

Research Article

New Results on the Geometric-Arithmetic Index

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Received 18 June 2021; Accepted 20 July 2021; Published 31 July 2021

Academic Editor: Ali Ahmad

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Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let d_u denote the degree of vertex $u \in V(G)$. The geometric-arithmetic index of G is defined as $GA(G) = \sum_{uv \in E(G)} (2\sqrt{d_u d_v} / (d_u + d_v))$. In this paper, we obtain some new lower and upper bounds for the geometric-arithmetic index and improve some known bounds. Moreover, we investigate the relationships between geometric-arithmetic index and several other topological indices.

1. Introduction

Let G be a simple graph (i.e., graph without loops and multiple edges) with vertex set $V(G)$ and edge set $E(G)$. The integers $n = |V(G)|$ and $m = |E(G)|$ are the *order* and the *size* of the graph G , respectively. For $u \in V(G)$, we denote by d_u the degree of vertex u in G . The minimum and maximum degrees of a graph are denoted by δ and Δ , respectively.

Graph theory has provided chemists with a variety of useful tools, such as topological indices. A topological index $Top(G)$ of a graph G is a number with the property that, for every graph H isomorphic to G , $Top(H) = Top(G)$.

Molecular descriptors play a significant role in mathematical chemistry, especially in QSPR/QSAR investigations. Among them, special place is reserved for so-called topological descriptors. A topological index is a numeric quantity from the structural graph of a molecule.

Usage of topological indices in chemistry began in 1947 when Wiener [1] developed the most widely known topological descriptor, namely, the Wiener index, and used it to determine physical properties of types of alkanes known as paraffin (see, for instance, [2, 3]). The interest of topological indices lies in the fact that they synthesize some of the

properties of a molecule into a single number. With this in mind, hundreds of topological indices have been introduced and studied. Topological indices based on the vertex degree play a vital role in mathematical chemistry, and some of them are recognized as tools in chemical research.

Authors are studying various topological descriptors, such as Zagreb indices [4–6], general sum-connectivity index [7, 8], hyper-Zagreb index [9], and harmonic index [10, 11]. Besides the abovementioned ones, there are other topological descriptors based on end vertex degrees of edges of graphs that have found some applications in QSPR/QSAR research [2, 12, 13].

The geometric-arithmetic index of a graph is defined in [13] as

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}. \quad (1)$$

The geometric-arithmetic index has a number of interesting properties, e.g., see [13]. The lower and upper bounds of the geometric-arithmetic index of connected graphs and the characterizations of graphs for which these bounds are best possible can be found in [13–16].

The aim of this paper is to investigate new relationships between the geometric-arithmetic index and other topological indices. In particular, we obtain some lower and upper bounds for the geometric-arithmetic index. Moreover, we improve some known bounds.

2. Preliminaries

Let us recall some remarkable lemmas which will be used in this paper.

The first one is a very straightforward observation.

$$\frac{n}{(1/x_1) + (1/x_2) + \dots + (1/x_n)} \leq \sqrt[n]{\prod_{i=1}^n x_i} \leq \frac{x_1 + x_2 + \dots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n}}. \quad (3)$$

Lemma 3 (see [19]). Let $a = (a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be two sequences of positive numbers. For any $r \geq 0$,

$$\sum_{i=1}^n \frac{a_i^{r+1}}{b_i^r} \geq \frac{(\sum_{i=1}^n a_i)^{r+1}}{(\sum_{i=1}^n b_i)^r}. \quad (4)$$

Lemma 4 (see [20]). Let $r \leq a_i \leq R$ for $1 \leq i \leq m$ and r and R be some positive constants. Then,

$$\sum_{i=1}^m a_i \sum_{i=1}^m \frac{1}{a_i} \leq m^2 \left(1 + \frac{1}{4} \left(1 - \frac{1 + (-1)^{m+1}}{2m^2} \left(\sqrt{\frac{R}{r}} - \sqrt{\frac{r}{R}} \right)^2 \right) \right). \quad (5)$$

Lemma 5 (see [21]). If a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n are positive numbers, where $m_1 \leq a_i \leq N_1$ and $m_2 \leq b_i \leq N_2$ for each $1 \leq i \leq n$, then

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \frac{n^2}{4} (N_1 N_2 + m_1 m_2). \quad (6)$$

Lemma 6 (the Pólya–Szegő inequality, see p. 62 in [22]). Let $a = (a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be two sequences of positive numbers, where $0 < m_1 \leq a_i \leq M_1$ and $0 < m_2 \leq b_i \leq M_2$, for $i = 1, 2, \dots, n$. Then,

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \leq \frac{1}{4} \left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left(\sum_{i=1}^n a_i b_i \right)^2. \quad (7)$$

3. Upper Bounds for the Geometric-Arithmetic Index

In this section, we investigate the relationships between geometric-arithmetic index and some topological indices. Moreover, we obtain some upper bounds for the geometric-arithmetic index in terms of order, size, maximum degree, minimum degree, domination number, girth, number of cut edges, and number of pendent vertices.

Lemma 1 (see [17]). Let x and y be two positive numbers. Then,

$$\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{((x+y)/2) + \sqrt{xy}}{2} \leq \frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}}. \quad (2)$$

The following is the well-known inequality of arithmetic and geometric means.

Lemma 2 (inequality of arithmetic and geometric means, see [18]). Let x_1, \dots, x_n be positive numbers. Then,

The first and second Zagreb indices are vertex-degree-based graph invariants defined as

$$\begin{aligned} M_1(G) &= \sum_{uv \in E(G)} (d_u + d_v), \\ M_2(G) &= \sum_{uv \in E(G)} d_u d_v. \end{aligned} \quad (8)$$

The quantity M_1 was first considered in 1972 [6], whereas M_2 in 1975 [5]. The general Randić index is defined as follows [23]:

$$R_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha, \quad (9)$$

where α is a real number.

