Research Article

Planar, Outerplanar, and Toroidal Graphs of the Generalized Zero-Divisor Graph of Commutative Rings

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Let $\mathcal{A}$ be a commutative ring with unity and let set of all zero divisors of $\mathcal{A}$ be denoted by $\mathcal{Z}(\mathcal{A})$. An ideal $\mathcal{I}$ of the ring $\mathcal{A}$ is said to be essential if it has a nonzero intersection with every nonzero ideal of $\mathcal{A}$. It is denoted by $\mathcal{I} \leq_e \mathcal{A}$. The generalized zero-divisor graph denoted by $\Gamma_g(\mathcal{A})$ is an undirected graph with vertex set $\mathcal{Z}(\mathcal{A})^*$ (set of all nonzero zero-divisors of $\mathcal{A}$) and two distinct vertices $\mathfrak{x}_1$ and $\mathfrak{x}_2$ are adjacent if and only if $\text{ann}(\mathfrak{x}_1) + \text{ann}(\mathfrak{x}_2) \leq_e \mathcal{A}$. In this paper, first we characterized all the finite commutative rings $\mathcal{A}$ for which $\Gamma_g(\mathcal{A})$ is isomorphic to some well-known graphs. Then, we classify all the finite commutative rings $\mathcal{A}$ for which $\Gamma_g(\mathcal{A})$ is planar, outerplanar, or toroidal. Finally, we discuss about the domination number of $\Gamma_g(\mathcal{A})$.

1. Introduction

In this paper, all rings $\mathcal{A}$ are commutative with unity. For any element $\mathfrak{x}_1 \in \mathcal{A}$, the annihilator of $\mathfrak{x}_1$, which is denoted by $\text{ann}(\mathfrak{x}_1)$, is defined as $\text{ann}(\mathfrak{x}_1) = \{\mathfrak{x}_2 \in \mathcal{A} : \mathfrak{x}_1 \mathfrak{x}_2 = 0\}$. An ideal $\mathcal{I}_1$ of a ring $\mathcal{A}$ is called essential ideal of $\mathcal{A}$, which is denoted by $\mathcal{I}_1 \leq_e \mathcal{A}$, if $\mathcal{I}_1 \cap \mathcal{I}_2 \neq (0)$ for every nonzero ideal $\mathcal{I}_2$ of $\mathcal{A}$. The set of all nilpotent elements, set of all unit elements, and set of all zero-divisors of a ring $\mathcal{A}$ are denoted by $\mathcal{N}(\mathcal{A})$, $\mathcal{U}(\mathcal{A})$, and $\mathcal{Z}(\mathcal{A})$, respectively. For any $\mathcal{S} \subseteq \mathcal{A}$, $\mathcal{S}^*$ denotes the set $\mathcal{S} \setminus \{0\}$. For any undefined notation or terminology in ring theory, we refer the reader to [1].

In 1988, Beck [2] began to investigate the possibility of coloring a commutative ring $\mathcal{A}$ by associating to the ring a zero-divisor graph, defined as a simple graph, the vertices of which are the elements of the ring $\mathcal{A}$, with two distinct elements $\mathfrak{x}_1$ and $\mathfrak{x}_2$ being adjacent if and only if $\mathfrak{x}_1 \mathfrak{x}_2 = 0$. While Beck concentrated on the connection between the clique number and the chromatic number of the graph, various works inspired from this construction have concentrated on the interplay of commutative rings and their zero-divisor graphs. Retaining the original definition, the next decade brought little progress. However, in 1999,

Anderson and Livingston [3] modified and studied the zero-divisor graph whose vertices are the nonzero zero divisors of the commutative ring (see [4–7]). Further Redmond [8] has extended this concept to any arbitrary ring, whereas the same concept for semigroup was put forward by Demeyer et al. [9]. These graphs turn out to best exhibit the properties of the set of zero divisors of a ring. Actually zero-divisor graphs help us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language, and then the geometric properties of graphs help us to explore some interesting results in the algebraic structures of rings like the structure of zero-divisors.

In [10], Jahanbakhsh Basharlou et al. introduced and studied the generalized zero-divisor graph of a commutative ring $\mathcal{A}$, which is denoted by $\Gamma_g(\mathcal{A})$, whose vertex set is $\mathcal{Z}(\mathcal{A})^* = \mathcal{Z}(\mathcal{A}) \setminus \{0\}$ and $\mathfrak{x}_1 \sim \mathfrak{x}_2$ is an edge of $\Gamma_g(\mathcal{A})$ if and only if $\text{ann}(\mathfrak{x}_1) + \text{ann}(\mathfrak{x}_2) \leq_e \mathcal{A}$. The authors in [10] discussed some connections between the graph-theoretic properties of $\Gamma_g(\mathcal{A})$ and some algebraic properties of rings. Moreover, they investigated the affinity between generalized zero-divisor graph and zero-divisor graph associated with a ring and proved that $\Gamma(\mathcal{A})$ is a spanning subgraph of $\Gamma_g(\mathcal{A})$.

In this article, we characterized all the finite commutative rings $\mathcal{A}$ for which the generalized zero-divisor graph $\Gamma_g(\mathcal{A})$ is a unicycle, a tree, or a split graph. Further, we classify all the finite commutative rings $\mathcal{A}$ for which $\Gamma_g(\mathcal{A})$ is a planar graph, an outerplanar graph, or a toroidal graph. Finally, we discuss about the domination number of $\Gamma_g(\mathcal{A})$ and prove that the domination number of $\Gamma_g(\mathcal{A})$ can be any arbitrary number.

2. Basic Properties of Generalized Zero-Divisor Graph

In this section, we classify all the finite commutative rings $\mathcal{A}$ for which the generalized zero-divisor graph $\Gamma_g(\mathcal{A})$ is isomorphic to some well-known graphs such as a unicycle, a tree, or a split graph. We begin our discussion with the following result due to Basharlou et al. [10], which is essential to prove our theorems.

Lemma 1 (see [10, Lemma 2.1]). *For any ring $\mathcal{A}$, the following holds:*

- (1) If $\mathfrak{x} \in \mathcal{A}$, then $\mathcal{A}\mathfrak{x}^n + \text{ann}(\mathfrak{x})$ is an essential ideal of $\mathcal{A}$, for every integer $n \geq 0$.
- (2) $\mathfrak{x}$ is adjacent to all other vertices in $\Gamma_g(\mathcal{A})$, for every $\mathfrak{x} \in \mathcal{N}(\mathcal{A})^*$.
- (3) $\mathfrak{x} \sim \mathfrak{y}$ will be an edge of $\Gamma_g(\mathcal{A})$ if $\mathfrak{x} \sim \mathfrak{y}$ is an edge of $\Gamma(\mathcal{A})$.

Theorem 1. *For any ring $\mathcal{A}$, the following holds:*

- (1) If $(\mathcal{A}, \mathfrak{m})$ is a local ring, then $\Gamma_g(\mathcal{A})$ is a complete graph.
- (2) $\Gamma(\mathcal{A})$ is a spanning subgraph of $\Gamma_g(\mathcal{A})$.

Proof. Clear from parts (2) and (3) of Lemma 1. $\square$

A graph $\mathcal{G}$, which contains a unique cycle, is called a unicycle graph. In the following result, the finite rings $\mathcal{A}$ are classified for which $\Gamma_g(\mathcal{A})$ is unicycle.

