Research Article

A High-Order Melnikov Method for Heteroclinic Orbits in Planar Vector Fields and Heteroclinic Persisting Perturbations

Yi Zhong

School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China

Correspondence should be addressed to Yi Zhong; yizhong@zju.edu.cn

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This work extends the high-order Melnikov method established by FJ Chen and QD Wang to heteroclinic orbits, and it is used to prove, under a certain class of perturbations, the heteroclinic orbit in a planar vector field that remains unbroken. Perturbations which have this property together form the heteroclinic persisting space. The Van der Pol system is analysed as an application.

1. Introduction

Since the pioneering work of Poincaré on celestial mechanics [1, 2], homoclinic tangles created by transversal intersection of the stable and unstable manifolds have been playing a prominent role in the progress of chaos theory. In 1967, Smale observed that horseshoe chaos occurs in all homoclinic tangles [3]. For the past few years, the studies in [4–7] presented that periodic sinks and Henonlike attractors are also residing in homoclinic and heteroclinic tangles. The Henonlike attractors provide observable chaos in the sense of Sinai–Ruelle–Bowen (SRB) measure [8, 9].

In general, the standard Melnikov function is enough to verify whether the homoclinic tangle or heteroclinic tangle occurs [10, 11]. However, there exist perturbations which make the standard Melnikov function identically equal to zero. In such a case, no simple zero exists. As a result, we need to calculate high-order Melnikov functions for further investigation. There have been many previous works to refer high-order Melnikov method (see [12–19]). We focus on the method presented in [13] and extends the high-order Melnikov functions for homoclinic orbits to heteroclinic orbits.

Naturally, another question arises: does there exist a perturbation such that all the high-order Melnikov functions identically equal to zero? This means the homoclinic or

heteroclinic solution remains unbroken. Through the high-order Melnikov method, we prove that the answer of the above question is positive. The union of such perturbations will be defined as heteroclinic persisting space.

This paper is built up as follows: In Section 2, we outline the main results. Section 3 is devoted to focus on the derivation of high-order Melnikov functions for heteroclinic orbits. Then, in Section 4, we prove the heteroclinic orbit remains unbroken under a certain class of autonomous perturbations. Finally, Van der Pol system is considered in Section 5.

2. Statement of Results

We start with a planar system:

$$\begin{aligned}\frac{dx}{dt} &= f(x, y), \\ \frac{dy}{dt} &= g(x, y).\end{aligned}\tag{1}$$

Assume that (1) has two hyperbolic saddle fixed points $O_1(x_1, y_1)$ and $O_2(x_2, y_2)$ with a heteroclinic solution

$$\ell = \{(a(t), b(t)): t \in (-\infty, +\infty)\},\tag{2}$$

such that

$$\begin{aligned}\lim_{t \rightarrow -\infty} (a(t), b(t)) &= (x_1, y_1), \\ \lim_{t \rightarrow +\infty} (a(t), b(t)) &= (x_2, y_2).\end{aligned}\quad (3)$$

The functions $f(x, y)$ and $g(x, y)$ are real analytic near $\ell \cup \{O_1, O_2\}$.

Adding time-periodic perturbations to (1) yields

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) + \varepsilon P(x, y, t), \\ \frac{dy}{dt} &= g(x, y) + \varepsilon Q(x, y, t),\end{aligned}\quad (4)$$

where ε is a small parameter. The periodic functions $P(x, y, t)$ and $Q(x, y, t)$ are real analytic with respect to

(x, y) , and they are terms of second order and higher at O_1 and O_2 . Let $W(O_1)$ be the unstable manifold of O_1 and $W_\varepsilon^s(O_2)$ be the stable manifold of O_2 . In general, $W_\varepsilon^u(O_1) \neq W_\varepsilon^s(O_2)$ for $\varepsilon \neq 0$. We let $D(\varepsilon, t_0)$ be the splitting distance, which can be expressed in the form of

$$D(\varepsilon, t_0) = M_0(t_0)\varepsilon + M_1(t_0)\varepsilon^2 + \dots + M_k(t_0)\varepsilon^{k+1} + \dots, \quad (5)$$

