_\Sigma \cap G_\Sigma = \Phi$. This ends the proof that $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_2 . $\square$

Proposition 7. *If $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tp -soft regular space, then for each $\nu \in \mathbb{Y}$ and infrasoft open set H_Σ partially containing ν , there exists an infrasoft open set G_Σ such that $\nu \in G_\Sigma \subseteq Cl_\delta(G_\Sigma) \subseteq H_\Sigma$.*

Proof. Let H_Σ be an infrasoft open set partially containing ν . Then H_Σ^c is an infrasoft closed set and $\nu \notin H_\Sigma^c$. By

assumption, there are disjoint infrasoft open sets E_Σ and G_Σ such that $H_\Sigma \subseteq E_\Sigma$ and $\nu \in G_\Sigma$. Obviously, $G_\Sigma \subseteq E_\Sigma^c \subseteq H_\Sigma$. Thus, $\text{Cl}_\delta(G_\Sigma) \subseteq E_\Sigma^c \subseteq H_\Sigma$. $\square$

Theorem 3. *If $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tp -soft regular space, then the following concepts are identical.*

- (i) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_2 .
- (ii) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_1 .
- (iii) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_0 .

Proof. The implications (i) $\longrightarrow$ (ii) $\longrightarrow$ (iii) follow from Proposition 4.

To prove (iii) $\longrightarrow$ (i), let $\nu \neq \mu \in \mathbb{Y}$. Since $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_0 , then we have an infrasoft open set E_Σ such that $\nu \in E_\Sigma$ and $\mu \notin E_\Sigma$, or $\mu \in E_\Sigma$ and $\nu \notin E_\Sigma$. Say, $\nu \in E_\Sigma$ and $\mu \notin E_\Sigma$. It is clear that $\nu \notin E_\Sigma^c$ and $\mu \in E_\Sigma^c$. Since $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft regular, there exist two disjoint infrasoft open sets U_Σ and V_Σ such that $\nu \in U_\Sigma$ and $\mu \in V_\Sigma$. Since U_Σ and V_Σ are disjoint, $\mu \notin U_\Sigma$ and $\nu \notin V_\Sigma$. This ends the proof that $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_2 . $\square$

Proposition 8. *A finite infra- tp -soft T_2 -space $(\mathbb{Y}, \delta, \Sigma)$ contains at least $|\mathbb{Y}| + 2$ infrasoft open sets, where $|\mathbb{Y}| > 1$.*

Proof. We prove the proposition using mathematical induction. When $|\mathbb{Y}| = 2$, we need two infrasoft open sets beside the null and absolute soft sets. Then the claim is true for $i = 2$. Suppose that the claim is true for $|\mathbb{Y}| = n$; i.e., $|\delta| = n + 2$. Without loss of generality, consider $(\mathbb{Y}, \delta, \Sigma)$ as the smallest infra- tp -soft T_2 -space. Now, let $|\mathbb{Y}| = n + 1$; i.e., $\mathbb{Y}' = \mathbb{Y} \cup \{\xi\}$, where $\xi \notin \mathbb{Y}$. Then $\delta' = \delta \cup \{\xi\}$ is the smallest infra- tp -soft T_2 -space on $\mathbb{Y}'$. Hence, we obtain the desired result.

In the following result we will study the properties and relationships between infrasoft topology and its parametric infratopologies. $\square$

Proposition 9. *If $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tp -soft T_i -space, then $(\mathbb{Y}, \delta_\sigma)$ is infra- T_i for each $\sigma \in \Sigma$, where $i = 2, 3$.*

Proof. Let $\nu \neq \mu$ in an infra- tp -soft T_2 -space $(\mathbb{Y}, \delta, \Sigma)$. Then there exist disjoint infrasoft open sets E_Σ and F_Σ totally containing ν and μ , respectively. Therefore, $E(\sigma)$ and $F(\sigma)$ are disjoint infraopen sets containing ν and μ , respectively. Hence, $(\mathbb{Y}, \delta_\sigma)$ is an infra- T_2 -space.

To prove the proposition in case of $i = 3$, it suffices to prove the axiom of infraregularity. To do that, let G be an infraclosed subset of $(\mathbb{Y}, \delta_\sigma)$ such that $\nu \notin G$. Since δ is infrasoft tp -regular, a soft set $H_\Sigma = \{(\sigma, G): \text{for each } \sigma \in \Sigma\}$ is an infrasoft closed subset of $(\mathbb{Y}, \delta, \Sigma)$ such that $\nu \notin H_\Sigma$. By assumption, there exist disjoint infrasoft open sets E_Σ and F_Σ such that $H_\Sigma \subseteq E_\Sigma$ and $\nu \in F_\Sigma$. Thus, $E(\sigma)$ and $F(\sigma)$ are disjoint infraopen subsets of $(\mathbb{Y}, \delta_\sigma)$ containing G and ν , respectively. Hence, $(\mathbb{Y}, \delta_\sigma)$ is infraregular, as required.