Theorem 2. *Let $\mathcal{A}$ be a finite ring (not a field). Then, $\Gamma_g(\mathcal{A})$ is unicycle $\iff \mathcal{A}$ is one of these rings:*

$\mathbb{Z}_8, (\mathbb{Z}_2[x]/\langle x^3 \rangle), (\mathbb{Z}_4[x]/\langle x^3, x^2 - 2 \rangle), (\mathbb{Z}_4[x]/\langle 2x, x^2 \rangle), (\mathbb{Z}_2[x, y]/\langle x^2, xy, y^2 \rangle), (\mathbb{F}_4[x]/\langle x^2 \rangle), (\mathbb{Z}_4[x]/\langle x^2 + x + 1 \rangle), \mathbb{Z}_3 \times \mathbb{Z}_3$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Suppose that $\Gamma_g(\mathcal{A})$ is unicycle. Since $\mathcal{A}$ is finite, $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$, where $\mathcal{A}_i$ is local ring for each i . Suppose $n \geq 4$. Consider four vertices of $\Gamma_g(\mathcal{A})$ given by $\alpha_1 = (1, 0, 0, \dots, 0)$, $\alpha_2 = (0, 1, 0, \dots, 0)$, $\alpha_3 = (0, 0, 1, 0, \dots, 0)$, and $\alpha_4 = (0, 0, 0, 1, 0, \dots, 0)$. Since $\text{ann}(\alpha_i) + \text{ann}(\alpha_j) \leq_e \mathcal{A}$ for each i, j , then $\alpha_1 \sim \alpha_2 \sim \alpha_3 \sim \alpha_1$ and $\alpha_1 \sim \alpha_2 \sim \alpha_4 \sim \alpha_1$ are two different cycles in $\Gamma_g(\mathcal{A})$, which contradicts our assumption. Hence, $n \leq 3$.

Suppose $n = 3$ with $|\mathcal{A}_1| \geq 3$. Consider four vertices of $\Gamma_g(\mathcal{A})$ given as $\beta_1 = (1, 0, 0)$, $\beta_2 = (0, 1, 0)$, $\beta_3 = (0, 0, 1)$, and $\beta_4 = (a, 0, 0)$, where $1 \neq a \in \mathcal{A}_1^*$. Since $\beta_1\beta_2 = \beta_1\beta_3 = \beta_2\beta_3 = \beta_2\beta_4 = \beta_3\beta_4 = 0$, then $\beta_1 \sim \beta_2 \sim \beta_3 \sim \beta_1$ and

$\beta_2 \sim \beta_3 \sim \beta_4 \sim \beta_2$ are two different cycles in $\Gamma_g(\mathcal{A})$, which contradicts our assumption. Hence, $|\mathcal{A}_i| = 2$ for each $i = 1, 2, 3$, which implies that $\mathcal{A} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose $n = 2$ and $\mathcal{A}_1$ is not a field with maximal ideal $\mathfrak{m} \neq 0$; then, there is $b \in \mathfrak{m}^*$ such that $\text{ann}(b) = \mathfrak{m}$. Consider $\delta_1 = (1, 0)$, $\delta_2 = (0, 1)$, $\delta_3 = (b, 0)$, $\delta_4 = (u, 0) \in \mathcal{Z}(\mathcal{A})^*$, where $u \in \mathcal{U}(\mathcal{A}_1)$. One can see that $\delta_1 \sim \delta_2 \sim \delta_3 \sim \delta_1$ and $\delta_2 \sim \delta_3 \sim \delta_4 \sim \delta_2$ are two different cycles in $\Gamma_g(\mathcal{A})$, which contradicts our assumption. Hence, $\mathcal{A}_1$ and $\mathcal{A}_2$ both are fields, and so $\Gamma_g(\mathcal{A})$ is a complete bipartite graph. Since $\Gamma_g(\mathcal{A})$ is unicycle, $|\mathcal{A}_1| = 3$ and $|\mathcal{A}_2| = 3$, which implies that $\mathcal{A} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Suppose $n = 1$. Then, $\Gamma_g(\mathcal{A})$ is a complete graph by Theorem 1. Since $\Gamma_g(\mathcal{A})$ is unicycle, $|\mathcal{Z}(\mathcal{A})^*| = 3$. Hence, $\mathcal{A} \cong \mathbb{Z}_8, (\mathbb{Z}_2[x]/\langle x^3 \rangle), (\mathbb{Z}_4[x]/\langle x^3, x^2 - 2 \rangle), (\mathbb{Z}_4[x]/\langle 2x, x^2 \rangle), (\mathbb{Z}_2[x, y]/\langle x^2, xy, y^2 \rangle), (\mathbb{F}_4[x]/\langle x^2 \rangle)$, or $(\mathbb{Z}_4[x]/\langle x^2 + x + 1 \rangle)$. $\square$

A graph $\mathcal{G}$, which contains no cycle, is called acyclic. A connected acyclic graph is said to be a tree. In the following result, we classify all the finite rings $\mathcal{A}$ whose generalized zero-divisor graph $\Gamma_g(\mathcal{A})$ is a tree.

Theorem 3. *Let $\mathcal{A}$ be a finite ring (not a field). Then, $\Gamma_g(\mathcal{A})$ is tree $\iff \mathcal{A}$ is one of these rings:*

$\mathbb{Z}_4, (\mathbb{Z}_2[x]/\langle x^2 \rangle), \mathbb{Z}_9, (\mathbb{Z}_3[x]/\langle x^2 \rangle)$, or $\mathbb{Z}_2 \times \mathcal{F}$, where $\mathcal{F}$ is a finite field.

Proof. Suppose that $\Gamma_g(\mathcal{A})$ is a tree. Since $\mathcal{A}$ is finite ring, $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$, where $\mathcal{A}_i$ is local ring for each i . Suppose $n \geq 3$. Consider three distinct vertices of $\Gamma_g(\mathcal{A})$ given by $\alpha_1 = (1, 0, 0, \dots, 0)$, $\alpha_2 = (0, 1, 0, \dots, 0)$, and $\alpha_3 = (0, 0, 1, 0, \dots, 0)$. Since $\text{ann}(\alpha_i) + \text{ann}(\alpha_j) \leq_e \mathcal{A}$ for each i, j , then $\alpha_1 \sim \alpha_2 \sim \alpha_3 \sim \alpha_1$ is a cycle in $\Gamma_g(\mathcal{A})$, which contradicts our assumption. Hence, $n \leq 2$.

Suppose $n = 2$. We claim that $\mathcal{A}_i$ is a field for each $i = 1, 2$. Suppose on the contrary that $\mathcal{A}_1$ is not a field with nonzero maximal ideal $\mathfrak{m}$. Then, there is $a \in \mathfrak{m}^*$ such that $\text{ann}(a) = \mathfrak{m}$. Consider $\beta_1 = (1, 0)$, $\beta_2 = (0, 1)$, $\beta_3 = (a, 0) \in \mathcal{Z}(\mathcal{A})^*$. Since $\text{ann}(\beta_i) + \text{ann}(\beta_j) \leq_e \mathcal{A}$ for each i, j , then $\beta_1 \sim \beta_2 \sim \beta_3 \sim \beta_1$ is a cycle in $\Gamma_g(\mathcal{A})$, which contradicts our assumption. Hence, $\mathcal{A}_1$ and $\mathcal{A}_2$ both are fields, which implies that $\Gamma_g(\mathcal{A})$ is a complete bipartite graph. Since $\Gamma_g(\mathcal{A})$ is a tree, $|\mathcal{A}_i| = 2$ for some $i = 1, 2$. Hence, $\mathcal{A} \cong \mathbb{Z}_2 \times \mathcal{F}$, where $\mathcal{F}$ is finite field.