where $M_0(t_0)$ is equivalent to the standard Melnikov function in [10, 11] and $M_k(t_0)$, $k = 1, 2, \dots$, are the high-order Melnikov functions. Moreover, we write $a = a(t)$, $b = b(t)$, $\dot{a} = da/dt$, $\dot{b} = db/dt$, $\ddot{a} = d\dot{a}/dt$, and $\ddot{b} = d\dot{b}/dt$ for easy of notations. We also introduce

$$\begin{aligned}A(t, 0) &= -\frac{A_0(t)}{\dot{a}^2 + \dot{b}^2}, \\ A_{1,0}(t) &= \frac{dA(t, 0)}{dt}, \\ A_{0,1}(t) &= -\frac{A_1(t)}{\dot{a}^2 + \dot{b}^2} + \frac{A_0(t)H(t)}{(\dot{a}^2 + \dot{b}^2)^2}, \\ B(t, 0) &= \frac{B_0(t)}{\dot{a}^2 + \dot{b}^2}, \\ U(t, 0, t + t_0) &= \frac{U_0(t, t_0)}{\dot{a}^2 + \dot{b}^2}, \\ V(t, 0, t + t_0) &= \frac{V_0(t, t_0)}{\dot{a}^2 + \dot{b}^2}, \\ U_{1,0}(t, t_0) &= \frac{dU(t, 0, t + t_0)}{dt}, \\ U_{0,1}(t, t_0) &= -\frac{U_1(t, t_0)}{\dot{a}^2 + \dot{b}^2} + \frac{U_0(t, t_0)H(t)}{(\dot{a}^2 + \dot{b}^2)^2}, \\ Z_0^\pm(t, t_0) &= \int_0^{\pm\infty} U(\tau, 0, \tau + t_0) e^{\int_0^\tau A(\omega, 0) d\omega} d\tau, \\ S_0^\pm(t, t_0) &= \int_0^t [B(\tau, 0)Z_0^\pm(\tau, t_0) + V(\tau, 0, \tau + t_0)] d\tau,\end{aligned}\quad (6)$$

where

$$\begin{aligned}
A_0(t) &= [f_x(a, b) - g_y(a, b)](\dot{b}^2 - \dot{a}^2) - 2[f_y(a, b) + g_x(a, b)]\dot{a}\dot{b}, \\
U_0(t, t_0) &= \dot{b}P(a, b, t + t_0) - \dot{a}Q(a, b, t + t_0), \\
B_0(t) &= [f_y(a, b) + g_x(a, b)](\dot{b}^2 - \dot{a}^2) + 2[f_x(a, b) - g_y(a, b)]\dot{a}\dot{b}, \\
V_0(t, t_0) &= \dot{a}P(a, b, t + t_0) + \dot{b}Q(a, b, t + t_0), \\
A_1(t, t_0) &= \frac{\dot{b}}{2} [f_{xx}(a, b)\dot{b}^2 + f_{yy}(a, b)\dot{a}^2 - 2f_{xy}(a, b)\dot{a}\dot{b}] - \frac{a}{2} [g_{xx}(a, b)\dot{b}^2 + g_{yy}(a, b)\dot{a}^2 - 2g_{xy}(a, b)\dot{a}\dot{b}] \\
&\quad - [f_x^2(a, b) - f_y^2(a, b) - g_y^2(a, b) + g_x^2(a, b)]\dot{a}\dot{b} + [f_x(a, b)f_y(a, b) + g_x(a, b)g_y(a, b)](\dot{a}^2 - \dot{b}^2), \\
U_1(t, t_0) &= \dot{a}^2 Q_y(a, b, t + t_0) + \dot{b}^2 P_x(a, b, t + t_0) - [P_y(a, b, t + t_0) + Q_x(a, b, t + t_0)]\dot{a}\dot{b} \\
&\quad - \ddot{a}P(a, b, t + t_0) - \ddot{b}Q(a, b, t + t_0), \\
H(t) &= \ddot{a}\dot{b} - \ddot{b}\dot{a}.
\end{aligned} \tag{7}$$

In what follows, we give the formulas of $M_0(t_0)$ and $M_1(t_0)$.