We begin with the establishment of an upper bound for the geometric-arithmetic index in terms of the first Zagreb index and the general Randić index.

Theorem 1. Let G be a graph. Then,

$$GA(G) \leq \frac{M_1(G) + 2R_{1/2}(G)}{4}. \quad (10)$$

Proof. By Lemma 1, we have

$$\begin{aligned} GA(G) &= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \\ &\leq \sum_{uv \in E(G)} \frac{2d_u d_v}{d_u + d_v} \\ &\leq \sum_{uv \in E(G)} \frac{((d_u + d_v)/2) + \sqrt{d_u d_v}}{2} \\ &= \sum_{uv \in E(G)} \frac{d_u + d_v + 2\sqrt{d_u d_v}}{4} \\ &= \frac{M_1(G) + 2R_{1/2}(G)}{4}, \end{aligned} \quad (11)$$

as desired.

Using Lemma 1 and an argument similar to the proof of Theorem 1, we can obtain the next result. \square

Corollary 1. *Let G be a graph. Then,*

$$GA(G) \leq R_{1/2}(G). \quad (12)$$

From Lemma 1, we get

$$R_{1/2}(G) = \sum_{uv \in E(G)} \sqrt{d_u d_v} \leq \sum_{uv \in E(G)} \frac{d_u + d_v}{2} = \frac{M_1(G)}{2}. \quad (13)$$

Again by Lemma 1, we have

$$\begin{aligned} \frac{M_1(G) + 2R_{1/2}(G)}{4} &= \sum_{uv \in E(G)} \frac{d_u + d_v + 2\sqrt{d_u d_v}}{4} \\ &= \sum_{uv \in E(G)} \frac{((d_u + d_v)/2) + \sqrt{d_u d_v}}{2} \\ &\leq \sum_{uv \in E(G)} \frac{d_u + d_v}{2} = \frac{M_1(G)}{2}. \end{aligned} \quad (14)$$

Hence, we can see that the bounds in Theorem 1 and Corollary 1 improve the bound:

$$GA(G) \leq \frac{M_1(G)}{2}, \quad (15)$$

established in [15].

The proof of the following result can be found in [23].

Lemma 7 (see [23]). *Let G be a graph of size m . Then,*

$$R_\alpha(G) \leq m \left(\frac{\sqrt{8m+1} - 1}{2} \right)^{2\alpha}, \quad (16)$$

for $0 < \alpha \leq 1$.

Using Corollary 1 and Lemma 7, we can drive the next result.

Corollary 2. *Let G be a graph of size m . Then,*

$$GA(G) \leq \frac{m(\sqrt{8m+1} - 1)}{2}. \quad (17)$$

Lemma 8. *Let x and y be two positive numbers. Then,*

$$\begin{aligned} \frac{2\sqrt{xy}}{x+y} &\leq 1, \\ \frac{x+y}{\sqrt{xy}} &\geq 2. \end{aligned} \quad (18)$$

Now, we obtain an upper bound for the geometric-arithmetic index in terms of the first Zagreb index.

Theorem 2. *Let G be a graph of order $n \geq 2$, size m , and minimum degree δ . Then,*

$$GA(G) \leq m - n + \frac{M_1(G)}{\delta^2}. \quad (19)$$

Proof. Notice that

$$\sum_{uv \in E(G)} \frac{d_u + d_v}{d_u d_v} = \sum_{uv \in E(G)} \left(\frac{1}{d_u} + \frac{1}{d_v} \right) = n. \quad (20)$$

By Lemma 8, we have

$$\begin{aligned} GA(G) + n &= \sum_{uv \in E(G)} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{d_u + d_v}{d_u d_v} \right) \\ &\leq \sum_{uv \in E(G)} \left(1 + \frac{d_u + d_v}{d_u d_v} \right) \\ &= \sum_{uv \in E(G)} 1 + \sum_{uv \in E(G)} \frac{d_u + d_v}{d_u d_v} \\ &\leq m + \frac{M_1(G)}{\delta^2}, \end{aligned} \quad (21)$$

and this implies the desired bound.

A dominating set of a graph is a vertex subset whose closed neighborhood includes all vertices of the graph. The domination number of a graph G is the size of a minimum dominating set. \square

Theorem 3 (see [24]). *Let T be a tree of order n with domination number γ . Then,*

$$M_1(T) \leq (n - \gamma)(n - \gamma + 1) + 4(\gamma - 1). \quad (22)$$

By Theorems 2 and 3, we have the following result for trees with the given domination number.

Corollary 3. *Let T be a tree of order $n \geq 2$ with domination number γ . Then,*

$$GA(T) \leq (n - \gamma)(n - \gamma + 1) + 4(\gamma - 1) - 1. \quad (23)$$

Since for every two real numbers x, y , and $xy \leq ((x + y)^2/4)$, we have the next observation.

Lemma 9. *Let x and y be two real numbers, where $x + y \neq 0$. Then, $(xy/(x + y)^2) \leq (1/4)$.*

Next, we establish an upper bound for the geometric-arithmetic index in terms of the second Zagreb index.

Theorem 4. *Let G be a graph of size m with maximum degree Δ . Then,*

$$GA(G) \leq \frac{5m}{4} - \frac{M_2(G)}{4\Delta^2}. \quad (24)$$

Proof. By Lemmas 8 and 9, we have

$$\begin{aligned}
 \text{GA}(G) + \frac{M_2(G)}{4\Delta^2} &\leq \sum_{uv \in E(G)} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{d_u d_v}{(d_u + d_v)^2} \right) \\
 &\leq \sum_{uv \in E(G)} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{1}{4} \right) \\
 &\leq \sum_{uv \in E(G)} \left(1 + \frac{1}{4} \right) \\
 &= \frac{5m}{4},
 \end{aligned} \tag{25}$$

and this implies the desired bound. \square

In [25], it is proved that, for any tree T of order n , $M_2(T) \geq 4n - 8$. Using this and Theorem 4, we obtain the next result.