Suppose $n = 1$. Then, $\Gamma_g(\mathcal{A})$ is complete by Theorem 1. Since $\Gamma_g(\mathcal{A})$ is a tree, $\mathcal{A}$ has at most two nonzero zero-divisors. Hence, $\mathcal{A} \cong \mathbb{Z}_4, (\mathbb{Z}_2[x]/\langle x^2 \rangle), \mathbb{Z}_9$, or $(\mathbb{Z}_3[x]/\langle x^2 \rangle)$. $\square$

A graph $\mathcal{G}$, whose vertex set may be partitioned into an independent set and a clique, is called a split graph. The following result is useful to classify the finite commutative ring $\mathcal{A}$ whose generalized zero-divisor graph $\Gamma_g(\mathcal{A})$ is a split graph.

Theorem 4 (see [11]). *$\mathcal{G}$ is a split graph $\iff \mathcal{G}$ contains no induced subgraph which is isomorphic to $2K_2, C_4, C_5$, where $\mathcal{G}$ is a connected graph.*

Theorem 5. Let $\mathcal{A}$ be a finite nonlocal ring and $|\mathcal{Z}(\mathcal{A})^*| \geq 2$. Then, $\Gamma_g(\mathcal{A})$ is split graph $\iff \mathcal{A} \cong \mathbb{Z}_2 \times \mathcal{F}$, where $\mathcal{F}$ is a finite field or $\mathcal{A} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Assume that $\Gamma_g(\mathcal{A})$ is a split graph. Since $\mathcal{A}$ is a finite ring, $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$, where $\mathcal{A}_i$ is a local ring for each i and $n \geq 2$. Suppose $n \geq 4$. Consider four distinct vertices of $\Gamma_g(\mathcal{A})$ defined as $\alpha_1 = (1, 0, 0, \dots, 0)$, $\alpha_2 = (0, 1, 0, \dots, 0)$, $\alpha_3 = (0, 0, 1, 0, \dots, 0)$, and $\alpha_4 = (0, 0, 0, 1, 0, \dots, 0)$. Since $\text{ann}(\alpha_i) + \text{ann}(\alpha_j) \leq_e \mathcal{A}$ for each i, j , then $\Gamma_g(\mathcal{A})$ contains C_4 , which contradicts our assumption. Hence, $n \leq 3$.

Suppose $n = 3$ with $|\mathcal{A}_2| \geq 3$. Consider distinct vertices of $\Gamma_g(\mathcal{A})$ given by $\beta_1 = (1, 0, 0)$, $\beta_2 = (0, 1, 0)$, $\beta_3 = (0, 0, 1)$, and $\beta_4 = (0, a, 0)$, where $1 \neq a \in \mathcal{A}_2^*$. Since $\beta_1 \sim \beta_2 \sim \beta_3 \sim \beta_4 \sim \beta_1$, then $\Gamma_g(\mathcal{A})$ contains C_4 , which contradicts our assumption. Hence, $|\mathcal{A}_i| = 2$ for each $i = 1, 2, 3$. Hence, $\mathcal{A} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose $n = 2$ and $\mathcal{A}_1$ is not a field with maximal ideal $\mathfrak{m} \neq 0$; then, there is $b \in \mathfrak{m}^*$ such that $\text{ann}(b) = \mathfrak{m}$. Consider $\delta_1 = (1, 0)$, $\delta_2 = (b, 0)$, $\delta_3 = (u, 0)$, $\delta_4 = (0, 1) \in \mathcal{Z}(\mathcal{A})^*$, where $1 \neq u \in \mathcal{U}(\mathcal{A}_1)$. Since $\text{ann}(\delta_1) + \text{ann}(\delta_2) \leq_e \mathcal{A}$, $\text{ann}(\delta_2) + \text{ann}(\delta_3) \leq_e \mathcal{A}$, $\text{ann}(\delta_3) + \text{ann}(\delta_4) \leq_e \mathcal{A}$, and $\text{ann}(\delta_1) + \text{ann}(\delta_4) \leq_e \mathcal{A}$, then $\Gamma_g(\mathcal{A})$ contains a cycle of length four, which contradicts our assumption. Hence, $\mathcal{A}_i$ is a field for each $i = 1, 2$, which shows that $\Gamma_g(\mathcal{A})$ is a complete bipartite graph of order $(|\mathcal{A}_1| - 1, |\mathcal{A}_2| - 1)$. Since $\Gamma_g(\mathcal{A})$ is a split graph, then $|\mathcal{A}_1| = 2$ or $|\mathcal{A}_2| = 2$, which implies that $\mathcal{A} \cong \mathbb{Z}_2 \times \mathcal{F}$, where $\mathcal{F}$ is a finite field. $\square$

3. Planarity of Generalized Zero-Divisor Graph

If a graph $\mathcal{G}$ can be drawn in the plane in such a way that no two edges intersect each other except the end points, then it is called a planar graph. A planar graph $\mathcal{G}$ is said to be outerplanar if all the vertices of $\mathcal{G}$ lie on the unbounded face of the drawing.

In this section, we classify all the finite commutative rings whose generalized zero-divisor graph is planar or outerplanar graph. Before going to our main results, first we state some results which are useful for the proof of the subsequent results.

Theorem 6 (Kuratowski's theorem). A graph $\mathcal{G}$ is planar $\iff$ it contains no subdivision of K_5 or $K_{3,3}$.

Theorem 7 (see [10, Theorem 3.3]). Let $\mathcal{A}$ be a reduced ring and $\mathfrak{x}, \mathfrak{y} \in \mathcal{Z}(\mathcal{A})^*$. Then, these statements are equivalent:

- (1) $\Gamma(\mathcal{A}) = \Gamma_g(\mathcal{A})$.
- (2) If $\mathfrak{x}\mathfrak{y} \neq 0$, then $\text{ann}(\text{ann}(\mathfrak{x}) + \text{ann}(\mathfrak{y})) \neq (0)$.

