Research Article

On Laplacian Equienergetic Signed Graphs

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The Laplacian energy of a signed graph is defined as the sum of the distance of its Laplacian eigenvalues from its average degree. Two signed graphs of the same order are said to be Laplacian equienergetic if their Laplacian energies are equal. In this paper, we present several infinite families of Laplacian equienergetic signed graphs.

1. Introduction

A signed graph is an ordered pair $\Gamma = (G, \sigma)$, where $G = (V, E)$ is a simple unsigned graph called the underlying graph of Γ , and $\sigma: E \rightarrow \{+, -\}$, called a signing (or a signature), is a sign function from the edge set E to the set $\{+, -\}$ of signs.

The sign of a signed graph is defined as the product of signs of its edges. A signed graph is said to be positive (respectively, negative) if its sign is positive (respectively, negative), i.e., it contains an even (respectively, an odd) number of negative edges. A signed graph is said to be all-positive (respectively, all-negative) if all its edges are positive (respectively, negative), denoted by $(G, +)$ (respectively $(G, -)$). A signed graph is said to be balanced if each of its cycles is positive, otherwise unbalanced.

Let $A(G) = (a_{ij})_{n \times n}$ be adjacency matrix of unsigned graph G , where $a_{ij} = 1$ whenever vertices i and j are adjacent and $a_{ij} = 0$ otherwise. And, let $D(G)$ be its diagonal matrix of vertices degrees; then, $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called Laplacian and signless Laplacian matrix of G , respectively.

Similarly, for a signed graph $\Gamma = (G, \sigma)$, its adjacency matrix $A(\Gamma) = (a_{ij})_{n \times n}$ is defined as

$$a_{ij} = \begin{cases} \sigma_{ij}, & \text{if vertices } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

and $L(\Gamma) = D(\Gamma) - A(\Gamma)$ is called its Laplacian matrix, where $D(\Gamma)$ is the same as $D(G)$. Note that $L(G, +) = L(G)$ and $L(G, -) = Q(G)$.

Suppose $\Gamma = (G, \sigma)$ is a signed graph and $\theta: V \rightarrow \{+1, -1\}$ is any sign function. Switching Γ by θ means constructing a new signed graph $\Gamma^\theta = (G, \sigma^\theta)$ whose underlying graph is also G , while sign function specified on an edge $e = v_i v_j$ is $\sigma^\theta(e) = \theta(v_i)\sigma(e)\theta(v_j)$.

Let $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graphs on the same underlying graph G . We call Γ_1 and Γ_2 are switching equivalent; write $\Gamma_1 \sim \Gamma_2$, if there exists a switching function θ such that $\Gamma_2 = \Gamma_1^\theta$. Actually, switching equivalent signed graphs can be considered as switching isomorphic, and their signatures are thought to be equivalent. Switching leaves many signed-graphic invariant, such as the set of positive cycles.

Two matrices M_1 and M_2 of order n are called signature similar if there exists a signature matrix, i.e., a diagonal matrix $S = \text{diag}\{s_1, \dots, s_n\}$ with diagonal entries $s_i = \pm 1$ such that $M_2 = SM_1S$. Notice that two signature similar matrices have the same eigenvalues. From the definitions of switching equivalent and signature similar, we have the following.

Lemma 1. Suppose $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ be two signed graphs on the same underlying graph G ; then, $\Gamma_1 \sim \Gamma_2$ if and only if $L(\Gamma_1)$ and $L(\Gamma_2)$ are signature similar.

Signed graphs were first introduced by HARARY, in [1], in connection with the study of the theory of social balance in social psychology, see [2]. After that, there are many works on signed graph, see [3–5].

The energy of unsigned graph G [6–8] is defined as the sum of the absolute values of all eigenvalues of its adjacency matrix. The Laplacian energy of a graph G [9] is defined as $LE(G) = \sum_{i=1}^n |\lambda_i(G) - \bar{d}(G)|$, where $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G) = 0$ are the eigenvalues of Laplacian matrix of G , $\bar{d}(G) = (2m(G)/n)$ is the average degree of G . For its basic properties, see [10–15]. The signless Laplacian energy of a graph G is defined as $QE(G) = \sum_{i=1}^n |q_i(G) - \bar{d}(G)|$, where $q_1(G) \geq q_2(G) \geq \dots \geq q_n(G)$ are the eigenvalues of signless Laplacian matrix of G .