Corollary 4. *Let T be a tree of order n with maximum degree Δ . Then,*

$$\text{GA}(T) \leq \frac{5(n-1)}{4} - \frac{n-2}{\Delta^2}. \tag{26}$$

Here, we establish an upper bound for the geometric-arithmetic index in terms of the hyper-Zagreb index.

The hyper-Zagreb index is defined as follows [9]:

$$\text{HM}(G) = \sum_{uv \in E(G)} (d_u + d_v)^2. \tag{27}$$

Theorem 5. *Let G be a graph of order n , size m , and minimum degree δ . Then,*

$$\text{GA}(G) \leq m - n + \frac{\text{HM}(G)}{2\delta^2}. \tag{28}$$

Proof. By Inequality (21), we have

$$\begin{aligned}
 \text{GA}(G) + n &\leq \sum_{uv \in E(G)} 1 + \sum_{uv \in E(G)} \frac{d_u + d_v}{d_u d_v} \\
 &\leq \sum_{uv \in E(G)} 1 + \sum_{uv \in E(G)} \frac{d_u + d_v}{(2d_u d_v / (d_u + d_v))} \\
 &= \sum_{uv \in E(G)} 1 + \sum_{uv \in E(G)} \frac{(d_u + d_v)^2}{2d_u d_v} \\
 &\leq m + \frac{\text{HM}(G)}{2\delta^2}.
 \end{aligned} \tag{29}$$

It leads to the desired bound.

The next result is proven in [26]. \square

Theorem 6 (see [26]). *Let G be a graph with n vertices and m edges. Then,*

$$\text{HM}(G) \leq \frac{m^3 (n+1)^6}{16n^2 (n-1)^2}. \tag{30}$$

Theorems 5 and 6 lead to the desired result.

Corollary 5. *Let G be a graph of order n , size m , and minimum degree δ . Then,*

$$\text{GA}(G) \leq m - n + \frac{m^3 (n+1)^6}{32\delta^2 n^2 (n-1)^2}. \tag{31}$$

The redefined third Zagreb index is defined as follows [27]:

$$\text{ReZ}_3(G) = \sum_{uv \in E(G)} (d_u d_v)(d_u + d_v). \tag{32}$$

Now, we obtain an upper bound for the geometric-arithmetic index in terms of the second Zagreb index, the general Randić index, and the redefined third Zagreb index.

Theorem 7. *Let G be a graph with maximum degree Δ and minimum degree δ . Then,*

$$\text{GA}(G) \leq M_2(G) + \frac{R_{1/2}(G)}{\delta} - \frac{\text{ReZ}_3(G)}{2\Delta}. \tag{33}$$

Proof. It is easy to obtain

$$\begin{aligned}
 M_2(G) - \text{GA}(G) &= \sum_{uv \in E(G)} \left(d_u d_v - \frac{2\sqrt{d_u d_v}}{d_u + d_v} \right) \\
 &= \sum_{uv \in E(G)} \left(\frac{(d_u + d_v)d_u d_v - 2\sqrt{d_u d_v}}{d_u + d_v} \right) \\
 &= \sum_{uv \in E(G)} \frac{(d_u + d_v)d_u d_v}{d_u + d_v} - \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \\
 &\geq \frac{\text{ReZ}_3(G)}{2\Delta} - \frac{R_{1/2}(G)}{\delta}.
 \end{aligned} \tag{34}$$

The desired bound follows. \square

Theorem 8. *Let G be a graph of order n , size m , maximum degree Δ , and minimum degree δ . Then,*

$$\text{GA}(G) \leq \frac{2m^2}{n} \left(1 + \frac{1}{4} \left(1 - \frac{1 + (-1)^{m+1}}{2m^2} \left(\frac{\Delta}{\delta} - \frac{\delta}{\Delta} \right)^2 \right) \right). \tag{35}$$

Proof. Now, putting $a_{uv} = (2\sqrt{d_u d_v} / (d_u + d_v))$ for each edge $uv \in E(G)$, $R = (\Delta/\delta)$, and $r = (\delta/\Delta)$ in Lemma 4, we have

$$\sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \sum_{uv \in E(G)} \frac{d_u + d_v}{2\sqrt{d_u d_v}} \leq m^2 \left(1 + \frac{1}{4} \left(1 - \frac{1 + (-1)^{m+1} \left(\frac{\Delta}{\delta} - \frac{\delta}{\Delta} \right)^2}{2m^2} \right) \right). \quad (36)$$

On the contrary, we have

$$\frac{n}{2} = \sum_{uv \in E(G)} \frac{d_u + d_v}{2d_u d_v} \leq \sum_{uv \in E(G)} \frac{d_u + d_v}{2\sqrt{d_u d_v}} \quad (37)$$

Finally, we get the bound by using Inequalities (36) and (37).