Theorem 8 (see [6, Theorem 3.5.1]). Let $(\mathcal{A}, \mathfrak{m})$ be a finite local ring and $\Gamma(\mathcal{A})$ be the zero-divisor graph of $\mathcal{A}$. Then, $\Gamma(\mathcal{A})$ is planar $\iff \mathcal{A}$ is one of these rings:

$\mathbb{Z}_4, (\mathbb{Z}_2[x]/\langle x^2 \rangle), \mathbb{Z}_9, (\mathbb{Z}_3[x]/\langle x^2 \rangle), \mathbb{Z}_8, (\mathbb{Z}_2[x]/\langle x^3 \rangle), (\mathbb{Z}_4[x]/\langle x^3, x^2 - 2 \rangle), (\mathbb{Z}_4[x]/\langle 2x, x^2 \rangle), (\mathbb{Z}_2[x, y]/\langle x^2, xy, y^2 \rangle), (\mathbb{F}_4[x]/\langle x^2 \rangle), (\mathbb{Z}_4[x]/\langle x^2 + x + 1 \rangle), \mathbb{Z}_{25}, (\mathbb{Z}_5[x]/\langle x^2 \rangle), \mathbb{Z}_{16}, (\mathbb{Z}_2[x]/\langle x^4 \rangle), (\mathbb{Z}_4[x]/\langle x^2 - 2, x^4 \rangle), (\mathbb{Z}_2[x]/$

$\langle x^3 - 2, x^4 \rangle), (\mathbb{Z}_4[x]/\langle x^3 + x^2 - 2, x^4 \rangle), (\mathbb{Z}_2[x, y]/\langle x^3, xy, y^2 - x^2 \rangle), (\mathbb{Z}_2[x]/\langle x^3, x^2 - 2x \rangle), (\mathbb{Z}_8[x]/\langle x^2 - 4, 2x \rangle), (\mathbb{Z}_4[x, y]/\langle x^3, x^2 - 2, xy, y^2 - 2, y^3 \rangle), (\mathbb{Z}_4[x]/\langle x^2 \rangle), (\mathbb{Z}_4[x, y]/\langle x^2, y^2, xy - 2 \rangle), (\mathbb{Z}_2[x, y]/\langle x^2, y^2 \rangle), \mathbb{Z}_{27}, (\mathbb{Z}_3[x]/\langle x^3 \rangle), (\mathbb{Z}_9[x]/\langle x^2 - 3, x^3 \rangle), (\mathbb{Z}_9[x]/\langle x^2 + 3, x^3 \rangle).$

Theorem 9 (see [5, Theorem 3.7]). Let $\mathcal{A} = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n$ be a finite ring, where each $\mathcal{F}_i$ is a field and $n \geq 2$. Then, $\Gamma(\mathcal{A})$ is planar $\iff \mathcal{A}$ is one of these rings:

$\mathbb{Z}_2 \times \mathcal{F}, \mathbb{Z}_3 \times \mathcal{F}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathcal{F}$ is a finite field.

The following result characterized all the finite local commutative rings $\mathcal{A}$ for which $\Gamma_g(\mathcal{A})$ is planar.

Theorem 10. Let $\mathcal{A}$ be a finite local ring. Then, $\Gamma_g(\mathcal{A})$ is planar $\iff \mathcal{A}$ is one of these rings:

$\mathbb{Z}_4, (\mathbb{Z}_2[x]/\langle x^2 \rangle), \mathbb{Z}_9, (\mathbb{Z}_3[x]/\langle x^2 \rangle), \mathbb{Z}_8, (\mathbb{Z}_2[x]/\langle x^3 \rangle), (\mathbb{Z}_4[x]/\langle x^3, x^2 - 2 \rangle), (\mathbb{Z}_4[x]/\langle 2x, x^2 \rangle), (\mathbb{Z}_2[x, y]/\langle x^2, xy, y^2 \rangle), (\mathbb{F}_4[x]/\langle x^2 \rangle), (\mathbb{Z}_4[x]/\langle x^2 + x + 1 \rangle), \mathbb{Z}_{25}$ or $(\mathbb{Z}_5[x]/\langle x^2 \rangle).$

Proof. It is clear from Theorem 1 and 6. $\square$

Now, we are in a position to classify all the finite nonlocal rings $\mathcal{A}$ for which $\Gamma_g(\mathcal{A})$ is planar.

Theorem 11. Let $\mathcal{A}$ be a finite reduced ring (not a field). Then, $\Gamma_g(\mathcal{A})$ is planar $\iff \mathcal{A}$ is one of these rings:

$\mathbb{Z}_2 \times \mathcal{F}, \mathbb{Z}_3 \times \mathcal{F}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathcal{F}$ is a finite field.

Proof. Since $\mathcal{A}$ is a finite reduced ring, $\mathcal{A} \cong \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n$, where $\mathcal{F}_i$ is a field for each i and $n \geq 2$. Thus, by Theorem 7, $\Gamma(\mathcal{A}) = \Gamma_g(\mathcal{A})$. Hence, the result is clear from Theorem 9. $\square$

Remark 1. If $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$ is a finite commutative ring, where $(\mathcal{A}_i, \mathfrak{m}_i)$ is local ring with $\mathfrak{m}_i \neq 0$ for each i and $n \geq 2$, then $\Gamma_g(\mathcal{A})$ contains a copy of $K_{3,3}$, which shows that $\Gamma_g(\mathcal{A})$ is not planar. Hence, for $\Gamma_g(\mathcal{A})$ to be planar, at least one of $\mathcal{A}_i$ must be a field.

Theorem 12. Let $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n \times \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_m$ be a finite ring, where $(\mathcal{A}_i, \mathfrak{m}_i)$ is local ring with $\mathfrak{m}_i \neq 0$ for each i , and $\mathcal{F}_j$ is a field for each j and $n, m \geq 1$. Then, $\Gamma_g(\mathcal{A})$ is planar if and only if $\mathcal{A} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathcal{A} \cong (\mathbb{Z}_2[x]/\langle x^2 \rangle) \times \mathbb{Z}_2$.

Proof. Suppose that $\Gamma_g(\mathcal{A})$ is planar. If $n \geq 2$, then by Remark 1, $\Gamma_g(\mathcal{A})$ is not planar, which contradicts our assumption. Hence, $n = 1$.

Suppose $m \geq 2$. Since $\mathcal{A}_1$ is local ring with $\mathfrak{m}_1 \neq 0$, then there is $a \in \mathfrak{m}_1^*$ such that $\text{ann}(a) = \mathfrak{m}_1$. Consider $\mathfrak{x}_1 = (a, 0, 0, \dots, 0)$, $\mathfrak{x}_2 = (1, 0, 0, \dots, 0)$, $\mathfrak{x}_3 = (u, 0, 0, \dots, 0)$, $\mathfrak{y}_1 = (0, 1, 0, \dots, 0)$, $\mathfrak{y}_2 = (0, 0, 1, 0, \dots, 0)$, $\mathfrak{y}_3 = (0, 1, 1, 0, \dots, 0) \in \mathcal{Z}(\mathcal{A})^*$, where $1 \neq u \in \mathcal{U}(\mathcal{A}_1)$. Since $\text{ann}(\mathfrak{x}_i) + \text{ann}(\mathfrak{y}_j) \leq_e \mathcal{A}$ for each i, j , then $\Gamma_g(\mathcal{A})$ contains a

copy of $K_{3,3}$, which contradicts our assumption. Hence, $m = 1$.

Suppose $|m_1^*| \geq 2$, then $|\mathcal{U}(\mathcal{A}_1)| \geq 3$. Consider $\mathbf{a}_1 = (u_1, 0)$, $\mathbf{a}_2 = (u_2, 0)$, $\mathbf{a}_3 = (u_3, 0)$, $\mathbf{b}_1 = (0, 1)$, $\mathbf{b}_2 = (a, 0)$, $\mathbf{b}_3 = (a, 1) \in \mathcal{Z}(\mathcal{A})^*$, where $u_i \in \mathcal{U}(\mathcal{A}_1)$ for each $i = 1, 2, 3$. Since $\text{ann}(\mathbf{a}_i) + \text{ann}(\mathbf{b}_j) \leq_e \mathcal{A}$ for each i, j , then $\Gamma_g(\mathcal{A})$ contains $K_{3,3}$ as a subgraph, which contradicts our assumption. Hence, $|m_1^*| = 1$, which implies that $\mathcal{A}_1 \cong \mathbb{Z}_4$ or $(\mathbb{Z}_2[x]/\langle x^2 \rangle)$.