Theorem 1. In (5), we have the following:

(i) Integral formula for $M_0(t_0)$:

$$M_0(t_0) = \int_{-\infty}^{+\infty} U(t, 0, t + t_0) e^{\int_0^t A(\omega, 0) d\omega} dt. \tag{8}$$

(ii) Integral formula for $M_1(t_0)$:

$$\begin{aligned}
M_1(t_0) &= \int_0^{+\infty} [A_{1,0}(t)Z_0^+(t, t_0)S_0^+(t, t_0) + U_{1,0}(t, t_0)S_0^+(t, t_0) \\
&\quad + A_{0,1}(t)(Z_0^+(t, t_0))^2 + U_{0,1}(t, t_0)Z_0^+(t, t_0)] e^{\int_0^t A(\omega, 0) d\omega} dt \\
&\quad - \int_0^{-\infty} [A_{1,0}(t)Z_0^-(t, t_0)S_0^-(t, t_0) + U_{1,0}(t, t_0)S_0^-(t, t_0) \\
&\quad + A_{0,1}(t)(Z_0^-(t, t_0))^2 + U_{0,1}(t, t_0)Z_0^-(t, t_0)] e^{\int_0^t A(\omega, 0) d\omega} dt.
\end{aligned} \tag{9}$$

On the other hand, we consider the question whether there exist any perturbations such that the heteroclinic solution remains unbroken, which implies the splitting distance $D(\varepsilon, t_0)$ vanishes identically. By using the high-order Melnikov functions above, such class of perturbations are obtained. Before giving our main theorem, we first introduce some definitions.

Definition 1. Let X be a smooth manifold and F and G be vector fields on it. We assume there exists a heteroclinic orbit ℓ of F and a heteroclinic orbit ℓ_ε of $F + \varepsilon G$, where ε is a small parameter. We call G a heteroclinic persisting perturbation of ℓ if

$$\lim_{\varepsilon \rightarrow 0} \ell_\varepsilon = \ell, \tag{10}$$

in the sense of Hausdorff limit.

Definition 2. All heteroclinic persisting perturbations of ℓ are referred to as the heteroclinic persisting space of ℓ , which we denote by $HP(\ell)$.

It is not difficult to find a heteroclinic persisting perturbation. Let $(P(x, y, t), Q(x, y, t))$ in (4) be satisfying

$$P(a(t), b(t), t) = Q(a(t), b(t), t) = 0, \quad t \in (-\infty, +\infty). \tag{11}$$

Then, we have

$$\frac{d(a(t))}{dt} = f(a(t), b(t)) + \varepsilon P(a(t), b(t), t), \quad (12)$$

$$\frac{d(b(t))}{dt} = g(a(t), b(t)) + \varepsilon Q(a(t), b(t), t).$$

It follows that $(a(t), b(t))$ is also a heteroclinic solution of (4). By Definition 1, we claim $(P(x, y, t), Q(x, y, t))$ is a heteroclinic persisting perturbation.

However, this heteroclinic persisting perturbation is trivial, since the heteroclinic orbit $(a(t), b(t))$ is just the previous one. We would like to quest for more general ones. For this purpose, we focus on an autonomous perturbed system:

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) + \varepsilon P(x, y), \\ \frac{dy}{dt} &= g(x, y) + \varepsilon Q(x, y). \end{aligned} \quad (13)$$

Then, splitting distance (5) can be rewritten as

$$D(\varepsilon, P, Q) = M_0(P, Q)\varepsilon + M_1(P, Q)\varepsilon^2 + \dots + M_k(P, Q)\varepsilon^{k+1} \dots, \quad (14)$$

where $M_k(P, Q)$ is the $(k+1)$ -th order Melnikov integral, $k = 0, 1, 2, \dots$. The following fact is trivial.

The fact is that $(P(x, y), Q(x, y))$ is a heteroclinic persisting perturbation if and only if $D(\varepsilon, P, Q) = 0$ for all $|\varepsilon| < \varepsilon_0$, where $\varepsilon_0 > 0$ is sufficiently small.

Note that the geometric shape of heteroclinic orbits in planar vector fields are often symmetric which contributes to the computations of high-order Melnikov integrals. We obtain the following theorem.