The sigma index of G is defined in [28] as

$$\sigma(G) = \sum_{uv \in E(G)} (d_u - d_v)^2. \quad (38)$$

Here, we obtain an upper bound for the geometric-arithmetic index in terms of the first Zagreb index and the sigma index. \square

Theorem 9. Let G be a nontrivial graph with maximum degree Δ . Then,

$$GA(G) \leq \frac{M_1(G)}{2} - \frac{\sigma(G)}{4\Delta}. \quad (39)$$

Proof. For two real numbers x and y , we have that

$$xy = \frac{1}{4} \left((x+y)^2 - (x-y)^2 \right). \quad (40)$$

By (40), we obtain

$$\begin{aligned} GA(G) &= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \leq \sum_{uv \in E(G)} \frac{2d_u d_v}{d_u + d_v} \\ &= \sum_{uv \in E(G)} \frac{(d_u + d_v)^2 - (d_u - d_v)^2}{2(d_u + d_v)} \\ &= \frac{1}{2} \sum_{uv \in E(G)} (d_u + d_v) - \sum_{uv \in E(G)} \frac{(d_u - d_v)^2}{2(d_u + d_v)} \quad (41) \\ &\leq \frac{1}{2} \sum_{uv \in E(G)} (d_u + d_v) - \sum_{uv \in E(G)} \frac{(d_u - d_v)^2}{4\Delta} \\ &= \frac{M_1(G)}{2} - \frac{\sigma(G)}{4\Delta}, \end{aligned}$$

and this implies the desired bound.

The general first F -index of a graph G is defined in [29] as

$$F_1^a(G) = \sum_{uv \in E(G)} (d_u^2 + d_v^2)^a, \quad (42)$$

where a is a real number. In particular, $F_1^1(G) = F(G)$.

Since for every two real numbers x and y , $(x-y)^2 \geq 0$, and we deduce that, for any graph G ,

$$\begin{aligned} F(G) &\geq 2M_2(G), \\ \sigma(G) &= F(G) - 2M_2(G). \end{aligned} \quad (43)$$

Using these and Theorem 9, we obtain the next result. \square

Corollary 6. Let G be a nontrivial graph with maximum degree Δ . Then,

$$GA(G) \leq \frac{M_1(G)}{2} - \frac{F(G) - 2M_2(G)}{4\Delta}. \quad (44)$$

From $F(G) \geq 2M_2(G)$, we would like to indicate that the above new bound improves the known bound:

$$GA(G) \leq \frac{M_1(G)}{2}, \quad (45)$$

which was established in [15].

Now, by using the following result, we want to obtain an upper bound for trees.

Theorem 10 (see [30]). Let T be a tree of order n with independence number α . Then,

$$M_1(T) \leq \alpha^2 - 3\alpha + 4n - 4. \quad (46)$$

Here, by Theorems 9 and 10, we obtain the next result.

Corollary 7. Let T be a tree of order n with independence number α and maximum degree Δ . Then,

$$GA(T) \leq \frac{\alpha^2 - 3\alpha + 4n - 4}{2} - \frac{\sigma(G)}{4\Delta}. \quad (47)$$

4. Lower Bounds for the Geometric-Arithmetic Index

In this section, we first investigate the relationships between the geometric-arithmetic index and some other topological indices, and then, we obtain some lower bounds for the geometric-arithmetic index which improve some well-known bounds.

Theorem 11. Let G be a graph of size m with minimum degree δ . Then,

$$GA(G) \geq \frac{4\delta^2 m^2}{HM(G)}. \quad (48)$$

Proof. By Lemmas 1 and 2, we have

$$\begin{aligned}
\frac{m^2}{\text{GA}(G)} &= \frac{m^2}{\sum_{uv \in E(G)} \left(2\sqrt{d_u d_v} / (d_u + d_v) \right)} \\
&\leq \sum_{uv \in E(G)} \frac{d_u + d_v}{2\sqrt{d_u d_v}} \\
&\leq \sum_{uv \in E(G)} \frac{d_u + d_v}{(4d_u d_v / (d_u + d_v))} \\
&= \sum_{uv \in E(G)} \frac{(d_u + d_v)^2}{4d_u d_v} \\
&\leq \frac{1}{4\delta^2} \sum_{uv \in E(G)} (d_u + d_v)^2 \\
&= \frac{\text{HM}(G)}{4\delta^2}.
\end{aligned} \tag{49}$$

The result follows. \square

Here, by Theorems 11 and 6, we have the next result.

Corollary 8. *Let G be a graph of order n and size m , with minimum degree δ . Then,*

$$\text{GA}(G) \geq \frac{64n^2\delta^2(n-1)^2}{m(n+1)^6}. \tag{50}$$

Since for any real numbers x and y , it holds that $((x+y)^2/4) \leq ((x^2+y^2)/2)$; hence, by this fact and Inequality (49), we can obtain the following result.

Corollary 9. *Let G be a graph of size m with minimum degree δ . Then,*

$$\text{GA}(G) \geq \frac{2\delta^2 m^2}{F(G)}. \tag{51}$$

We start with a lower bound for the geometric-arithmetic index in terms of the general F -index.

Theorem 12. *Let G be a nontrivial graph of size m with minimum degree δ . Then,*

$$\text{GA}(G) \geq \frac{\sqrt{2}\delta m^2}{F_1^{1/2}(G)}. \tag{52}$$

Proof. Set $r = 1$, $a_{uv} = \sqrt[4]{2d_u d_v}$, and $b_{uv} = \sqrt{d_u^2 + d_v^2}$ for each $uv \in E(G)$. By Lemmas 1 and 3, we have

$$\begin{aligned}
\text{GA}(G) &= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \\
&\geq \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{2\sqrt{(d_u^2 + d_v^2)/2}} \\
&= \sum_{uv \in E(G)} \frac{\sqrt{2d_u d_v}}{\sqrt{d_u^2 + d_v^2}} \\
&= \sum_{uv \in E(G)} \frac{\left(\sqrt[4]{2d_u d_v} \right)^2}{\sqrt{d_u^2 + d_v^2}} \\
&\geq \frac{\left(\sum_{uv \in E(G)} \sqrt[4]{2d_u d_v} \right)^2}{\sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}} \\
&\geq \frac{\sqrt{2}\delta m^2}{F_1^{1/2}(G)}.
\end{aligned} \tag{53}$$

The proof is completed. \square

The harmonic index is defined as follows [11]:

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}. \tag{54}$$

Theorem 13. *Let G be a nontrivial graph of order n , size m , and minimum degree δ . Then,*

$$\text{GA}(G) \geq \delta(H(G) + n) - 2m. \tag{55}$$

Proof. Notice that

$$\begin{aligned}
\text{GA}(G) + 2m &= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \sum_{u \in V(G)} d_u \\
&\leq \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \sum_{u \in V(G)} \delta \\
&= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + n\delta \\
&\leq \delta H(G) + n\delta.
\end{aligned} \tag{56}$$

The result follows.