Now, suppose that $|\mathcal{F}_1| \geq 3$ and $x \in m_1^*$ such that $x^2 = 0$. Consider $\mathbf{r}_1 = (1, 0)$, $\mathbf{r}_2 = (x, 0)$, $\mathbf{r}_3 = (u, 0)$, $\mathbf{s}_1 = (0, 1)$, $\mathbf{s}_2 = (0, y)$, $\mathbf{s}_3 = (x, y) \in \mathcal{Z}(\mathcal{A})^*$, where $1 \neq u \in \mathcal{U}(\mathcal{A}_1)$ and $1 \neq y \in \mathcal{F}_1^*$. Since $\text{ann}(\mathbf{r}_i) + \text{ann}(\mathbf{s}_j) \leq_e \mathcal{A}$ for each i, j , then $\Gamma_g(\mathcal{A})$ contains a subdivision of $K_{3,3}$, which contradicts our assumption. Hence, $|\mathcal{F}_1| = 2$, which implies that $\mathcal{F}_1 \cong \mathbb{Z}_2$.

Converse is clear from Figure 1. $\square$

Theorem 13 (see [12]). *A graph $\mathcal{G}$ is outerplanar $\iff$ there is no subdivision of K_4 or $K_{2,3}$ in $\mathcal{G}$.*

We end this section with the following result which characterized all the finite commutative rings $\mathcal{A}$ for which $\Gamma_g(\mathcal{A})$ is outerplanar.

Theorem 14. *Let $\mathcal{A}$ be a finite ring (not a field). Then, $\Gamma_g(\mathcal{A})$ is outerplanar $\iff \mathcal{A}$ is one of these rings:*

$\mathbb{Z}_4$, $(\mathbb{Z}_2[x]/\langle x^2 \rangle)$, $\mathbb{Z}_9$, $(\mathbb{Z}_3[x]/\langle x^2 \rangle)$, $\mathbb{Z}_8$, $(\mathbb{Z}_2[x]/\langle x^3 \rangle)$, $(\mathbb{Z}_4[x]/\langle x^3, x^2 - 2 \rangle)$, $(\mathbb{Z}_4[x]/\langle 2x, x^2 \rangle)$, $(\mathbb{Z}_2[x, y]/\langle x^2, xy, y^2 \rangle)$, $(\mathbb{F}_4[x]/\langle x^2 \rangle)$, $(\mathbb{Z}_4[x]/\langle x^2 + x + 1 \rangle)$, $\mathbb{Z}_2 \times \mathcal{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathcal{F}$ is a finite field.

Proof. Assume that $\Gamma_g(\mathcal{A})$ is outerplanar. Since $\mathcal{A}$ is finite, $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n$, where $\mathcal{A}_i$ is local ring for each i . Suppose $n \geq 4$. Consider $\mathbf{v}_1 = (1, 0, \dots, 0)$, $\mathbf{v}_2 = (0, 1, 0, \dots, 0)$, $\mathbf{w}_1 = (0, 0, 1, 0, \dots, 0)$, $\mathbf{w}_2 = (0, 0, 0, 1, 0, \dots, 0)$, $\mathbf{w}_3 = (0, 0, 1, 1, 0, \dots, 0) \in \mathcal{Z}(\mathcal{A})^*$. Since $\text{ann}(\mathbf{v}_i) + \text{ann}(\mathbf{w}_j) \leq_e \mathcal{A}$ for each i, j , then $\Gamma_g(\mathcal{A})$ contains a copy of $K_{2,3}$, which contradicts our assumption. Hence, $n \leq 3$.

Suppose $n = 3$ and $|\mathcal{A}_1| \geq 3$. Consider $\mathbf{a}_1 = (1, 0, 0)$, $\mathbf{a}_2 = (x, 0, 0)$, $\mathbf{b}_1 = (0, 1, 0)$, $\mathbf{b}_2 = (0, 0, 1)$, $\mathbf{b}_3 = (0, 1, 1) \in \mathcal{Z}(\mathcal{A})^*$, where $1 \neq x \in \mathcal{A}_1^*$. Since $\mathbf{a}_i \mathbf{b}_j = 0$ for each i, j , then $\Gamma_g(\mathcal{A})$ contains $K_{2,3}$ as a subgraph. Hence, $|\mathcal{A}_i| = 2$ for each $i = 1, 2, 3$. This implies that $\mathcal{A} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose $n = 2$ and $\mathcal{A}_1$ is not a field with nonzero maximal ideal $\mathbf{m}_1$. Then, there is $a \in \mathbf{m}_1^*$ such that $\text{ann}(a) = \mathbf{m}_1$. Consider $\mathbf{u}_1 = (1, 0)$, $\mathbf{u}_2 = (a, 0)$, $\mathbf{u}_3 = (u, 0)$, $\mathbf{v}_1 = (0, 1)$, $\mathbf{v}_2 = (a, 1) \in \mathcal{Z}(\mathcal{A})^*$, where $1 \neq u \in \mathcal{U}(\mathcal{A}_1)$. Since $\text{ann}(\mathbf{u}_i) + \text{ann}(\mathbf{v}_j) \leq_e \mathcal{A}$ for each i, j , then $\Gamma_g(\mathcal{A})$ contains a copy of $K_{2,3}$, which contradicts our assumption. Hence, $\mathcal{A}_1$ and $\mathcal{A}_2$ both are fields, which shows that $\Gamma_g(\mathcal{A})$ is a complete bipartite graph with order $(|\mathcal{A}_1| - 1, |\mathcal{A}_2| - 1)$. Since $\Gamma_g(\mathcal{A})$ is outerplanar, $\mathcal{A} \cong \mathbb{Z}_2 \times \mathcal{F}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, where $\mathcal{F}$ is a finite field.

Suppose $n = 1$. Then, by Theorem 1, $\Gamma_g(\mathcal{A})$ is a complete graph. Since $\Gamma_g(\mathcal{A})$ is outerplanar, $|\mathcal{Z}(\mathcal{A})^*| \leq 3$. Hence, $\mathcal{A} \cong \mathbb{Z}_4$, $(\mathbb{Z}_2[x]/\langle x^2 \rangle)$, $\mathbb{Z}_9$, $(\mathbb{Z}_3[x]/\langle x^2 \rangle)$, $\mathbb{Z}_8$, $(\mathbb{Z}_2[x]/\langle x^3 \rangle)$, $(\mathbb{Z}_4[x]/\langle x^3, x^2 - 2 \rangle)$, $(\mathbb{Z}_4[x]/\langle 2x, x^2 \rangle)$, $(\mathbb{Z}_2[x, y]/\langle x^2, xy, y^2 \rangle)$, $(\mathbb{F}_4[x]/\langle x^2 \rangle)$ or $(\mathbb{Z}_4[x]/\langle x^2 + x + 1 \rangle)$.

Converse is clear. $\square$

FIGURE 1: Planar embedding of $\Gamma_g(\mathbb{Z}_4 \times \mathbb{Z}_2) \cong \Gamma_g((\mathbb{Z}_2[x]/\langle x^2 \rangle) \times \mathbb{Z}_2)$.

4. Genus of Generalized Zero-Divisor Graph

If a graph $\mathcal{G}$ can be embedded on the sphere with k handles without crossing itself, then the minimality of such integer k is called the genus of $\mathcal{G}$. It is denoted by $\lambda(\mathcal{G})$.