Theorem 2. *For the given vector field defined by (1), we assume $a(t)$ is odd and $b(t)$ is even in t and $f(x, y)$ is even and $g(x, y)$ is odd in x . Then, we have*

$$\{(P, Q): P(x, y) \text{ is even in } x, Q(x, y) \text{ is odd in } x\} \subset HP(\ell). \quad (15)$$

The example considered in Section 5 will show that the parity assumption made in Theorem 2 is without loss of generality.

3. High-Order Melnikov Method

This section is intended to derive the high-order Melnikov functions for equation (4). The method here is based on the homoclinic case introduced in [13], but there still exist some nonnegligible differences between them.

3.1. Canonical Equation around the Heteroclinic Orbit. We introduce new phase variables (s, z) by letting

$$(x, y) = (a(s), b(s)) + z(\dot{b}(s), -\dot{a}(s)). \quad (16)$$

Differentiating both sides of (16) with respect to t gives

$$\begin{aligned} \frac{dx}{dt} &= (\dot{a}(s) + z\ddot{b}(s))\frac{ds}{dt} + \dot{b}(s)\frac{dz}{dt}, \\ \frac{dy}{dt} &= (\dot{b}(s) - z\ddot{a}(s))\frac{ds}{dt} - \dot{a}(s)\frac{dz}{dt}. \end{aligned} \quad (17)$$

By Cramer's Rule and (4), we have

$$\begin{aligned} \frac{dz}{dt} &= -A(s, z)z - \varepsilon U(s, z, t), \\ \frac{ds}{dt} &= 1 + B(s, z)z + \varepsilon V(s, z, t), \end{aligned} \quad (18)$$

where

$$\begin{aligned} A(s, z) &= -\frac{1}{z} \frac{(\dot{b} - z\ddot{a})f(a + z\dot{b}, b - z\dot{a}) - (\dot{a} + z\ddot{b})g(a + z\dot{b}, b - z\dot{a})}{\dot{a}^2 + \dot{b}^2 + z(\ddot{a}\dot{b} - \ddot{b}\dot{a})} \\ U(s, z, t) &= -\frac{(\dot{b} - z\ddot{a})P(a + z\dot{b}, b - z\dot{a}, t) - (\dot{a} + z\ddot{b})Q(a + z\dot{b}, b - z\dot{a}, t)}{\dot{a}^2 + \dot{b}^2 + z(\ddot{a}\dot{b} - \ddot{b}\dot{a})} \\ B(s, z) &= \frac{1}{z} \left(\frac{\dot{a}f(a + z\dot{b}, b - z\dot{a}) + \dot{b}g(a + z\dot{b}, b - z\dot{a})}{\dot{a}^2 + \dot{b}^2 + z(\ddot{a}\dot{b} - \ddot{b}\dot{a})} - 1 \right) \\ V(s, z, t) &= \frac{\dot{a}P(a + z\dot{b}, b - z\dot{a}, t) + \dot{b}Q(a + z\dot{b}, b - z\dot{a}, t)}{\dot{a}^2 + \dot{b}^2 + z(\ddot{a}\dot{b} - \ddot{b}\dot{a})}. \end{aligned} \quad (19)$$

We suppose $(\hat{s}(t), \hat{z}(t))$ to be a stable solution or an unstable solution of system (18) such that $\hat{s}(t_0) = 0$ and $\hat{z}(t_0) = z_0$. Let $s(t) = \hat{s}(t + t_0)$ and $z(t) = \hat{z}(t + t_0)$;

then, $(s(t), z(t))$ is well-defined for all $t \in [0, +\infty)$ or $t \in (-\infty, 0]$, and it is the solution of the system

$$\begin{aligned}\frac{dz}{dt} &= -A(s, z)z - \varepsilon U(s, z, t + t_0), \\ \frac{ds}{dt} &= 1 + B(s, z)z + \varepsilon V(s, z, t + t_0),\end{aligned}\quad (20)$$

satisfying $s(0) = 0$ and $z(0) = z_0$. To differ from the other solutions, we call them the primary stable and unstable solutions.