To demonstrate that the converse of the proposition above fails, we supply the next example. $\square$

Example 2. Consider $\mathbb{Y}$ and Σ as the same as in Example 1. Then $\delta = \{\Phi, \tilde{\mathbb{Y}}, E_\Sigma, F_\Sigma\}$ is an infrasoft topology on $\mathbb{Y}$, where

$$\begin{aligned} E_\Sigma &= \{(\sigma_1, \{\nu\}), (\sigma_2, \{\mu\})\}, \\ F_\Sigma &= \{(\sigma_1, \{\mu\}), (\sigma_2, \{\nu\})\}. \end{aligned} \quad (4)$$

It is clear that $(\mathbb{Y}, \delta, \Sigma)$ is not an infra- tp -soft T_0 -space. On the other hand, $(\mathbb{Y}, \delta_{\sigma_1})$ and $(\mathbb{Y}, \delta_{\sigma_2})$ are infra- T_3 and infra- T_4 .

From the above example and the next two examples, we elucidate that there is not a relationship between infra- tp -soft T_i -spaces and their parametric infratopological spaces when $i = 0, 1, 4$.

Example 3. In Example 1, note that $(\mathbb{Y}, \delta_{\sigma_2})$ is not infra- T_0 in spite of the fact that $(\mathbb{Y}, \delta_2, \Sigma)$ is infra- tp -soft T_1 .

Example 4. The collection $\delta = \{\Phi, \tilde{\mathbb{Y}}, E_{1\Sigma}, E_{3\Sigma}, E_{3\Sigma}, E_{4\Sigma}\}$ is an infrasoft topology on $\mathbb{Y} = \{\nu_1, \nu_2, \nu_3\}$ with $\Sigma = \{\sigma_1, \sigma_2\}$ as a set of parameters, where

$$\begin{aligned} E_{1\Sigma} &= \{(\sigma_1, \{\nu_1\}), (\sigma_2, \mathbb{Y})\}; \\ E_{2\Sigma} &= \{(\sigma_1, \{\nu_2\}), (\sigma_2, \mathbb{Y})\}; \\ E_{3\Sigma} &= \{(\sigma_1, \{\nu_3\}), (\sigma_2, \mathbb{Y})\}, \\ E_{4\Sigma} &= \{(\sigma_1, \emptyset), (\sigma_2, \mathbb{Y})\}. \end{aligned} \quad (5)$$

One can check that $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_4 . But $(\mathbb{Y}, \delta_{\sigma_2})$ is not infra- T_4 .

Theorem 4. *Let $(\mathbb{Y}, \delta, \Sigma)$ be stable. Then, $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_i iff $(\mathbb{Y}, \delta_\sigma)$ is infra- T_i for each $i = 0, 1, 2, 3, 4$.*

Proof. Since $(\mathbb{Y}, \delta, \Sigma)$ is stable, then E is an infraopen subset of $(\mathbb{Y}, \delta_\sigma)$ if and only if $\{(\sigma, E): \sigma \in \Sigma\}$ is an infrasoft open subset of $(\mathbb{Y}, \delta, \Sigma)$. Hence, the desired result is proved. $\square$

Remark 2. Any extended infrasoft topological space is not infra- tp -soft regular; hence, it is not infra- tp -soft T_3 .

Theorem 5. *Let $(\mathbb{Y}, \delta, \Sigma)$ be extended. If there exists $\sigma \in \Sigma$ such that $(\mathbb{Y}, \delta_\sigma)$ is infra- T_i , then $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_i for $i = 0, 1$.*

Proof. When $i = 1$, let $\nu \neq \mu \in \mathbb{Y}$. Since $(\mathbb{Y}, \delta_\sigma)$ is infra- T_1 , there are two infraopen sets U, V in δ_σ such that $\nu \in U$, $\mu \in V$. Since δ is extended, there are two infrasoft open sets G_Σ, H_Σ in δ such that

$$\begin{aligned} G(\sigma) &= U \text{ and } G(\sigma') = \mathbb{Y}, \quad \text{for each } \sigma' \neq \sigma, \\ H(\sigma) &= V \text{ and } H(\sigma') = \mathbb{Y}, \quad \text{for each } \sigma' \neq \sigma. \end{aligned} \quad (6)$$

Now, $\nu \in G_\Sigma$, $\mu \notin G_\Sigma$, and $\mu \in H_\Sigma$, $\nu \notin H_\Sigma$. Hence, (X, δ, Σ) is infra- tp -soft T_1 .