Applying (56), we obtain the next results. \square

Corollary 10. *Let G be a nontrivial graph of order n , size m , and minimum degree δ . Then,*

$$GA(G) \geq \frac{R_{1/2}(G)}{\Delta} + \delta n - 2m. \quad (57)$$

Corollary 11. Let G be a nontrivial graph of order n , size m , and minimum degree δ . Then,

$$GA(G) \geq \frac{\delta m}{\Delta} + \delta n - 2m. \quad (58)$$

Theorem 14 (see [31]). Let G be a connected graph of order $n \geq 3$. Then,

$$H(G) \geq \frac{2(n-1)}{n}. \quad (59)$$

A cut edge of a graph is an edge whose removal increases the number of connected components of the graph.

Lemma 10 (see [32]). Let G be a connected graph of order n and k' cut edges. Then,

$$m \leq \frac{(n-k')(n-k'-1)}{2} + k'. \quad (60)$$

Now, by Theorems 13 and 14, and Lemma 10, we can obtain the next result.

Corollary 12. Let G be a connected graph of order n , k' cut edges, and minimum degree δ . Then,

$$GA(G) \geq \delta \left(\frac{2(n-1)}{n} + n \right) - 2 \left(\frac{(n-k')(n-k'-1)}{2} + k' \right). \quad (61)$$

Here, we will use the following particular case of Jensen's inequality.

Lemma 11. Let $f(x)$ be a convex function defined in $x > 0$. For $x_1, x_2, \dots, x_m > 0$,

$$f\left(\frac{x_1 + x_2 + \dots + x_m}{m}\right) \leq \frac{1}{m} (f(x_1) + f(x_2) + \dots + f(x_m)). \quad (62)$$

The general sum-connectivity index is defined as follows [8]:

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d_u + d_v)^\alpha. \quad (63)$$

Now, we obtain a lower bound for the geometric-arithmetic index in terms of the general sum connectivity index.

Theorem 15. Let G be a graph of size m and minimum degree δ . Then,

$$GA(G) \geq \frac{4\delta^2 \sqrt{m^3}}{\sqrt{\chi_4(G)}}. \quad (64)$$

Proof. Since $f(x) = (1/x^2)$ is a convex function for $x > 0$, from Lemmas 1 and 11, we have

$$\begin{aligned} \left(\frac{m}{GA(G)} \right)^2 &= \left(\frac{m}{\sum_{uv \in E(G)} (2\sqrt{d_u d_v} / (d_u + d_v))} \right)^2 \\ &\leq \frac{1}{m} \sum_{uv \in E(G)} \left(\frac{d_u + d_v}{2\sqrt{d_u d_v}} \right)^2 \\ &\leq \frac{1}{m} \sum_{uv \in E(G)} \left(\frac{d_u + d_v}{(4d_u d_v / (d_u + d_v))} \right)^2 \\ &= \frac{1}{m} \sum_{uv \in E(G)} \left(\frac{(d_u + d_v)^2}{4d_u d_v} \right)^2 \\ &\leq \frac{1}{16m\delta^4} \sum_{uv \in E(G)} (d_u + d_v)^4 \\ &= \frac{\chi_4(G)}{16m\delta^4}, \end{aligned} \quad (65)$$

as desired. \square

Now, we obtain an upper bound for the geometric-arithmetic index in terms of the sigma index.

Theorem 16. Let G be a simple connected graph of size m with maximum degree Δ , p pendent vertices, and minimum nonpendent vertex degree δ_1 . Then,

$$GA(G) \geq \frac{2p\sqrt{\Delta}}{1+\Delta} + \frac{\sqrt{4(m-p)^2 - (m-p/\delta_1^2)(\sigma(G) - p(\delta_1 - 1)^2)}}{\sqrt{(\Delta + \delta_1/2\sqrt{\Delta\delta_1})} + \sqrt{(2\sqrt{\Delta\delta_1}/\Delta + \delta_1)}}. \quad (66)$$

Proof. We partition all the edges into two parts: pendent edges and nonpendent edges, so

$$GA(G) = \sum_{\substack{uv \in E(G) \\ d_u=1}} \frac{2\sqrt{d_v}}{1+d_v} + \sum_{\substack{uv \in E(G) \\ d_u, d_v \neq 1}} \frac{2\sqrt{d_u d_v}}{d_u + d_v}. \quad (67)$$

On one hand, for the pendent edges, it is not hard to check that $(2\sqrt{d_v}/(1+d_v))$ decreases in $2 \leq d_v \leq \Delta$; thus,

$$\sum_{\substack{uv \in E(G) \\ d_u=1}} \frac{2\sqrt{d_v}}{1+d_v} \geq \frac{2p\sqrt{\Delta}}{1+\Delta}. \quad (68)$$