3.2. Integral of Primary Solutions. We let

$$\begin{aligned}Z &= \varepsilon^{-1}z, \\ S &= \varepsilon^{-1}(s - t),\end{aligned}\quad (21)$$

as one more change of variables for system (20). Then, we have

$$\begin{aligned}\frac{dZ}{dt} &= -A(t + \varepsilon S, \varepsilon Z)Z - U(t + \varepsilon S, \varepsilon Z, t + t_0), \\ \frac{dS}{dt} &= B(t + \varepsilon S, \varepsilon Z)Z + V(t + \varepsilon S, \varepsilon Z, t + t_0).\end{aligned}\quad (22)$$

Expanding $A(t + \varepsilon S, t + \varepsilon Z)$, $U(t + \varepsilon S, t + \varepsilon Z, t + t_0)$, $B(t + \varepsilon S, t + \varepsilon Z)$, and $V(t + \varepsilon S, t + \varepsilon Z, t + t_0)$ at $(t, 0)$, we obtain

$$\begin{aligned}A(t + \varepsilon S, t + \varepsilon Z) &= A(t, 0) + \sum_{n+m \geq 1} A_{n,m}(t) \varepsilon^{n+m} S^n Z^m, \\ U(t + \varepsilon S, t + \varepsilon Z, t + t_0) &= U(t, 0, t + t_0) + \sum_{n+m \geq 1} U_{n,m}(t, t_0) \varepsilon^{n+m} S^n Z^m, \\ B(t + \varepsilon S, t + \varepsilon Z) &= B(t, 0) + \sum_{n+m \geq 1} B_{n,m}(t) \varepsilon^{n+m} S^n Z^m, \\ V(t + \varepsilon S, t + \varepsilon Z, t + t_0) &= V(t, 0, t + t_0) + \sum_{n+m \geq 1} V_{n,m}(t, t_0) \varepsilon^{n+m} S^n Z^m.\end{aligned}\quad (23)$$

It then follows that

$$\begin{aligned}\frac{dZ}{dt} &= -A(t, 0)Z - U(t, 0, t + t_0) - \sum_{n+m \geq 1} \varepsilon^{n+m} [A_{n,m}(t) S^n Z^{m+1} + U_{n,m}(t, t_0) S^n Z^m], \\ \frac{dS}{dt} &= B(t, 0)Z + V(t, 0, t + t_0) + \sum_{n+m \geq 1} \varepsilon^{n+m} [B_{n,m}(t) S^n Z^{m+1} + V_{n,m}(t, t_0) S^n Z^m].\end{aligned}\quad (24)$$

We solve (24) to obtain the primary stable solution

$$\begin{aligned}Z^+(t, t_0) &= e^{-\int_0^t A(\tau, 0) d\tau_1} \left(Z_0^+ - \int_0^t U(\tau, 0, \tau + t_0) e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau \right. \\ &\quad - \sum_{n+m=1}^{+\infty} \varepsilon^{n+m} \int_0^t [A_{n,m}(\tau) Z^+(\tau, t_0) \\ &\quad \left. + U_{n,m}(\tau, t_0)] (S^+(\tau, t_0))^n (Z^+(\tau, t_0))^m e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau \right), \\ S^+(t, t_0) &= \int_0^t [V(\tau, 0, \tau + t_0) + B(\tau, 0) Z^+(\tau, t_0)] d\tau \\ &\quad + \sum_{n+m=1}^{+\infty} \varepsilon^{n+m} \int_0^t [B_{n,m}(\tau) Z^+(\tau, t_0) \\ &\quad \left. + V_{n,m}(\tau, t_0)] (S^+(\tau, t_0))^n (Z^+(\tau, t_0))^m d\tau,\end{aligned}\quad (25)$$

and the primary unstable solution

$$\begin{aligned} Z^-(t, t_0) = & e^{-\int_0^t A(\omega, 0) d\omega} \left(Z_0^- - \int_0^t U(\tau, 0, \tau + t_0) e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau \right. \\ & - \sum_{n+m=1}^{+\infty} \varepsilon^{n+m} \int_0^t [A_{n,m}(\tau) Z^-(\tau, t_0) \\ & \left. + U_{n,m}(\tau, t_0)] (S^-(\tau, t_0))^n (Z^-(\tau, t_0))^m e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau \right), \end{aligned} \quad (26)$$

$$\begin{aligned} S^-(t, t_0) = & \int_0^t [V(\tau, 0, \tau + t_0) + B(\tau, 0) Z^-(\tau, t_0)] d\tau \\ & + \sum_{n+m=1}^{+\infty} \varepsilon^{n+m} \int_0^t [B_{n,m}(\tau) Z^-(\tau, t_0) \\ & + V_{n,m}(\tau, t_0)] (S^-(\tau, t_0))^n (Z^-(\tau, t_0))^m d\tau, \end{aligned}$$

in which Z_0^+ and Z_0^- are the initial values.