The case $i = 0$ can be proved similarly. $\square$

Theorem 6. *Let $(\mathbb{Y}, \delta, \Sigma)$ be extended. If all $(\mathbb{Y}, \delta_\sigma)$ are infra- T_i , then (X, δ, Σ) is infra- tp -soft T_i for $i = 0, 1, 2, 4$.*

Proof. We only prove the theorem when $i = 4$.

The case of $i = 1$ was proved in the above theorem. So, it suffices to prove the axioms of infrasoftware normality. To do this, let H_Σ, L_Σ be two disjoint infrasoftware closed sets. Then $H(\sigma)$ and $L(\sigma)$ are two disjoint infrasoftware closed sets for each $\sigma \in \Sigma$. Since (X, δ_σ) is infranormal, there are two disjoint infrasoftware open sets U_σ and V_σ such that $H(\sigma) \subseteq U_\sigma$ and $L(\sigma) \subseteq V_\sigma$. Since $(\mathbb{Y}, \delta, \Sigma)$ is extended, there are disjoint infrasoftware open subsets G_Σ, F_Σ of $(\mathbb{Y}, \delta, \Sigma)$ such that

$$\begin{aligned} G(\sigma) &= U_\sigma, & \text{for each } \sigma \in \Sigma, \\ F(\sigma) &= V_\sigma, & \text{for each } \sigma \in \Sigma. \end{aligned} \quad (7)$$

Thus, (X, δ, Σ) is infrasoftware normal. Hence, it is infra- tp -soft T_4 . $\square$

Theorem 7. Every soft subspace $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ of an infra- tp -soft T_i -space $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tp -soft T_i -space, for $i = 0, 1, 2, 3$.

Proof. We give a proof for $i = 3$. The other cases are made similarly.

First, we prove that $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ is infra- tp -soft T_1 . To do this, let $\nu \neq \mu \in \mathbb{Z} \subseteq \mathbb{Y}$. Since $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_1 , δ contains infrasoftware open sets E_Σ and F_Σ such that $\nu \in E_\Sigma, \mu \notin E_\Sigma$ and $\mu \in F_\Sigma, \nu \notin F_\Sigma$. Now, $U_\Sigma = \widetilde{\mathbb{Y}} \cap E_\Sigma$ and $V_\Sigma = \widetilde{\mathbb{Y}} \cap F_\Sigma$ are infrasoftware open subsets of $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ such that $\nu \in U_\Sigma, \mu \notin U_\Sigma$ and $\mu \in V_\Sigma, \nu \notin V_\Sigma$. Thus, $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ is infra- tp -soft T_1 .

Second, we prove that $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ is infra- tp -soft regular. To do this, let L_Σ be an infrasoftware closed subset of $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ and $\mu \in \mathbb{Z}$ such that $\mu \notin L_\Sigma$. Then there exists an infrasoftware closed subset H_Σ of $(\mathbb{Y}, \delta, \Sigma)$ such that $L_\Sigma = \widetilde{\mathbb{Y}} \cap H_\Sigma$. By assumption, there exist disjoint infrasoftware open sets E_Σ and F_Σ such that $H_\Sigma \subseteq E_\Sigma$ and $\mu \in F_\Sigma$. Now, $\widetilde{\mathbb{Y}} \cap E_\Sigma$ and $\widetilde{\mathbb{Y}} \cap F_\Sigma$ are infrasoftware open subsets of $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ such that $L_\Sigma \subseteq \widetilde{\mathbb{Y}} \cap E_\Sigma$ and $\mu \in \widetilde{\mathbb{Y}} \cap F_\Sigma$. Thus, $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ is infra- tp -soft regular. This ends the proof that $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ is infra- tp -soft T_3 .

Following similar technique given above, one can prove the following result. $\square$

Theorem 8. Every infrasoftware closed subspace $(\mathbb{Z}, \delta_\Sigma, \Sigma)$ of an infra- tp -soft T_4 -space $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tp -soft T_4 -space.

Theorem 9. The concepts of infra- tp -soft T_i -spaces are preserved under finite product spaces, where $i = 0, 1, 2$.

Proof. We give a proof for the theorem when $j = 2$. The remaining two cases follow similar lines.