The Laplacian energy of a signed graph Γ , see [16], is defined similarly as $LE(\Gamma) = \sum_{i=1}^n |\lambda_i(\Gamma) - \bar{d}(\Gamma)|$, where $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \dots \geq \lambda_n(\Gamma)$ are the eigenvalues of Laplacian matrix of Γ and $\bar{d}(\Gamma)$ is the average degree of Γ , which is the same as that of G , so we will denote it by $\bar{d}$ simply.

Two (signed) graphs of the same order are said to be Laplacian equienergetic if they have the same Laplacian energy. Two (signed) graphs are said to be Laplacian cospectral if they have the same Laplacian eigenvalues. From Lemma 1, we know that switching equivalent signed graphs must be Laplacian cospectral and Laplacian equienergetic.

Finding Laplacian equienergetic graphs is interesting, and there are so many research works on Laplacian equienergetic unsigned graphs already, see [17–20], for example, but it is very few on signed ones. In this paper, we present several infinite families of Laplacian equienergetic signed graphs which are not Laplacian cospectral.

2. Main Results

It seems more difficult to find Laplacian equienergetic signed graphs than unsigned ones, since we know too little about Laplacian spectrum of signed graphs. It is known that an unsigned graph could be considered as an all-positive signed graph; therefore, two Laplacian equienergetic unsigned graphs is also a pair of Laplacian equienergetic signed ones. And, we will not deal with this case.

In the paper, we try to find Laplacian equienergetic signed graph pairs with common underlying graph. That is, we are interested in graphs G which satisfy $LE(G, \sigma_1) = LE(G, \sigma_2)$. At first, we consider graphs which satisfy $LE(G, +) = LE(G, -)$; in other words, we consider graphs which satisfy $LE(G) = QE(G)$. In fact, there are such graphs trivially. It is well known that, for a regular graph, $LE(G) = QE(G)$ holds. Besides, for bipartite graphs, the Laplacian spectrum coincides with the signless Laplacian spectrum; obviously, in this case, we have $LE(G) = QE(G)$

also. Now, we present two infinite families of connected nonregular and nonbipartite such graphs.

Let $G_1 \nabla G_2$ denote the join of graphs G_1 and G_2 , obtained from the union of G_1 and G_2 by joining every vertex of G_1 with every vertex of G_2 . The following lemma is from [21].

Lemma 2. *Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. Let L_1 and L_2 be the Laplacian matrices for G_1 and G_2 , respectively, and let L be the Laplacian matrix for $G_1 \nabla G_2$. If $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{n_1} = 0$ and $\beta_1 \geq \beta_2 \geq \dots \geq \beta_{n_2} = 0$ are the eigenvalues of L_1 and L_2 , respectively, then the eigenvalues of L are $\{n_1 + n_2, n_1 + \beta_1, n_1 + \beta_2, \dots, n_1 + \beta_{n_2-1}, n_2 + \alpha_1, n_2 + \alpha_2, \dots, n_2 + \alpha_{n_1-1}, 0\}$.*

The first family of such graphs is from [22].

Theorem 3. *For each integer $p \geq 1$, unsigned graph $G \cong K_1 \nabla (K_3 \cup pK_2)$ satisfying $LE(G) = QE(G)$.*

The second family of such graphs is as follows.

Theorem 4. *For each integer $p \geq 3$, unsigned graph $G \cong K_2 \nabla (pK_2)$ satisfying $LE(G) = QE(G)$.*

Proof. On the one hand, by Lemma 2, the Laplacian spectrum of $K_2 \nabla (pK_2)$ is $\{2p + 2^{(2)}, 4^{(p)}, 2^{(p-1)}, 0\}$. Noting that its average degree is $\bar{d} = ((5p + 1)/(p + 1))$, it can be easily verified that $LE(K_2 \nabla (pK_2)) = 4((2p^2 - p + 1)/(p + 1))$.