Before proceeding further, we need to consider the asymptotic behaviors of $a(t)$ and $b(t)$ as $t \rightarrow \pm\infty$ which are crucial to get the integral equations for primary solutions. Observe that

$$\begin{aligned} \lim_{t \rightarrow -\infty} (a(t), b(t)) &= (x_1, y_1), \\ \lim_{t \rightarrow +\infty} (a(t), b(t)) &= (x_2, y_2), \end{aligned} \quad (27)$$

and due to the local linearization theory, there exists a small number $r > 0$ such that

$$\begin{aligned} \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} &= \begin{pmatrix} f_x(x_1, y_1) & f_y(x_1, y_1) \\ g_x(x_1, y_1) & g_y(x_1, y_1) \end{pmatrix} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} + \begin{pmatrix} f_1(x, y) \\ g_1(x, y) \end{pmatrix}, \left\| \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} \right\| < 2r; \\ \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} &= \begin{pmatrix} f_x(x_2, y_2) & f_y(x_2, y_2) \\ g_x(x_2, y_2) & g_y(x_2, y_2) \end{pmatrix} \begin{pmatrix} x - x_2 \\ y - y_2 \end{pmatrix} + \begin{pmatrix} f_2(x, y) \\ g_2(x, y) \end{pmatrix}, \left\| \begin{pmatrix} x - x_2 \\ y - y_2 \end{pmatrix} \right\| < 2r, \end{aligned} \quad (28)$$

where $f_1(x, y)$ and $g_1(x, y)$ are terms of second order and higher at $O_1(x_1, y_1)$ and $f_2(x, y)$ and $g_2(x, y)$ are terms of second order and higher at $O_2(x_2, y_2)$. We assume that $\lambda_1 < 0 < \mu_1$ are the eigenvalues of $O_1(x_1, y_1)$ and $\lambda_2 < 0 < \mu_2$ are the eigenvalues of $O_2(x_2, y_2)$. There exist two invertible matrices $\Gamma_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$ and $\Gamma_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$ such that

$$\Gamma_1 \begin{pmatrix} f_x(x_1, y_1) & f_y(x_1, y_1) \\ g_x(x_1, y_1) & g_y(x_1, y_1) \end{pmatrix} \Gamma_1^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix}, \quad (29)$$

$$\Gamma_2 \begin{pmatrix} f_x(x_2, y_2) & f_y(x_2, y_2) \\ g_x(x_2, y_2) & g_y(x_2, y_2) \end{pmatrix} \Gamma_2^{-1} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \mu_2 \end{pmatrix}. \quad (30)$$

In what follows, we introduce two coordinate transformations:

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \Gamma_1 \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix}, \quad (31)$$

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \Gamma_2 \begin{pmatrix} x - x_2 \\ y - y_2 \end{pmatrix}.$$

Unperturbed system (1) is transformed to

$$\frac{du_1}{dt} = \lambda_1 u_1 + \tilde{f}(u_1, v_1),$$

$$\frac{dv_1}{dt} = \mu_1 v_1 + \tilde{g}(u_1, v_1),$$

$$\frac{du_2}{dt} = \lambda_2 u_2 + \tilde{f}(u_2, v_2),$$

$$\frac{dv_2}{dt} = \mu_2 v_2 + \tilde{g}(u_2, v_2).$$

In the case $t \rightarrow -\infty$, our problem is about the unstable solution in a small neighborhood of O_1 . Let $T_0 > 0$ be

sufficiently large. For $t < -T_0$, we can find a near-identity coordinate transformation:

$$\begin{aligned} u_1 &= X + \sum_{j=2}^{+\infty} \phi_j^1(X, Y), \\ v_1 &= Y + \sum_{j=2}^{+\infty} \varphi_j^1(X, Y), \end{aligned} \quad (33)$$

$$\begin{aligned} \frac{dX}{dt} &= \lambda_1 X, \\ \frac{dY}{dt} &= \mu_1 Y. \end{aligned} \quad (34)$$