Without loss of generality, we consider $(\mathbb{Y}_1, \delta_1, \Sigma_1)$ and $(\mathbb{Y}_2, \delta_2, \Sigma_2)$ as two infra- tp -soft T_2 -spaces. Let $(\nu_1, \mu_1) \neq (\nu_2, \mu_2)$ in $\mathbb{Y}_1 \times \mathbb{Y}_2$. Then $\nu_1 \neq \nu_2$ or $\mu_1 \neq \mu_2$. Say, $\nu_1 \neq \nu_2$. By assumption, δ_1 contains two disjoint infrasoftware open sets $E_{\Sigma_1}, F_{\Sigma_1}$ such that $\nu_1 \in E_{\Sigma_1}$ and $\nu_2 \notin E_{\Sigma_1}$, and $\nu_2 \in F_{\Sigma_1}$ and $\nu_1 \notin F_{\Sigma_1}$. Now, $E_{\Sigma_1} \times \widetilde{\mathbb{Y}_2}$ and $F_{\Sigma_1} \times \widetilde{\mathbb{Y}_2}$ are disjoint infrasoftware open sets such that $(\nu_1, \mu_1) \in E_{\Sigma_1} \times \widetilde{\mathbb{Y}_2}$ and $(\nu_2, \mu_2) \notin E_{\Sigma_1} \times \widetilde{\mathbb{Y}_2}$, and $(\nu_2, \mu_2) \in F_{\Sigma_1} \times \widetilde{\mathbb{Y}_2}$ and $(\nu_1, \mu_1) \notin F_{\Sigma_1} \times \widetilde{\mathbb{Y}_2}$. Hence, $\mathbb{Y}_1 \times \mathbb{Y}_2$ is infra- tp -soft T_2 .

We close this section by studying the behaviors of infra- tp -soft T_i -spaces under some types of soft maps. $\square$

Proposition 10. The inverse image of infra- tp -soft T_i -spaces under an infrasoftware continuous map f_η is infra- tp -soft T_i for $i = 0, 1, 2$ provided that f is injective and η is surjective.

Proof. When $i = 2$, consider $f_\eta: (\mathbb{Y}, \delta, \Sigma) \longrightarrow (\mathbb{Z}, \theta, \Omega)$ is an infrasoftware continuous map. Let $\nu \neq \mu \in \mathbb{Y}$. By the injectivity of f , there are only two distinct points $\zeta, \xi \in \mathbb{Z}$ such that $f(\nu) = \zeta$ and $f(\mu) = \xi$. By assumption, there are two disjoint infrasoftware open sets E_Ω and G_Ω such that $\zeta \in E_\Omega, \xi \notin E_\Omega$ and $\xi \in G_\Omega, \zeta \notin G_\Omega$. Now, $f_\eta^{-1}(E_\Omega)$ and $f_\eta^{-1}(G_\Omega)$ are disjoint infrasoftware open sets. By the surjectivity of η we obtain from Proposition 1 that $\nu \in f_\eta^{-1}(E_\Omega), \mu \notin f_\eta^{-1}(E_\Omega)$ and $\mu \in f_\eta^{-1}(G_\Omega), \nu \notin f_\eta^{-1}(G_\Omega)$. Hence, we get the coveted result.

In a similar manner, one can prove the next results. $\square$

Proposition 11. The inverse image of infra- tp -soft T_i -spaces under a bijective infrasoftware continuous map f_η is infra- tp -soft T_i for $i = 0, 1, 2, 3, 4$.

Proposition 12. The image of infra- tp -soft T_i -spaces under a bijective infrasoftware open map f_η is infra- tp -soft T_i for $i = 0, 1, 2, 3, 4$.

Proposition 13. The property of being an infra- tp -soft T_i -space ($i = 0, 1, 2, 3, 4$) is preserved under an infrasoftware homeomorphism map.

4. Infra- tt -Soft T_i -Spaces ($i = 0, 1, 2, 3, 4$)

In this part, we apply the relations of total belong and total nonbelong to define the concepts of infra- tt -soft T_i -spaces ($i = 0, 1, 2, 3, 4$), where tt is the abbreviation for the first letter of total belong and total nonbelong relations that are used to define these spaces. We provide some counterexamples to illustrate the relationships between these concepts and the interrelations between them and infra- tp -soft T_i -spaces.