On the other hand, the signless Laplacian matrix of $K_2 \nabla (pK_2)$ with suitable labeling has the form (with zeros omitted)

$$Q(K_2 \nabla (pK_2)) = \begin{pmatrix} 2p+1 & 1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 2p+1 & 1 & \dots & \dots & 1 & 1 \\ 1 & 1 & 3 & 1 & & & \\ \vdots & \vdots & 1 & 3 & & & \\ & & & & 3 & 1 & \\ & & & & 1 & 3 & \\ & & & & & & \ddots \\ \vdots & \vdots & & & & & 3 & 1 \\ 1 & 1 & & & & & 1 & 3 \end{pmatrix}. \quad (2)$$

Thus, its signless Laplacian characteristic polynomial is $|Q(K_2 \nabla (pK_2)) - xI| =$

$$\begin{vmatrix} 2p+1-x & 1 & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 2p+1-x & 1 & \cdots & \cdots & 1 & 1 \\ 1 & 1 & 3-x & 1 & & & \\ \vdots & \vdots & 1 & 3-x & & & \\ & & 3-x & 1 & & & \\ & & 1 & 3-x & & & \\ & & & \ddots & & & \\ \vdots & \vdots & & & 3-x & 1 & \\ 1 & 1 & & & 1 & 3-x \end{vmatrix}. \quad (3)$$

Performing $C_1 - (1/(4-x))(C_3 + \cdots + C_n)$ and $C_2 - (1/(4-x))(C_3 + \cdots + C_n)$ where C_i is the i -th column of the above determinant, then according to the Laplace theorem, expand the resulting determinant along the first two columns we obtain $|Q(K_2 \nabla(pK_2)) - xI| = (2p+4-x)(2p-x)(4-x)^{p-1}(2-x)^{p+1}$. Therefore its signless Laplacian spectrum is $\{2p+4, 2p, 4^{(p-1)}, 2^{(p+1)}\}$. By direct computation, we get $QE(K_2 \nabla(pK_2)) = 4((2p^2 - p + 1)/(p+1))$. Thus, $LE(K_2 \nabla(pK_2)) = QE(K_2 \nabla(pK_2))$. This completes the proof. $\square$

In the sequel, we consider graphs satisfying $LE(G, \sigma) = LE(G, +)$. We should point out that, by the definition of Laplacian energy of a (signed) graph, when computing its Laplacian energy, it suffices to know its Laplacian eigenvalues greater than its average degree.

Lemma 5. Suppose Γ has exactly k Laplacian eigenvalues greater than average degree $\bar{d}$, that is, $\lambda_1(\Gamma) \geq \cdots \geq \lambda_k(\Gamma) > \bar{d} \geq \lambda_{k+1}(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$; then, $LE(\Gamma) = 2 \sum_{i=1}^k (\lambda_i(\Gamma) - \bar{d})$. Besides, if two signed graphs $\Gamma_1 = (G, \sigma_1)$ and $\Gamma_2 = (G, \sigma_2)$ both have k Laplacian eigenvalues greater than $\bar{d}$ and the same sum of first k -greatest Laplacian eigenvalues, then Γ_1 and Γ_2 are Laplacian equienergetic.

Proof. Notice that $\sum_{i=1}^n \lambda_i(\Gamma) = 2m(\Gamma)$ and $\bar{d} = (2m(\Gamma)/n)$; by the definition of Laplacian energy,

$$\begin{aligned} LE(\Gamma) &= \sum_{i=1}^n |\lambda_i(\Gamma) - \bar{d}| \\ &= \sum_{i=1}^n |\lambda_i(\Gamma) - \bar{d}| + \sum_{i=k+1}^n (\bar{d} - \lambda_i(\Gamma)) \\ &= \sum_{i=1}^n |\lambda_i(\Gamma) - \bar{d}| + (n-k)\bar{d} - \sum_{i=k+1}^n \lambda_i(\Gamma) \\ &= \sum_{i=1}^n |\lambda_i(\Gamma) - \bar{d}| + n\bar{d} - k\bar{d} - \left(2m(\Gamma) - \sum_{i=1}^k \lambda_i(\Gamma)\right) \\ &= 2 \sum_{i=1}^n |\lambda_i(\Gamma) - \bar{d}|. \end{aligned} \quad (4)$$

The rest of the lemma is obvious. $\square$

Theorem 6. For each odd integer $n = 2k+1$, $k \geq 2$, let $G \cong K_2 \nabla(H_k \cup (k-1)K_1)$, where H_k is an arbitrary graph on k vertices (connected or disconnected), the sign function σ specifies exactly k edges to be negative: the only edge of K_2 ; $k-1$ edges connect one vertex of K_2 to $k-1$ distinct isolated vertices K_1 s. And, all remaining edges are positive; then, G satisfies $LE(G, \sigma) = LE(G, +)$.