In this section, we classify all the finite commutative rings for which the generalized zero-divisor graph is toroidal.

In order to characterized the rings with toroidal generalized zero-divisors graph, we need the following results, which deal with genus properties of graphs.

Lemma 2 (see [12]). $\lambda(K_n) = \lceil (1/12)(n-3)(n-4) \rceil$, where $\lceil \mathfrak{x} \rceil$ is the least integer, that is, $\geq \mathfrak{x}$. In particular, $\lambda(K_n) = 1$ if $n = 5, 6, 7$.

Lemma 3 (see [12]). $\lambda(K_{m,n}) = \lceil (1/4)(m-2)(n-2) \rceil$, where $\lceil \mathfrak{x} \rceil$ is the least integer, that is, $\geq \mathfrak{x}$. In particular, $\lambda(K_{4,4}) = \lambda(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also, $\lambda(K_{5,4}) = \lambda(K_{6,4}) = \lambda(K_{3,m}) = 2$, if $m = 7, 8, 9, 10$.

Theorem 15 (see [6], Theorem 3.5.2). *Let $(\mathcal{A}, \mathbf{m})$ be a finite local ring and $\Gamma(\mathcal{A})$ be the zero-divisor graph of $\mathcal{A}$. Then, $\lambda(\Gamma(\mathcal{A})) = 1 \iff \mathcal{A}$ is one of these rings:*

$\mathbb{Z}_{49}$, $(\mathbb{Z}_7[x]/\langle x^2 \rangle)$, $(\mathbb{Z}_2[x, y]/\langle x^3, xy, y^2 \rangle)$, $(\mathbb{Z}_4[x]/\langle x^3, 2x \rangle)$, $(\mathbb{Z}_4[x, y]/\langle x^3, x^2 - 2, xy, y^2 \rangle)$, $(\mathbb{Z}_8[x]/\langle x^2, 2x \rangle)$, $(\mathbb{F}_8[x]/\langle x^2 \rangle)$, $(\mathbb{Z}_4[x]/\langle x^3 + x + 1 \rangle)$, $(\mathbb{Z}_4[x, y]/\langle 2x, 2y, x^2, xy, y^2 \rangle)$, $(\mathbb{Z}_2[x, y, z]/\langle x, y, z \rangle^2)$, $\mathbb{Z}_{32}$, $(\mathbb{Z}_2[x]/\langle x^5 \rangle)$, $(\mathbb{Z}_4[x]/\langle x^3 - 2, x^5 \rangle)$, $(\mathbb{Z}_4[x]/\langle x^4 - 2, x^5 \rangle)$, $(\mathbb{Z}_8[x]/\langle x^2 - 2, x^5 \rangle)$, $(\mathbb{Z}_8[x]/\langle x^2 - 2x + 2, x^5 \rangle)$, $(\mathbb{Z}_8[x]/\langle x^2 + 2x - 2, x^5 \rangle)$.

Theorem 16 (see [7], Theorem 3.1). *Let $\mathcal{A} = \mathcal{F}_1 \times \mathcal{F}_2 \times \dots \times \mathcal{F}_n$ be a finite ring, where each $\mathcal{F}_i$ is a field and $n \geq 2$. Then, $\lambda(\Gamma(\mathcal{A})) = 1 \iff \mathcal{A}$ is one of these rings:*

$\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\mathbb{F}_4 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The following result classifies all the finite local commutative rings $\mathcal{A}$ for which $\Gamma_g(\mathcal{A})$ is toroidal.

Theorem 17. *Let $\mathcal{A}$ be a finite local ring (not a field). Then, $\lambda(\Gamma_g(\mathcal{A})) = 1 \iff \mathcal{A}$ is one of these rings:*

$\mathbb{Z}_{49}, (\mathbb{Z}_7[x]/\langle x^2 \rangle), \mathbb{Z}_{16}, (\mathbb{Z}_2[x]/\langle x^4 \rangle), (\mathbb{Z}_4[x]/\langle x^2 - 2, x^4 \rangle), (\mathbb{Z}_2[x]/\langle x^3 - 2, x^4 \rangle), (\mathbb{Z}_4[x]/\langle x^3 + x^2 - 2, x^4 \rangle), (\mathbb{Z}_2[x, y]/\langle x^3, xy, y^2 - x^2 \rangle), (\mathbb{Z}_2[x]/\langle x^3, x^2 - 2x \rangle), (\mathbb{Z}_8[x]/\langle x^2 - 4, 2x \rangle), (\mathbb{Z}_4[x, y]/\langle x^3, x^2 - 2, xy, y^2 - 2, y^3 \rangle), (\mathbb{Z}_4[x]/\langle x^2 \rangle), (\mathbb{Z}_4[x, y]/\langle x^2, y^2, xy - 2 \rangle), (\mathbb{Z}_2[x, y]/\langle x^2, y^2 \rangle), (\mathbb{Z}_2[x, y]/\langle x^2, y^2, xy \rangle), (\mathbb{Z}_4[x]/\langle x^3, 2x \rangle), (\mathbb{Z}_4[x, y]/\langle x^3, x^2 - 2, xy, y^2 \rangle), (\mathbb{Z}_8[x]/\langle x^2, 2x \rangle), (\mathbb{F}_8[x]/\langle x^2 \rangle), (\mathbb{Z}_4[x]/\langle x^3 + x + 1 \rangle), (\mathbb{Z}_4[x, y]/\langle 2x, 2y, x^2, y^2, xy \rangle) \text{ or } (\mathbb{Z}_2[x, y, z]/\langle x, y, z \rangle^2).$

Proof. Since $\mathcal{A}$ is finite local ring, $\Gamma_g(\mathcal{A})$ is complete graph by Theorem 1. Thus, by Lemma 2, $\lambda(\Gamma_g(\mathcal{A})) = 1$ if and only if $\mathcal{A}$ has at least 5 and at most seven nonzero zero-divisors. This completes the proof. $\square$

Now, we are in a position to characterize all the finite nonlocal commutative rings $\mathcal{A}$ for which $\Gamma_g(\mathcal{A})$ is toroidal.

Theorem 18. *Let $\mathcal{A}$ be finite reduced ring (not a field). Then, $\lambda(\Gamma_g(\mathcal{A})) = 1 \iff \mathcal{A}$ is one of these rings:*

$\mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{Z}_7, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4 \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$

Proof. Since $\mathcal{A}$ is finite reduced ring, $\mathcal{A} \cong \mathcal{F}_1 \times \mathcal{F}_2 \times \dots \times \mathcal{F}_n$, where $\mathcal{F}_i$ is field for each i and $n \geq 2$. Thus, by Theorem 7, $\Gamma_g(\mathcal{A}) = \Gamma(\mathcal{A})$. Hence, the proof follows from Theorem 16. $\square$

Remark 2. If $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n$ is a finite commutative ring, where $(\mathcal{A}_i, \mathbf{m}_i)$ is local ring with $\mathbf{m}_i \neq 0$ for each i and $n \geq 2$, then $\Gamma_g(\mathcal{A})$ contains a copy of $K_{5,5}$, which shows that $\lambda(\Gamma_g(\mathcal{A})) > 1$. Hence, for $\Gamma_g(\mathcal{A})$ to be toroidal, at least one of $\mathcal{A}_i$ must be a field.