Definition 19. $(\mathbb{Y}, \delta, \Sigma)$ is said to be

- (i) An infra- tt -soft T_0 -space if for every $\nu \neq \mu \in \mathbb{Y}$, there is an infrasoftware open set E_Σ such that $\nu \in E_\Sigma, \mu \notin E_\Sigma$, or $\mu \in E_\Sigma, \nu \notin E_\Sigma$;
- (ii) An infra- tt -soft T_1 -space if for every $\nu \neq \mu \in \mathbb{Y}$, there are infrasoftware open sets E_Σ and F_Σ such that $\nu \in E_\Sigma, \mu \notin E_\Sigma$ and $\mu \in F_\Sigma, \nu \notin F_\Sigma$. In other words, $\nu \in E_\Sigma, \mu \notin E_\Sigma$ and $\mu \in F_\Sigma, \nu \notin F_\Sigma$;
- (iii) An infra- tt -soft T_2 -space if for every $\nu \neq \mu \in \mathbb{Y}$, there are two disjoint infrasoftware open sets E_Σ and F_Σ such that $\nu \in E_\Sigma \setminus F_\Sigma$ and $\mu \in F_\Sigma \setminus E_\Sigma$;
- (iv) An infra- tt -soft regular space if for every infrasoftware closed set H_Σ such that $\nu \notin H_\Sigma$, there are disjoint infrasoftware open sets E_Σ and F_Σ such that $H_\Sigma \subseteq E_\Sigma$ and $\nu \in F_\Sigma$;

- (v) An infra- tt -soft T_3 (resp., infra- tt -soft T_4) -space if it is infra- tt -soft regular (resp., infrasoftware normal) and infra- tt -soft T_1 .

This section begins with the next result which describes the relationships between infra- tt -soft T_i -spaces as well as the interrelations between them and infra- tp -soft T_i -spaces.

Proposition 14

- (i) Every infra- tt -soft T_i -space is infra- tt -soft T_{i-1} for $i = 1, 2, 3$.
- (ii) The concepts of infra- tt -soft T_2 -space and infra- tp -soft T_2 -space are identical.
- (iii) Every infra- tt -soft T_i -space is infra- tp -soft T_i for $i = 0, 1, 4$.
- (iv) Every infra- tp -soft T_3 -space is infra- tt -soft T_3 .

Proof

- (i) The proof of the cases $i = 1, 2$ follows directly from the above definition. To prove case $i = 3$, let ν, μ be two distinct elements in an infra- tt -soft T_3 -space. Then there exists an infrasoftware open set E_Σ such that $\nu \in E_\Sigma$ and $\mu \notin E_\Sigma$. Now, E_Σ^c is infrasoftware closed such that $\nu \notin E_\Sigma^c$. By hypothesis, there exist disjoint infrasoftware open sets U_Σ and V_Σ such that $E_\Sigma^c \subseteq U_\Sigma$ and $\nu \in V_\Sigma$. Obviously, $\mu \in U_\Sigma$; hence, we obtain the desired result.
- (ii) Let E_Σ and F_Σ be two disjoint soft sets such that $\nu \in E_\Sigma$ and $\mu \in F_\Sigma$. Then $\mu \notin E_\Sigma$ iff $\mu \notin E_\Sigma$, and $\nu \notin F_\Sigma$ iff $\nu \notin F_\Sigma$. This proves the required equivalence.
- (iii) obvious.
- (iv) It is clear that every infra- tp -soft regular space is infra- tt -soft regular. According to Lemma 1, the concepts of infra- tt -soft T_1 -space and infra- tp -soft T_1 -space are identical. Hence, we get the coveted result.

To elucidate that the converses of the above proposition fail, we provide the next three counterexamples. $\square$

Example 5. Let $\mathbb{Y}$ be any infinite set and $\nu, \mu \in \mathbb{Y}$. We define the following three infrasoftware topologies on $\mathbb{Y}$ with Σ as set of parameters such that $|\Sigma| \neq 1$.

$$\begin{aligned} \delta_1 &= \{\tilde{\mathbb{Y}}\} \cup \{E_\Sigma \subseteq \tilde{\mathbb{Y}} : \nu \notin E_\Sigma\}; \\ \delta_2 &= \{\Phi\} \cup \{E_\Sigma \subseteq \tilde{\mathbb{Y}} : E_\Sigma^c \text{ is finite}\}, \\ \delta_3 &= \{\tilde{\mathbb{Y}}\} \cup \{E_\Sigma \subseteq \tilde{\mathbb{Y}} : \nu \notin E_\Sigma \text{ or } \mu \notin E_\Sigma\}. \end{aligned} \quad (8)$$

Now, we can note that $(\mathbb{Y}, \delta_1, \Sigma)$, $(\mathbb{Y}, \delta_2, \Sigma)$, and $(\mathbb{Y}, \delta_3, \Sigma)$ are, respectively, infra- tt -soft T_0 , infra- tt -soft T_1 , and infra- tt -soft T_2 . On the other hand, we can check that $(\mathbb{Y}, \delta_1, \Sigma)$ is not infra- tt -soft T_1 and $(\mathbb{Y}, \delta_2, \Sigma)$ is not infra- tt -soft T_2 . Also, $(\mathbb{Y}, \delta_3, \Sigma)$ is not infra- tt -soft T_3 because there does not exist an infrasoftware open set (except for the absolute soft set) containing an infrasoftware closed set $E_\Sigma = \{(\sigma, \{\nu, \mu\}) : \text{for each } \sigma \in \Sigma\}$.