Proof. With suitable labeling of vertices of (G, σ) , its Laplacian matrix has the form (with zeros omitted)

$$L(G, \sigma) = \begin{pmatrix} n-1 & 1 & 1 & \cdots & 1 & -1 & \cdots & -1 \\ 1 & n-1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\ 1 & -1 & 2 & & & & & \\ \vdots & \vdots & & \ddots & & & & \\ 1 & -1 & & & 2 & & & \\ -1 & -1 & & & & & & \\ \vdots & \vdots & & & & & 2I_k + L(H_k) & \\ -1 & -1 & & & & & & \end{pmatrix}. \quad (5)$$

Thus, its Laplacian characteristic polynomial is
 $|L(G, \sigma) - xI| =$

$$\begin{vmatrix} n-1-x & 1 & 1 & \cdots & 1 & -1 & \cdots & -1 \\ 1 & n-1-x & -1 & \cdots & -1 & -1 & \cdots & -1 \\ 1 & -1 & 2-x & & & & & \\ \vdots & \vdots & & \ddots & & & & \\ 1 & -1 & & & 2-x & & & \\ -1 & -1 & & & & & & \\ \vdots & \vdots & & & & & & \\ & & & & & & (2-x)I_k + L(H_k) & \\ -1 & -1 & & & & & & \end{vmatrix}. \quad (6)$$

Noting that the row sum of $(2-x)I_k + L(H_k)$ is $2-x$, performing $C_1 + C_2$ and $C_1 + (2/(2-x))(C_{k+2} + \cdots + C_n)$, where C_i is the i th column of the determinant, and

performing $C_2 + (1/(2-x))(C_3 + \cdots + C_n)$, the resulting determinant is

$$\begin{vmatrix} n-x-\frac{n-1}{2-x} & \frac{1-x}{2-x} & 1 & \cdots & 1 & -1 & \cdots & -1 \\ n-x-\frac{n-1}{2-x} & n-1-x-\frac{n-2}{2-x} & -1 & \cdots & -1 & -1 & \cdots & -1 \\ 0 & 0 & 2-x & & & & & \\ \vdots & \vdots & & \ddots & & & & \\ & & & & 2-x & & & \\ \vdots & \vdots & & & & & (2-x)I_k + L(H_k) & \\ 0 & 0 & & & & & & \end{vmatrix}. \quad (7)$$

Now, according to the Laplace theorem, expanding the above determinant along the first two columns, we obtain

$$\begin{vmatrix} n-x-\frac{n-1}{2-x} & \frac{1-x}{2-x} \\ n-x-\frac{n-1}{2-x} & n-1-x-\frac{n-2}{2-x} \end{vmatrix} \times (-1)^{1+2+1+2} \times (2-x)^{k-1} \times |(2-x)I_k + L(H_k)|. \quad (8)$$

Suppose the Laplacian spectrum of H_k is $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k = 0$; then, we have $|(2-x)I_k + L(H_k)| = \prod_{i=1}^k (2 + \alpha_i - x)$; thus, we obtain $|L(G, \sigma) - xI| = (n+1-x)(n-1-x)(1-x)^2(2-x)^{(n-7)/2} \prod_{i=1}^k (2 + \alpha_i - x)$. Therefore, the Laplacian spectrum of (G, σ) is $\{n+1, n-1, 2 + \alpha_1, \dots, 2 + \alpha_{k-1}, 2^{((n-5)/2)}, 1^{(2)}\}$.

On the contrary, by Lemma 2, we can get the Laplacian spectrum of $(G, +)$ is $\{n^{(2)}, 2 + \alpha_1, \dots, 2 + \alpha_{k-1}, 2^{(k-1)}, 0\}$. Notice that $(G, +)$ and (G, σ) have the same average degree, which is $\bar{d} \geq ((4n-6)/n) > 2$, and both have i ; for some $2 \leq i \leq k+1$, Laplacian eigenvalues are greater than $\bar{d}$ and the same sum of first i -greatest Laplacian eigenvalues; by Lemma 5, we can conclude that $LE(G, \sigma) = LE(G, +)$. This completes the proof. $\square$

Theorem 7

(1) For each even integer $n = 4k + 2$, $k \geq 1$, let $G \cong K_3 \nabla (H_{k-1} \cup 3kK_1)$, where H_{k-1} is an arbitrary

graph on $k-1$ vertices (connected or disconnected); the sign function σ specifies exactly $3k$ edges to be negative, where k edges connecting the i th vertex of K_3 to k distinct isolated vertices K_1 s, $i = 1, 2, 3$, and all remaining edges are positive; then, G satisfies $LE(G, \sigma) = LE(G, +)$.