Assume that the power series in (33) are convergent on $\|(X, Y)\| < 2r$. It is not difficult to verify that

where $\phi_j^1(X, Y)$ and $\varphi_j^1(X, Y)$ are homogeneous polynomials of degree j in X, Y , such that the unperturbed system near O_1 is linearized to

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \begin{pmatrix} a(t) - x_1 \\ b(t) - y_1 \end{pmatrix} = \begin{pmatrix} re^{\mu_1(t-t_0)} + \sum_{j=2}^{+\infty} \phi_j^1(re^{\mu_1(t-t_0)}, 0) \\ \sum_{j=2}^{+\infty} \varphi_j^1(re^{\mu_1(t-t_0)}, 0) \end{pmatrix}, \quad (35)$$

where t_0 be such that $X(t_0) = r$ and $Y(t_0) = 0$. We solve (35) to obtain

$$\begin{aligned} a(t) - x_1 &= \frac{1}{\det(\Gamma_1)} \left[-\beta_1 re^{\mu_1(t-t_0)} + \sum_{j=2}^{+\infty} \left(\delta_1 \phi_j^1(re^{\mu_1(t-t_0)}, 0) - \beta_1 \varphi_j^1(re^{\mu_1(t-t_0)}, 0) \right) \right], \\ b(t) - y_1 &= \frac{1}{\det(\Gamma_1)} \left[\alpha_1 re^{\mu_1(t-t_0)} - \sum_{j=2}^{+\infty} \left(\gamma_1 \phi_j^1(re^{\mu_1(t-t_0)}, 0) - \alpha_1 \varphi_j^1(re^{\mu_1(t-t_0)}, 0) \right) \right]. \end{aligned} \quad (36)$$

On the other hand, let $t > T_0$. The problem is then about the stable solution in a small neighborhood of O_2 . We similarly have

$$\begin{aligned} a(t) - x_2 &= \frac{1}{\det(\Gamma_2)} \left[\delta_2 re^{\lambda_2(t-t_0)} + \sum_{j=2}^{+\infty} \left(\delta_2 \phi_j^2(re^{\lambda_2(t-t_0)}, 0) - \beta_2 \varphi_j^2(re^{\lambda_2(t-t_0)}, 0) \right) \right], \\ b(t) - y_2 &= \frac{1}{\det(\Gamma_2)} \left[-\gamma_2 re^{\lambda_2(t-t_0)} - \sum_{j=2}^{+\infty} \left(\gamma_2 \phi_j^2(re^{\lambda_2(t-t_0)}, 0) - \alpha_2 \varphi_j^2(re^{\lambda_2(t-t_0)}, 0) \right) \right], \end{aligned} \quad (37)$$

where ϕ_j^2 and φ_j^2 are homogeneous polynomials of degree j .

Lemma 1. For the above heteroclinic solution ℓ , we have

$$\begin{aligned} \lim_{t \rightarrow -\infty} A(t, 0) &> 0, \\ \lim_{t \rightarrow +\infty} A(t, 0) &< 0. \end{aligned} \quad (38)$$

Proof. We recall that

$$A(s, z) = -\frac{1}{z} \frac{(\dot{b} - z\ddot{a})f(a + z\dot{b}, b - z\dot{a}) - (\dot{a} + z\dot{b})g(a + z\dot{b}, b - z\dot{a})}{\dot{a}^2 + \dot{b}^2 + z(\ddot{a}\dot{b} - \ddot{b}\dot{a})}, \quad (39)$$

and let

Then,

$$\mathcal{A}(s, z) = -z \left[\dot{a}^2 + \dot{b}^2 + z(\ddot{a}\dot{b} - \ddot{b}\dot{a}) \right] A(s, z). \quad (40)$$