Example 6. Consider $(\mathbb{Y}, \delta, \Sigma)$ as the same as given in Example 4. Then $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tp -soft T_4 -space but it is not infra- tt -soft T_0 .

Example 7. Any discrete soft topological space is infra- tt -soft T_3 . On the other hand, it contains some infrasoftware open sets which are not stable, so it is not infra- tp -soft regular. Hence, it is not infra- tp -soft T_3 .

Remark 3. We show that $(\mathbb{Y}, \delta_3, \Sigma)$ given in Example 5 is not infra- tt -soft T_3 . In contrast, one can check that it is infra- tt -soft T_4 . According to Remark 1 there is an infra- tt -soft T_3 -space which is not infra- tt -soft T_4 . Hence, the concepts of infra- tt -soft T_3 and infra- tt -soft T_4 -spaces are independent of each other.

Theorem 10. Let $(\mathbb{Y}, \delta, \Sigma)$ be a stable space. Then the following statements hold.

- (i) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft $T_0 \Leftrightarrow (\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_0 .
- (ii) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft $T_1 \Leftrightarrow (\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_1 .
- (iii) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft $T_2 \Leftrightarrow (\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_2 .
- (iv) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft $T_3 \Leftrightarrow (\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_3 .
- (v) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft $T_4 \Leftrightarrow (\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_4 .

Proof. It comes from the fact that partial nonbelong and total nonbelong relations are identical with respect to the stable soft sets.

In what follows, we study and investigate some properties of infra- tt -soft T_i -spaces. $\square$

Proposition 15. If every singleton soft subset of $(\mathbb{Y}, \delta, \Sigma)$ is infrasoftware closed, then $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tt -soft T_1 -space.

Proof. Let $\nu \neq \mu \in \mathbb{Y}$. By assumption, $(\nu_\Sigma)^c$ and $(\mu_\Sigma)^c$ are infrasoftware open sets. Obviously, $\mu \in (\nu_\Sigma)^c$, $\nu \notin (\nu_\Sigma)^c$, and $\nu \in (\mu_\Sigma)^c$, $\mu \notin (\mu_\Sigma)^c$. Hence, $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft T_1 .

The converse of the above proposition is not always true as the next counterexample explains. $\square$

Example 8. Let $\delta = \{\tilde{\mathbb{N}}\} \cup \{F_\Sigma \subseteq \tilde{\mathbb{N}} : F_\Sigma \text{ is finite}\}$ be an infrasoftware topology on the set of natural numbers $\mathbb{N}$ with Σ as any set of parameters. It is clear that $(\mathbb{N}, \delta, \Sigma)$ is infra- tt -soft T_1 . But every singleton soft set is not infrasoftware closed.

Proposition 16. Let $\mathbb{Y}$ be a finite set. Then, $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft T_1 iff every singleton soft set is infrasoftware open.

Proof. To prove the necessary condition, let ν_Σ be a singleton soft set. Then for each $\mu_i \neq \nu \in \mathbb{Y}$ there exists an infrasoftware open set E_Σ such that $\nu \in E_\Sigma$ and $\mu_i \notin E_\Sigma$. Now,

$\bigcap_{i=1}^{n-1} E_{i\Sigma} = \nu_\Sigma$ is an infrasoft open set, where n is the cardinality number of $\mathbb{Y}$.

The sufficient condition is obvious. $\square$

Corollary 2. If $(\mathbb{Y}, \delta, \Sigma)$ is a finite infra- tt -soft T_1 -space, then $|\delta| \geq |\mathbb{Y}| + 2$.

Corollary 3. If $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tt -soft T_1 -space, then $\nu_\Sigma = \bigcap \{E_\Sigma : \nu \in E_\Sigma \in \delta\}$ for all $\nu \in \mathbb{Y}$.

Proposition 17. $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tt -soft T_2 -space iff for every $\nu \neq \mu$, there exists an infrasoft open set E_Σ such that $\nu \in E_\Sigma$ and $\mu \notin Cl_\delta(E_\Sigma)$.