Now, we consider the nonpendent edges. It is easy to see that the function $x + (1/x)$ gets its maximum value when x attains the maximum or minimum value. From $(\Delta/\delta) \geq (d_u/d_v) \geq (\delta/\Delta)$ for all u and $v \in V(G)$, we have

$$\sqrt{\frac{d_u}{d_v}} + \sqrt{\frac{d_v}{d_u}} \leq \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}, \quad (69)$$

which is equivalent to

$$\frac{2\sqrt{\Delta\delta_1}}{\Delta + \delta_1} \leq \frac{2\sqrt{d_u d_v}}{d_u + d_v} \leq 1. \quad (70)$$

Set $a_{uv} = 1$ and $b_{uv} = (2\sqrt{d_u d_v}/d_u + d_v)$ for each edge $uv \in E(G)$, $M_1 = m_1 = M_2 = 1$, and $m_2 = (2\sqrt{\Delta\delta_1}/\Delta + \delta_1)$ in Lemma 6, and we have

$$\begin{aligned} & \sum_{\substack{uv \in E(G) \\ d_v \neq 1}} 1^2 \sum_{\substack{uv \in E(G) \\ d_v \neq 1}} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} \right)^2 \\ &= (m - p) \sum_{\substack{uv \in E(G) \\ d_v \neq 1}} \left(1 - \left(\frac{d_u - d_v}{d_u + d_v} \right)^2 \right) \\ &\leq \frac{1}{4} \left(\sqrt{\frac{1}{(2\sqrt{\Delta\delta_1}/\Delta + \delta_1)}} + \sqrt{\frac{2\sqrt{\Delta\delta_1}}{\Delta + \delta_1}} \right)^2 \left(\sum_{\substack{uv \in E(G) \\ d_v \neq 1}} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \right)^2, \end{aligned} \quad (71)$$

which implies that

$$\begin{aligned} \sum_{\substack{uv \in E(G) \\ d_v \neq 1}} \frac{2\sqrt{d_u d_v}}{d_u + d_v} &\geq \frac{\sqrt{4(m-p) \sum_{\substack{uv \in E(G) \\ d_v \neq 1}} (1 - (d_u - d_v/d_u + d_v)^2)}}{\sqrt{(\Delta + \delta_1/2\sqrt{\Delta\delta_1})} + \sqrt{(2\sqrt{\Delta\delta_1}/\Delta + \delta_1)}} \\ &\geq \frac{\sqrt{4(m-p)^2 - (m-p/\delta_1^2) \sum_{\substack{uv \in E(G) \\ d_v \neq 1}} (d_u - d_v)^2}}{\sqrt{(\Delta + \delta_1/2\sqrt{\Delta\delta_1})} + \sqrt{(2\sqrt{\Delta\delta_1}/\Delta + \delta_1)}} \\ &= \frac{\sqrt{4(m-p)^2 - (m-p/\delta_1^2)(\sigma(G) - \sum_{uv \in E(G) d_u=1} (d_v - 1)^2)}}{\sqrt{(\Delta + \delta_1/2\sqrt{\Delta\delta_1})} + \sqrt{(2\sqrt{\Delta\delta_1}/\Delta + \delta_1)}} \\ &\geq \frac{\sqrt{4(m-p)^2 - (m-p/\delta_1^2)(\sigma(G) - p(\delta_1 - 1)^2)}}{\sqrt{(\Delta + \delta_1/2\sqrt{\Delta\delta_1})} + \sqrt{(2\sqrt{\Delta\delta_1}/\Delta + \delta_1)}}. \end{aligned} \quad (72)$$

Finally, the result follows from (67), (68), and (72).

Next, results are immediate consequences of Theorem 16 with the setting $p = 0$. \square

Corollary 13. For a graph G of size m with maximum degree Δ and minimum degree $\delta \geq 2$,

$$\text{GA}(G) \geq \frac{\sqrt{4m^2 - (m/\delta^2)\sigma(G)}}{\sqrt{(\Delta + \delta/2\sqrt{\Delta\delta})} + \sqrt{(2\sqrt{\Delta\delta}/\Delta + \delta)}}. \quad (73)$$

Now, we obtain a lower bound for the geometric-arithmetic index in terms of the second Zagreb index and the general sum connectivity index.

Theorem 17. Let G be a graph of size m , maximum degree Δ , and minimum degree δ . Then,

$$\text{GA}(G) \geq \sqrt{4M_2(G)\chi_{-2}(G) - \frac{m^2}{4} \cdot \frac{\Delta^2 + \delta^2}{\Delta\delta}}. \quad (74)$$

Proof. By Lemma 5 and putting $a_{uv} = 2\sqrt{d_u d_v}$, $b_{uv} = (1/d_u + d_v)$, $m_1 = 2\delta$, $N_1 = 2\Delta$, $m_2 = (1/2\Delta)$, and $N_2 = (1/2\delta)$, we have

$$\sum_{i=1}^n 4d_u d_v \sum_{i=1}^n \frac{1}{(d_u + d_v)^2} - \left(\sum_{i=1}^n \frac{2\sqrt{d_u d_v}}{d_u + d_v} \right)^2 \leq \frac{m^2}{4} \cdot \frac{\Delta^2 + \delta^2}{\Delta\delta}. \quad (75)$$

This implies that

$$GA(G)^2 \geq 4M_2(G)\chi_{-2}(G) - \frac{m^2}{4} \cdot \frac{\Delta^2 + \delta^2}{\Delta\delta}. \quad (76)$$

The result follows.

Now, we obtain a lower bound for the geometric-arithmetic index in terms of the harmonic index. \square

Theorem 18. *Let G be a graph without isolated edges. Then,*

$$GA(G) \geq \sqrt{2}H(G). \quad (77)$$

Proof. Since for each $uv \in E(G)$, $d_u d_v \geq 2$, we obtain

$$GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \geq \sum_{uv \in E(G)} \frac{2\sqrt{2}}{d_u + d_v} = \sqrt{2}H(G), \quad (78)$$

as desired.