Theorem 19. *Let $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n \times \mathcal{F}_1 \times \mathcal{F}_2 \times \dots \times \mathcal{F}_m$ be a finite ring, where $(\mathcal{A}_i, \mathbf{m}_i)$ is local ring with $\mathbf{m}_i \neq 0$ for each i , and $\mathcal{F}_j$ is a field for each j and $n, m \geq 1$. Then, $\lambda(\Gamma_g(\mathcal{A})) = 1$ if and only if $\mathcal{A}$ is one of these rings:*

$\mathbb{Z}_4 \times \mathbb{Z}_3, (\mathbb{Z}_2[x]/\langle x^2 \rangle) \times \mathbb{Z}_3, \mathbb{Z}_4 \times \mathbb{F}_4, \text{ or } (\mathbb{Z}_2[x]/\langle x^2 \rangle) \times \mathbb{F}_4.$

Proof. Assume that $\lambda(\Gamma_g(\mathcal{A})) = 1$. If $n \geq 2$, then by Remark 2, $\lambda(\Gamma_g(\mathcal{A})) > 1$, which is a contradiction to our assumption. Hence, $n = 1$.

Suppose $m \geq 2$ and there is $a \in \mathbf{m}_1^*$ such that $\text{ann}(a) = \mathbf{m}_1$. Consider the set $S = \{a_1, a_2, \dots, a_{10}\}$, where $a_1 = (1, 0, 0, \dots, 0)$, $a_2 = (a, 0, 0, \dots, 0)$, $a_3 = (u, 0, 0, \dots, 0)$, $a_4 = (0, 1, 0, \dots, 0)$, $a_5 = (0, 0, 1, 0, \dots, 0)$, $a_6 = (0, 1, 1, 0, \dots, 0)$, $a_7 = (a, 1, 0, \dots, 0)$, $a_8 = (a, 0, 1, 0, \dots, 0)$, $a_9 = (a, 1, 1, 0, \dots, 0)$, $a_{10} = (1, 0, 1, 0, \dots, 0) \in \mathcal{Z}(\mathcal{A})^*$, where $1 \neq u \in \mathcal{U}(\mathcal{A}_1)$. Let G_1 be the graph induced by the set S . Then, G_1 is a subgraph of $\Gamma_g(\mathcal{A})$ containing G' as shown in Figure 2. From Figure 2, it is clear that G' contains $K_{3,6}$, which implies that $\lambda(G') \geq 1$.

Assume that $\lambda(G') = 1$. Consider an embedding of $K_{3,6}$ on torus; then, by Euler's formula, the embedding of $K_{3,6}$ has 9 faces, say $\{F'_1, F'_2, \dots, F'_9\}$. Let f_i denote the length of the face F'_i for each i . Then, $\sum_{i=1}^9 f_i = 2e = 36$ and $f_i \geq 4$ for each

FIGURE 2: The graph of G' .

i . This implies that $f_i = 4$ for each i . Let $P = \{a_7, a_8, [a_4, a_{10}]\} \subset V(G')$; then, the subgraph G'' induced by P is K_3 such that $E(G'') \cap E(K_{3,6}) = \emptyset$. Since K_3 cannot be embedded in the torus along with an embedding of G'' on $K_{3,6}$ together in torus, this implies that $\lambda(G') \geq 2$. Since G' is a subgraph of $\Gamma_g(\mathcal{A})$, $\lambda(\Gamma_g(\mathcal{A})) \geq 2$, which is a contradiction to our assumption. Hence, $m = 1$.

We claim that $|\mathbf{m}_1^*| = 1$. Suppose on the contrary that $|\mathbf{m}_1^*| = 2$; then, $\mathcal{A}_1 \cong \mathbb{Z}_9$ or $(\mathbb{Z}_3[x]/\langle x^2 \rangle)$, and hence $|\tilde{\mathcal{U}}(\mathcal{A}_1)| = 6$. Let $a, b \in \mathbf{m}_1^*$ such that $ab = 0$ and $\text{ann}(a) = \mathbf{m}_1$. Consider $\mathbf{p}_1 = (a, 0)$, $\mathbf{p}_2 = (b, 0)$, $\mathbf{p}_3 = (u_1, 0)$, $\mathbf{p}_4 = (u_2, 0)$, $\mathbf{p}_5 = (u_3, 0)$, $\mathbf{p}_6 = (u_4, 0)$, $\mathbf{p}_7 = (u_5, 0)$, $\mathbf{q}_1 = (0, 1)$, $\mathbf{q}_2 = (a, 1)$, $\mathbf{q}_3 = (b, 1) \in \mathcal{Z}(\mathcal{A})^*$, where $u_i \in \mathcal{U}(\mathcal{A}_1)$. Since $\text{ann}(\mathbf{p}_i) + \text{ann}(\mathbf{q}_j) \leq_e \mathcal{A}$ for each i, j , then $K_{3,7}$ is a subgraph of $\Gamma_g(\mathcal{A})$. Thus, by Lemma 3, $\lambda(\Gamma_g(\mathcal{A})) > 1$, which is a contradiction to our assumption. Hence, $|\mathbf{m}_1^*| \neq 2$.

Suppose $|\mathbf{m}_1^*| \geq 3$; then, $|\mathcal{U}(\mathcal{A}_1)| \geq 4$. Let $a, b, c \in \mathbf{m}_1^*$ such that $ab = bc = 0$ and $\text{ann}(a) = \mathbf{m}_1$. Consider $\mathbf{w}_1 = (u_1, 0)$, $\mathbf{w}_2 = (u_2, 0)$, $\mathbf{w}_3 = (u_3, 0)$, $\mathbf{w}_4 = (u_4, 0)$, $\mathbf{w}_5 = (a, 0)$, $\mathbf{w}_6 = (b, 0)$, $\mathbf{w}_7 = (c, 0)$, $\mathbf{z}_1 = (0, 1)$, $\mathbf{z}_2 = (a, 1)$, $\mathbf{z}_3 = (b, 1) \in \mathcal{Z}(\mathcal{A})^*$, where $u_i \in \mathcal{U}(\mathcal{A}_1)$. Since $\text{ann}(\mathbf{w}_i) + \text{ann}(\mathbf{z}_j) \leq_e \mathcal{A}$ for each i, j , then $\Gamma_g(\mathcal{A})$ contains $K_{3,7}$ as a subgraph, which is a contradiction by Lemma 3. Hence, $|\mathbf{m}_1^*| = 1$, which shows that $\mathcal{A}_1 \cong \mathbb{Z}_4$ or $(\mathbb{Z}_2[x]/\langle x^2 \rangle)$.

Now, suppose $|\mathcal{F}_1| \geq 5$ and $a \in \mathbf{m}_1^*$ such that $a^2 = 0$ and $\text{ann}(a) = \mathbf{m}_1$. Consider $\mathbf{r}_1 = (1, 0)$, $\mathbf{r}_2 = (a, 0)$, $\mathbf{r}_3 = (u, 0)$, $\mathbf{s}_1 = (0, x_1)$, $\mathbf{s}_2 = (0, x_2)$, $\mathbf{s}_3 = (0, x_3)$, $\mathbf{s}_4 = (0, x_4)$, $\mathbf{s}_5 = (a, x_1)$, $\mathbf{s}_6 = (a, x_2)$, $\mathbf{s}_7 = (a, x_3) \in \mathcal{Z}(\mathcal{A})^*$, where $1 \neq u \in \mathcal{U}(\mathcal{A}_1)$ and $x_i \in \mathcal{F}_1^*$. Since $\text{ann}(\mathbf{r}_i) + \text{ann}(\mathbf{s}_j) \leq_e \mathcal{A}$ for each i, j , then $K_{3,7}$ is a subgraph of $\Gamma_g(\mathcal{A})$, which is a contradiction to our assumption. Hence, $|\mathcal{F}_1| \leq 4$. From Theorem 12, it is clear that $\mathcal{F}_1 \not\cong \mathbb{Z}_2$. Hence, $\mathcal{F}_1 \cong \mathbb{Z}_3$ or $\mathbb{F}_4$.