Proof. It comes from Proposition 6 and item (ii) of Proposition 14. $\square$

Proposition 18. A finite infra- tp -soft T_2 -space $(\mathbb{Y}, \delta, \Sigma)$ contains at least $|\mathbb{Y}| + 2$ infrasoft open sets, where $|\mathbb{Y}| > 1$.

Proof. It comes from Proposition 8 and item (ii) of Proposition 14. $\square$

Proposition 19. If $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tt -soft regular space, then for each $\nu \in \mathbb{Y}$ and infrasoft open set H_Σ totally containing ν , there exists an infrasoft open set G_Σ such that $\nu \in G_\Sigma \subseteq Cl_\delta(G_\Sigma) \subseteq H_\Sigma$.

Proof. It follows similar arguments given in the proof of Proposition 7. $\square$

Theorem 11. Let $(\mathbb{Y}, \delta, \Sigma)$ be infra- tp -soft regular. Then $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft T_2 iff it is infra- tp -soft T_0 .

Proof. It comes from Theorem 3 and item (ii) of Proposition 14. $\square$

Corollary 4. The following concepts are equivalent under an infra- tp -soft regular space $(\mathbb{Y}, \delta, \Sigma)$.

- (i) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft T_2 ;
- (ii) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft T_1 ;
- (iii) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft T_0 ;
- (iv) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_0 ;
- (v) $(\mathbb{Y}, \delta, \Sigma)$ is infra- tp -soft T_1 .

Now, we determine the conditions under which the property infra- tt -soft T_i -spaces is transmitted from infrasoft topology to crisp infratopology and vice versa.

Proposition 20. If $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tt -soft T_i -space, then $(\mathbb{Y}, \delta_\sigma)$ is infra- T_i for each $\sigma \in \Sigma$, where $i = 0, 1, 2$.

Proof. Let $\nu \neq \mu$ in an infra- tt -soft T_0 -space $(\mathbb{Y}, \delta, \Sigma)$. Without loss of generality, there is an infrasoft open set E_Σ such that $\nu \in E_\Sigma$ and $\mu \notin E_\Sigma$. Therefore, $E(\sigma)$ is an infraopen set such that $\nu \in E(\sigma)$ and $\mu \notin E(\sigma)$. Hence, $(\mathbb{Y}, \delta_\sigma)$ is an infra- T_0 -space.

The cases $i = 1, 2$ are achieved similarly.

Example 2 illustrates that the converse of the above proposition fails.

The next remark and example elucidate that there is not a relationship between infra- tt -soft T_i -spaces and their parametric infratopological spaces when $i = 3, 4$. $\square$

Remark 4. $(\mathbb{Y}, \delta_3, \Sigma)$ given in Example 5 is not infra- tt -soft T_3 but all its parametric infratopological spaces are infra- T_3 . Also, $(\mathbb{Y}, \delta_1, \Sigma)$ given in Example 5 is not infra- tt -soft T_4 but all its parametric infratopological spaces are infra- T_4 .

Example 9. Let $\mathbb{Y} = \{\nu_1, \nu_2\}$ be the universal set and $\Sigma = \{\sigma_1, \sigma_2\}$ be a parameters set. We define the following soft sets $E_{i\Sigma}$ over $\mathbb{Y}$ as follows:

$$\begin{aligned} E_{1\Sigma} &= \{(\sigma_1, \{\nu_1\}), (\sigma_2, \mathbb{Y})\}; \\ E_{2\Sigma} &= \{(\sigma_1, \{\nu_2\}), (\sigma_2, \mathbb{Y})\}, \\ E_{3\Sigma} &= \{(\sigma_1, \emptyset), (\sigma_2, \mathbb{Y})\}. \end{aligned} \quad (9)$$

Then $\delta = \{\Phi, \tilde{\mathbb{Y}}, E_{1\Sigma}, E_{2\Sigma}, E_{3\Sigma}\}$ is an infrasoft topology on $\mathbb{Y}$. Obviously, $(\mathbb{Y}, \delta, \Sigma)$ is not infra- tt -soft T_0 . But its parametric infratopological space $(\mathbb{Y}, \delta_{\sigma_1})$ is infra- T_4 and infra- T_3 .

Theorem 12. Let $(\mathbb{Y}, \delta, \Sigma)$ be stable (infra- tp -soft regular). Then, $(\mathbb{Y}, \delta_\sigma)$ is infra- T_i iff $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft T_i for each $i = 0, 1, 2, 3, 4$.

Proof. It is straightforward. $\square$

Theorem 13. Let $(\mathbb{Y}, \delta, \Sigma)$ be extended. If all $(\mathbb{Y}, \delta_\sigma)$ are infra- T_i , then (X, δ, Σ) is infra- tt -soft T_i for $i = 0, 1, 2, 3, 4$.