(2) For each even integer $n = 4k$, $k \geq 2$, let $G \cong K_3 \nabla (H_k \cup (3k-3)K_1)$, where H_k is an arbitrary graph on k vertices (connected or disconnected), the sign function σ specifies $3k$ edges to be negative, where $k-1$ edges connect the i th vertex of K_3 to $k-1$ distinct isolated vertices K_1 s, $i = 1, 2, 3$, the 3 edges of K_3 . And, all remaining edges are positive; then, G satisfies $LE(G, \sigma) = LE(G, +)$.

Proof. With suitable labeling of vertices of (G, σ) , its Laplacian characteristic polynomial is $|L(G, \sigma) - xI| =$

$$\begin{vmatrix} n-1-x & -1 & -1 & \overbrace{-1 \dots -1}^{k-1} & \overbrace{-1 \dots -1}^k & \overbrace{-1 \dots -1}^k & \overbrace{1 \dots 1}^k \\ -1 & n-1-x & -1 & -1 \dots -1 & -1 \dots -1 & 1 \dots 1 & -1 \dots -1 \\ -1 & -1 & n-1-x & -1 \dots -1 & 1 \dots 1 & -1 \dots -1 & -1 \dots -1 \\ -1 & -1 & -1 & & & & \\ \vdots & \vdots & \vdots & (3-x)I_{k-1} + L(H_{k-1}) & & & \\ -1 & -1 & -1 & & & & \\ -1 & -1 & 1 & & 3-x & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ -1 & -1 & 1 & & & & \\ -1 & 1 & -1 & & & & \\ \vdots & \vdots & \vdots & & & & \\ -1 & 1 & -1 & & & & \\ 1 & -1 & -1 & & & & \\ \vdots & \vdots & \vdots & & & & \\ 1 & -1 & -1 & & & & \ddots \\ & & & & & & 3-x \end{vmatrix}. \quad (9)$$

Performing, in turn, $C_1 + (1/(3-x))(C_4 + \dots + C_{3k+2} - C_{3k+3} - \dots - C_n)$, $C_2 + (1/(3-x))(C_4 + \dots + C_{2k+2} - C_{2k+3} - \dots - C_{3k+2} + C_{3k+3} + \dots + C_n)$, and $C_3 + (1/(3-x))(C_4 +$

$\dots + C_{k+2} - C_{k+3} - \dots - C_{2k+2} + C_{2k+3} + \dots + C_n)$, where C_i is the i th column of the determinant, up to now, except for the first three rows, all elements of the first three columns are

FIGURE 1: Examples in theorems.

TABLE 1 The Laplacian spectra in examples.

$(G_1, \sigma_1): \{8, 6, 5, 3, 2, 1^{(2)}\}$	$(G_1, +): \{7^{(2)}, 5, 3, 2^{(2)}, 0\}$	LE = $(110/7)$
$(G_2, \sigma_2): \{15^{(2)}, 12, 5, 3^{(7)}, 2^{(3)}\}$	$(G_2, +): \{14^{(3)}, 5, 3^{(9)}, 0\}$	LE = $(366/7)$
$(G_3, \sigma_3): \{14, 11^{(2)}, 6^{(2)}, 3^{(4)}, 2^{(3)}\}$	$(G_3, +): \{12^{(3)}, 6^{(2)}, 3^{(6)}, 0\}$	LE = 41
$(G_4, \sigma_4): \{14, 12^{(2)}, 10, 7, 5, 4^{(4)}, 2^{(2)}\}$	$(G_4, +): \{12^{(4)}, 7, 5, 4^{(5)}, 0\}$	LE = $(130/3)$

simplified to zero. Then, according to the Laplace theorem, expanding the resulting determinant along the first three columns, we obtain

$$\begin{vmatrix}
 n-2-x-\frac{4k-2}{3-x} & \frac{2x-4}{3-x} & \frac{x-2}{3-x} \\
 n-2-x-\frac{4k-2}{3-x} & n-2-x-\frac{4k-2}{3-x} & \frac{x-2}{3-x} \\
 \frac{2x-4}{3-x} & n-2-x-\frac{4k-2}{3-x} & n-1-x-\frac{4k-1}{3-x}
 \end{vmatrix} \times |(3-x)I_{k-1} + L(H_{k-1})| \times (3-x)^{3k}. \quad (10)$$