The proof of next results can be found in [33]. \square

Theorem 19 (see [33]). *Let G be a triangle-free graph of order n and the minimum degree $\delta \geq k$ ($k \leq (n/2)$). Then,*

$$H(G) \geq \frac{2k(n-k)}{n}. \quad (79)$$

Theorem 20 (see [33]). *Let G be a triangle-free graph of order n and size m . Then,*

$$H(G) \geq \frac{2m}{n}. \quad (80)$$

Applying Theorems 18–20, it leads to the next results.

Corollary 14. *Let G be a triangle-free graph of order n without isolated edges, and the minimum degree $\delta \geq k$ ($k \leq (n/2)$). Then,*

$$GA(G) \geq \frac{2\sqrt{2}k(n-k)}{n}, \quad (81)$$

$$GA(G) \geq \frac{2\sqrt{2}m}{n}. \quad (82)$$

We can see that Inequality (82) improves the next well-known result for triangle-free graphs [13]. Let G be a graph of order n and size m without isolated vertex. Then,

$$GA(G) \geq \frac{2m}{n}. \quad (83)$$

The eccentricity $\varepsilon(v)$ of v is defined as

$$\varepsilon(v) = \max\{d(v, w) : w \in V(G)\}, \quad (84)$$

where $d(v, w)$ is the length of a shortest path connecting v and w . The radius r and diameter D are defined as the minimum and maximum values among $\varepsilon(v)$ over all vertices $v \in V(G)$, respectively.

Xu [34] showed that, for any nontrivial connected graph G of order n , size m , and radius r , $H(G) \geq (m/n - r)$. Using this and Theorem 18, we obtain the next result.

Corollary 15. *Let G be a nontrivial connected graph of order n , size m , and radius r . Then,*

$$GA(G) \geq \frac{\sqrt{2}m}{n-r}. \quad (85)$$

Theorem 21. *Let G be a nontrivial connected graph of size m and radius r . Then,*

$$GA(G) \geq \frac{R_{1/2}(G)}{n-r}. \quad (86)$$

Proof. Note that, for each vertex $u \in V(G)$, we have $d_u \leq n - \varepsilon(u)$. Thus, for each edge $uv \in E(G)$,

$$\begin{aligned} GA(G) &= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \geq \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{2n - \varepsilon(u) - \varepsilon(v)} \\ &\geq \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{2n - 2r} = \frac{R_{1/2}(G)}{n-r}, \end{aligned} \quad (87)$$

as desired. \square

Theorem 22. *Let G be a nontrivial graph of order n , size m , and p pendent edges without isolated vertex. Then,*

$$GA(G) \geq \frac{p}{\sqrt{n-1}} + \frac{m-p}{n-1-(p/2)}. \quad (88)$$

Proof. Since $0 < (1/d_u)$ and $(1/d_v) \leq 1$, therefore we deduce that

$$\begin{aligned} GA(G) &= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \geq \sum_{uv \in E(G)} \frac{((1/d_u) + (1/d_v))\sqrt{d_u d_v}}{d_u + d_v} \\ &= \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}} \end{aligned} \quad (89)$$

For each pendent edge $e = uv$, we clearly have $(1/\sqrt{d_u d_v}) \geq (1/\sqrt{n-1})$. If $e = uv$ is a nonpendent edge, then $d_u + d_v \leq 2(n-1) - p$, as any pendent vertex is adjacent to at most one of u and v . So, $\sqrt{d_u d_v} \leq (d_u + d_v/2) \leq n-1-(p/2)$; hence,

$$\frac{1}{\sqrt{d_u d_v}} \geq \frac{1}{n-1-(p/2)}. \quad (90)$$

Thus,

$$GA(G) \geq \frac{p}{\sqrt{n-1}} + \frac{m-p}{n-1-(p/2)}. \quad (91)$$

The desired result follows.

In [35], Kulli et al. defined the first and second generalized multiplicative Zagreb indices:

$$\begin{aligned} MZ_1^a(G) &= \prod_{uv \in E(G)} (d_u + d_v)^a, \\ MZ_2^a(G) &= \prod_{uv \in E(G)} (d_u d_v)^a. \end{aligned} \quad (92)$$

Here, we obtain a lower bound in terms of the first and second generalized multiplicative Zagreb indices. \square

Theorem 23. Let G be a nontrivial graph of size m . Then,

$$GA(G) \geq 2m \sqrt{\frac{MZ_2^{1/2}(G)}{MZ_1^1(G)}}. \quad (93)$$

Proof. By Lemma 2, we obtain

$$\begin{aligned} \frac{GA(G)}{2m} &= \frac{1}{m} \sum_{uv \in E(G)} \frac{\sqrt{d_u d_v}}{d_u + d_v} \\ &\geq \sqrt{\prod_{uv \in E(G)} \frac{\sqrt{d_u d_v}}{d_u + d_v}} \\ &= \sqrt[m]{\frac{\prod_{uv \in E(G)} \sqrt{d_u d_v}}{\prod_{uv \in E(G)} (d_u + d_v)}} = \sqrt[m]{\frac{MZ_2^{1/2}(G)}{MZ_1^1(G)}}, \end{aligned} \quad (94)$$

as desired. \square

Theorem 24. Let G be a graph of size m and minimum degree δ . Then,

$$GA(G) \geq \frac{4\delta^2 m^2}{HM(G)}. \quad (95)$$

Proof. By Lemma 1, we get

$$\begin{aligned} \frac{GA(G)}{2m} &= \frac{1}{m} \sum_{uv \in E(G)} \frac{\sqrt{d_u d_v}}{d_u + d_v} \\ &\geq \frac{1}{m} \sum_{uv \in E(G)} \frac{(2d_u d_v / (d_u + d_v))}{d_u + d_v} = \frac{1}{m} \sum_{uv \in E(G)} \frac{2d_u d_v}{(d_u + d_v)^2} \\ &\geq \frac{m}{\sum_{uv \in E(G)} ((d_u + d_v)^2 / 2d_u d_v)} \\ &\geq \frac{m}{(1/2\delta^2) \sum_{uv \in E(G)} (d_u + d_v)^2} \\ &= \frac{2\delta^2 m}{HM(G)}, \end{aligned} \quad (96)$$

as desired.