Proof. The cases of $i = 0, 1, 2$ were proved in Theorem 1. The case of $i = 4$ can be made similar to the proof of Theorem 6. It remains to prove case $i = 3$. It suffices to prove the axiom of infra- tt -soft regularity. To do this, let H_Σ be an infrasoft closed set such that $\nu \notin H_\Sigma$. Then $\nu \notin H(\sigma)$ for each $\sigma \in \Sigma$. Since $H(\sigma)$ is an infraclosed set and $(\mathbb{Y}, \delta_\sigma)$ is infraregular, there exist disjoint infraopen sets U_σ and V_σ such that $H(\sigma) \subseteq U_\sigma$ and $\nu \in V_\sigma$. Since $(\mathbb{Y}, \delta, \Sigma)$ is extended, then there exist infrasoft open sets G_Σ and F_Σ such that $G(\sigma) = U_\sigma$ and $F(\sigma) = V_\sigma$ for each $\sigma \in \Sigma$. Obviously, $H_\Sigma \subseteq G_\Sigma$ and $\nu \in F_\Sigma$. The disjointness of G_Σ and F_Σ proves that $(\mathbb{Y}, \delta, \Sigma)$ is infra- tt -soft regular.

The following findings can be proved in a similar way to their counterparts in Section 3. $\square$

Theorem 14. Every soft subspace $(\mathbb{Z}, \delta_\mathbb{Z}, \Sigma)$ of an infra- tt -soft T_i -space $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tt -soft T_i -space for $i = 0, 1, 2, 3$.

Theorem 15. Every infrasoft closed subspace $(\mathbb{Z}, \delta_\mathbb{Z}, \Sigma)$ of an infra- tt -soft T_4 -space $(\mathbb{Y}, \delta, \Sigma)$ is an infra- tt -soft T_4 -space.

Theorem 16. The concepts of infra- tt -soft T_i -spaces are preserved under finite product spaces, where $i = 0, 1, 2$.

FIGURE 1: The relationships between infratopological separation axioms.

Proposition 21. The inverse image of infra- tt -soft T_i -spaces under an infratopological continuous map f_η is infra- tt -soft T_i for $i = 0, 1, 2$ provided that f is injective.

Proposition 22. The inverse image of infra- tt -soft T_i -spaces under a bijective infratopological continuous map f_η is infra- tt -soft T_i for $i = 0, 1, 2, 3, 4$.

Proposition 23. The image of infra- tt -soft T_i -spaces under a bijective infratopological open map f_η is infra- tt -soft T_i for $i = 0, 1, 2, 3, 4$.

Proposition 24. The property of being an infra- tt -soft T_i -space ($i = 0, 1, 2, 3, 4$) is preserved under an infratopological homeomorphism map.

In the end of this work, we summarize the relationships between the two introduced infratopological separation axioms in Figure 1.

5. Conclusion and Future Work

In this paper, we have introduced two types of separation axioms in the frame of infratopological spaces with respect to distinct ordinary points. The reasons beyond this study are the following.

- (1) Construct new classes of soft spaces and explore the (classical) topological properties that are valid in these spaces.
- (2) Investigate these soft spaces in the frame of rough sets models as those given [34–36] and show whether they have an influence on the rough approximations and accuracy measures.
- (3) Obtain new results induced from the interaction between these soft spaces and some celebrated concepts such as infratopological connectedness and infratopological compactness which we will introduce later.

We outline the achievements of this work in the following.

- (1) Formulate the concepts of infra- tp -soft T_i -spaces for $i = 0, 1, 2, 3, 4$ using total belong and partial nonbelong relations.
- (2) Formulate the concepts of infra- tt -soft T_i -spaces for $i = 0, 1, 2, 3, 4$ using total belong and total nonbelong relations.
- (3) Provide some counterexamples to show the relationships between the spaces in each type of them as well as the interrelationships between the two types.
- (4) Study the transmission property of these two types of soft spaces between infratopology and its parametric infratopologies.
- (5) Discuss the image and preimage of these two types of soft spaces under some infratopological maps.

In upcoming papers, we plan to do the following.

- (1) Establish novel types of soft spaces using the two pairs of relations $(\in, \notin)$ and $(\in, \notin)$.
- (2) Define the celebrated generalizations of open sets in the context of infratopological spaces such as infratopological α -open, infratopological semiopen, infratopological preopen, infratopological b -open, and infratopological β -open sets.
- (3) Carry out further investigations in the areas of rough set theory and infratopologies.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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