Noting that $n = 4k + 2$, the above expression can be simplified as $(n+1-x)^2 (n-2-x)(2-x)^3 |(3-x)I_{k-1} + L(H_{k-1})| (3-x)^{3k}$. Suppose the Laplacian spectrum of H_{k-1} is $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{k-1} = 0$; then, we have $|(3-x)I_{k-1} + L(H_{k-1})| = \prod_{i=1}^{k-1} (3 + \alpha_i - x)$; consequently, the Laplacian

spectrum of (G, σ) is $\{n+1^{(2)}, n-2, 3 + \alpha_1, \dots, 3 + \alpha_{k-1}, 3^{(3k-3)}, 2^{(3)}\}$.

By Lemma 2, we can easily get the spectrum of $L(G, +)$ is $\{n^{(3)}, 3 + \alpha_1, \dots, 3 + \alpha_{k-1}, 3^{(3k-1)}, 0\}$. Notice that $(G, +)$ and (G, σ) have the same average degree, which is

$\bar{d} \geq ((6n - 12)/n) \geq 4$, and both have i ; for some $3 \leq i \leq k + 2$, Laplacian eigenvalues are greater than $\bar{d}$ and the same sum of first i -greatest Laplacian eigenvalues; by Lemma 5, we can conclude that $LE(G, \sigma) = LE(G, +)$. Therefore, case (1) holds.

As for the proof of case (2), it is similar to that of case (1). We just present the Laplacian spectrum of (G, σ) , which is $\{n + 2, n - 1^{(2)}, 3 + \alpha_1, \dots, 3 + \alpha_k, 3^{(3k-6)}, 2^{(3)}\}$, and the spectrum of $L(G, +)$ is $\{n^{(3)}, 3 + \alpha_1, \dots, 3 + \alpha_k, 3^{(3k-4)}, 0\}$, where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k = 0$ is the Laplacian spectrum of H_k . And, it is not difficult to verify that $LE(G, \sigma) = LE(G, +)$ holds. $\square$

Theorem 8. For each even integer $n = 2k + 2$, $k \geq 2$, let $G \cong K_4 \nabla (H_{k-2} \cup kK_1)$, where H_{k-2} is an arbitrary graph on $k - 2$ vertices (connected or disconnected); the sign function σ specifies exactly $2k$ edges to be negative, where $2k$ edges connect two vertices of K_4 to k distinct isolated vertices K_1 s, and all remaining edges are positive; then, G satisfies $LE(G, \sigma) = LE(G, +)$.

Proof. The proof of Theorem 8 is similar to that of Theorem 6. We just present the Laplacian spectrum of (G, σ) , which is $\{n + 2, n^{(2)}, n - 2, 4 + \alpha_1, \dots, 4 + \alpha_{k-2}, 4^{(k-2)}, 2^{(2)}\}$, where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{k-2} = 0$, is the Laplacian spectrum of H_{k-2} . While the Laplacian spectrum of $(G, +)$ is $\{n^{(4)}, 4 + \alpha_1, \dots, 4 + \alpha_{k-2}, 4^{(k-1)}, 0\}$, and it is not difficult to verify that $LE(G, \sigma) = LE(G, +)$ holds. $\square$

Example 1. We give four instances with property $LE(G, \sigma) = LE(G, +)$ corresponding to Theorems 6, 7 (1) and (2), and 8, respectively, see Figure 1, where bold lines denote positive edges and dotted lines denote negative edges. $G_1 \cong K_2 \nabla (S_3 \cup 2K_1)$, $G_2 \cong K_3 \nabla (K_2 \cup 9K_1)$, $G_3 \cong K_3 \nabla (K_3 \cup 6K_1)$, and $G_4 \cong K_4 \nabla (S_3 \cup 5K_1)$. Their Laplacian spectra are shown in Table 1.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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