The converse follows from Figures 3 and 4. $\square$

5. Domination of Generalized Zero-Divisor Graph

Let $\mathcal{G}(V, E)$ be a graph and $\mathcal{S} \subseteq V$. Then, $\mathcal{S}$ is said to be a dominating set of $\mathcal{G}$ if for $u \in V \setminus \mathcal{S}$ there exists $v \in \mathcal{S}$ such that $u \sim v$. A dominating set with minimum cardinality is said to be γ -set of $\mathcal{G}$. The cardinality of γ -set of $\mathcal{G}$ is called the domination number of $\mathcal{G}$ and is denoted by $\gamma(\mathcal{G})$.

In this section, we discuss about the domination number of the generalized zero-divisor graph of a commutative ring.

FIGURE 3: Toroidal embedding of $\Gamma_g(\mathbb{Z}_4 \times \mathbb{Z}_3) \cong \Gamma_g((\mathbb{Z}_2[x]/\langle x^2 \rangle) \times \mathbb{Z}_3)$.

FIGURE 4: Toroidal embedding of $\Gamma_g(\mathbb{Z}_4 \times \mathbb{F}_4) \cong \Gamma_g((\mathbb{Z}_2[x]/\langle x^2 \rangle) \times \mathbb{F}_4)$.

The following result shows that the dominating number of $\Gamma_g(\mathcal{A})$ may be any arbitrary number.

Proposition 1. Let $\mathcal{A} = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n$ be a ring, where $\mathcal{F}_i$ is a field for each i and $n \geq 3$. Then, $\gamma(\Gamma_g(\mathcal{A})) = n$.

Proof. Since $\mathcal{A}$ is a reduced ring, then by Theorem 7, $\Gamma_g(\mathcal{A}) = \Gamma(\mathcal{A})$. Hence, by [6], (Proposition 8), $\gamma(\Gamma_g(\mathcal{A})) = \gamma(\Gamma(\mathcal{A})) = n$. $\square$

The following result gives the necessary and sufficient conditions for a commutative ring $\mathcal{A}$ for which $\gamma(\Gamma_g(\mathcal{A})) = 1$.

Theorem 20. Let $\mathcal{A}$ be a ring. Then, $\gamma(\Gamma_g(\mathcal{A})) = 1 \iff$ one of the following holds:

- (1). There exists $0 \neq \mathfrak{x} \in \mathcal{A}$ such that $\text{ann}(\mathfrak{x})$ is essential ideal in $\mathcal{A}$.

- (2) $\mathcal{A} \cong \mathbb{Z}_2 \times \mathcal{S}$, where $\mathcal{S}$ is a reduced ring.

Proof. Suppose there exists $0 \neq \mathfrak{x} \in \mathcal{A}$ such that $\text{ann}(\mathfrak{x}) \leq_e \mathcal{A}$. Then, by Lemma 1, $\mathfrak{x}$ is adjacent to every other vertex of $\Gamma_g(\mathcal{A})$. Hence, $\gamma(\Gamma_g(\mathcal{A})) = 1$. Also, if $\mathcal{A} \cong \mathbb{Z}_2 \times \mathcal{S}$, where $\mathcal{S}$ is a reduced ring, then $(1, 0)$ is adjacent to every other vertex of $\Gamma_g(\mathcal{A})$. Hence, $\gamma(\Gamma_g(\mathcal{A})) = 1$.

Conversely, assume that $\gamma(\Gamma_g(\mathcal{A})) = 1$. The following two cases occur:

Case (i): $\mathcal{A}$ is nonreduced ring. Let $\mathfrak{a}$ be the nonzero nilpotent element of $\mathcal{A}$. Then, by Lemma 1, $\text{ann}(\mathfrak{a}) \leq_e \mathcal{A}$.

Case (ii): $\mathcal{A}$ is reduced ring. Since $\gamma(\Gamma_g(\mathcal{A})) = 1$, then let $\{\mathfrak{x}\}$ be the dominating set of $\Gamma_g(\mathcal{A})$. If $\mathfrak{x}^2 \neq \mathfrak{x}$, then $\text{ann}(\mathfrak{x}^2) = \text{ann}(\mathfrak{x}) + \text{ann}(\mathfrak{x}^2) \leq_e \mathcal{A}$, and we are done. Thus, suppose that $\mathfrak{x} = \mathfrak{x}^2$; then, $\mathcal{A} \cong \mathcal{A}_1 \times \mathcal{A}_2$ with dominating set $\{(1, 0)\}$, where $\mathcal{A}_i$ is a reduced ring for each $i = 1, 2$. Let $1 \neq \mathfrak{a} \in \mathcal{A}_1^*$; then, $(\mathfrak{a}, 0) \in \mathcal{F}(\mathcal{A})^*$. Since $\text{ann}(\mathfrak{a}, 0) + \text{ann}(1, 0) = \text{ann}(\mathfrak{a}) \times \mathcal{A}_2$ and $\mathcal{A}$ is a reduced ring, then $(\text{ann}(\mathfrak{a}) \times \mathcal{A}_2) \cap (\mathfrak{a}\mathcal{A}_1 \times \mathcal{A}_2) = (0)$, which implies that $(\mathfrak{a}, 0)$ is not adjacent to $(1, 0)$, a contradiction. Hence, $\mathcal{A}_1 \cong \mathbb{Z}_2$. $\square$

We immediately get the following corollary from Theorem 20, which classifies the domination number of $\Gamma_g(\mathcal{A})$ for all finite commutative rings $\mathcal{A}$.

Corollary 1. Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$ be a finite ring, where $\mathcal{A}_i$ is a local ring for each i and $n \geq 1$. Then, the following holds:

- (1) If $\mathcal{A}_i$ is not a field for some i , then $\gamma(\Gamma_g(\mathcal{A})) = 1$.
- (2) If $\mathcal{A}_i$ is a field for each i and $n \geq 3$, then $\gamma(\Gamma_g(\mathcal{A})) = n$.
- (3) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are fields and $\min\{|\mathcal{A}_1|, |\mathcal{A}_2|\} \geq 3$, then $\gamma(\Gamma_g(\mathcal{A})) = 2$.
- (4) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are fields and $\min\{|\mathcal{A}_1|, |\mathcal{A}_2|\} = 2$, then $\gamma(\Gamma_g(\mathcal{A})) = 1$.

Proof. The proof of (1) and (4) follows from Theorem 20 and the proof of (2) is clear from Proposition 1. Now for part (3), the set $\{(1, 0), (0, 1)\}$ is a dominating set of $\Gamma_g(\mathcal{A})$. Hence, $\gamma(\Gamma_g(\mathcal{A})) = 2$. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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