In the sequel, we obtain a lower bound in terms of the first Zagreb index. \square

Theorem 25. Let G be a graph of size m , maximum degree Δ , and minimum degree δ . Then,

$$GA(G) \geq \frac{\delta m}{\Delta} + 2m - \frac{M_1(G)}{\delta}. \quad (97)$$

Proof. By Lemma 8, we have

$$\begin{aligned} GA(G) + \frac{M_1(G)}{\delta} &\geq \sum_{uv \in E(G)} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{d_u + d_v}{\sqrt{d_u d_v}} \right) \\ &\geq \sum_{uv \in E(G)} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} + 2 \right) \\ &= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \sum_{uv \in E(G)} 2 \\ &\geq \frac{\delta m}{\Delta} + 2m, \end{aligned} \quad (98)$$

and this implies the desired bound. \square

Zhou [36] proved that, for any triangle-free graph of order n and size m , $M_1(G) \leq mn$. Together with Theorem 25, we get the next result.

Corollary 16. Let G be a triangle-free graph of order n , size m , maximum degree Δ , and minimum degree δ . Then,

$$GA(G) \geq m \left(\frac{\delta}{\Delta} + 2 - \frac{n}{\delta} \right). \quad (99)$$

Inequality (98) leads to the following results.

Corollary 17. Let G be a graph of size m , maximum degree Δ , and minimum degree δ . Then,

$$\begin{aligned} \text{GA}(G) &\geq \delta H(G) + 2m - \frac{M_1(G)}{\delta}, \\ \text{GA}(G) &\geq \frac{R_{1/2}(G)}{\Delta} + 2m - \frac{M_1(G)}{\delta}. \end{aligned} \quad (100)$$

Note that, for every two real numbers x and y , $((x+y)^2/xy) \geq 4$. Applying this, we obtain a lower bound for the geometric-arithmetic index in terms of the hyper-Zagreb index.

Theorem 26. Let G be a graph of size m , maximum degree Δ , and minimum degree δ . Then,

$$\text{GA}(G) \geq \frac{\delta m}{\Delta} + 4m - \frac{\text{HM}(G)}{\delta^2}. \quad (101)$$

Proof. From the above inequality, we have

$$\begin{aligned} \text{GA}(G) + \frac{\text{HM}(G)}{\delta^2} &\geq \sum_{uv \in E(G)} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{(d_u + d_v)^2}{d_u d_v} \right) \\ &\geq \sum_{uv \in E(G)} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} + 4 \right) \\ &= \sum_{uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v} + \sum_{uv \in E(G)} 4 \\ &\geq \frac{\delta m}{\Delta} + 4m, \end{aligned} \quad (102)$$

and this implies the desired bound.

Here, we obtain a lower bound for the geometric-arithmetic index in terms of the first Zagreb index. \square

Theorem 27. Let G be a graph of size m and minimum degree δ . Then,

$$\text{GA}(G) \geq 2m - \frac{M_1(G)}{2\delta}. \quad (103)$$

Proof. From the fact that $x + (1/x) \geq 2$ for any $x > 0$, we have

$$\begin{aligned} \text{GA}(G) + \frac{M_1(G)}{2\delta} &\geq \sum_{uv \in E(G)} \left(\frac{2\sqrt{d_u d_v}}{d_u + d_v} + \frac{d_u + d_v}{2\sqrt{d_u d_v}} \right) \\ &\geq \sum_{uv \in E(G)} 2 = 2m, \end{aligned} \quad (104)$$

and this implies the desired bound. \square

Theorem 28 (see [37]). Let G be a graph of size m and diameter $D > 1$. Then,

$$M_1(G) \leq m^2 - m(D-3) + (D-2). \quad (105)$$

Now, by Theorems 27 and 28, we have the following result.

Corollary 18. Let G be a graph of size m , minimum degree δ , and diameter $D > 1$. Then,

$$\text{GA}(G) \geq 2m - \frac{m^2 - m(D-3) + (D-2)}{2\delta}. \quad (106)$$

Theorem 29 (see [38]). Let G be a graph of size m , with t triangles and pendent vertex p . Then,

$$M_1(G) \leq m(p+2) + 3t. \quad (107)$$

Again, by Theorems 27 and 29, we have the following result.

Corollary 19. Let G be a graph of size m , with t triangles, leaf number L , and minimum degree δ . Then,

$$\text{GA}(G) \geq 2m - \frac{m(p+2) + 3t}{2\delta}. \quad (108)$$

Theorem 30 (see [39]). Let G be a triangle- and quadrangle-free graph with $n > 1$ vertices. Then,

$$M_1(G) \leq n(n-1). \quad (109)$$

Also, by Theorems 27 and 30, we have the following result.

Corollary 20. Let G be a triangle- and quadrangle-free graph of order n , size m , and minimum degree δ . Then,

$$\text{GA}(G) \geq 2m - \frac{n(n-1)}{2\delta}. \quad (110)$$

Data Availability

The data used to support the findings of the study are provided within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

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