$$\begin{aligned} \mathcal{A}(s, z) &= (\dot{b} - z\ddot{a}) \left(f(a, b) + z\dot{b}f_x(a, b) - z\dot{a}f_y(a, b) + \mathcal{O}(z^2) \right) \\ &\quad - (\dot{a} + z\ddot{b}) \left(g(a, b) + z\dot{b}g_x(a, b) - z\dot{a}g_y(a, b) + \mathcal{O}(z^2) \right) \\ &= \dot{b}f(a, b) - \dot{a}'g(a, b) - z\ddot{a}f(a, b) - z\ddot{b}g(a, b) + z\dot{b}^2f_x(a, b) \\ &\quad - z\dot{a}\dot{b}f_y(a, b) - z\dot{a}\dot{b}g_x(a, b) + z\dot{a}^2g_y(a, b) + \mathcal{O}(z^2) \\ &= -z\ddot{a}f(a, b) - z\ddot{b}g(a, b) + z\dot{b}^2f_x(a, b) + z\dot{a}^2g_y(a, b) \\ &\quad - z\dot{a}\dot{b}f_y(a, b) - z\dot{a}\dot{b}g_x(a, b) + \mathcal{O}(z^2). \end{aligned} \quad (41)$$

This is to imply

$$\begin{aligned} A(s, 0) &= \frac{((d/ds)f(a, b))f(a, b) + ((d/ds)g(a, b))g(a, b) - g^2(a, b)f_x(a, b)}{f^2(a, b) + g^2(a, b)} \\ &\quad + \frac{-f^2(a, b)g_y(a, b) + f(a, b)g(a, b)[f_y(a, b) + g_x(a, b)]}{f^2(a, b) + g^2(a, b)} \\ &= \frac{(f_xf + f_yg)f + (g_xf + g_yg)g - g^2f_x - f^2g_y + fg(f_y + g_x)}{f^2 + g^2} \\ &= \frac{(f_x - g_y)(f^2 - g^2) + 2(f_y + g_x)fg}{f^2 + g^2}. \end{aligned} \quad (42)$$

As $s < -T_0$, we infer from (36) that

$$f(a, b) = -\frac{1}{\det(\Gamma_1)} \beta_1 \mu_1 r e^{\mu_1(s-t_0)} + \mathcal{O}(r^2),$$

$$g(a, b) = \frac{1}{\det(\Gamma_1)} \alpha_1 \mu_1 r e^{\mu_1(s-t_0)} + \mathcal{O}(r^2).$$

Accordingly, we have

(43)

$$\begin{aligned} \lim_{s \rightarrow -\infty} A(s, 0) &= \lim_{s \rightarrow -\infty} \frac{(f_x - g_y)(f^2 - g^2) + 2(f_y + g_x)fg}{f^2 + g^2}(a(s), b(s)) \\ &= \lim_{s \rightarrow -\infty} \left[\frac{(f_x(a(s), b(s)) - g_y(a(s), b(s)))(\beta_1^2 - \alpha_1^2) \mu_1^2 r^2 e^{2\mu_1(s-t_0)} + \mathcal{O}(r^3)}{(\alpha_1^2 + \beta_1^2) \mu_1^2 r^2 e^{2\mu_1(s-t_0)} + \mathcal{O}(e^{3\mu_1(s-t_0)})} \right. \\ &\quad \left. - \frac{2(f_y(a(s), b(s)) + g_x(a(s), b(s))) \alpha_1 \beta_1 \mu_1^2 r^2 e^{2\mu_1(s-t_0)} + \mathcal{O}(r^3)}{(\alpha_1^2 + \beta_1^2) \mu_1^2 r^2 e^{2\mu_1(s-t_0)} + \mathcal{O}(e^{3\mu_1(s-t_0)})} \right]. \end{aligned}$$
(44)

Recall (29), and we have

$$\begin{aligned} \lim_{s \rightarrow -\infty} \begin{pmatrix} f_x(a(s), b(s)) & f_y(a(s), b(s)) \\ g_x(a(s), b(s)) & g_y(a(s), b(s)) \end{pmatrix} &= \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix} \\ &= \frac{1}{\det(\Gamma_1)} \begin{pmatrix} \lambda_1 \alpha_1 \delta_1 - \mu_1 \beta_1 \gamma_1 & (\lambda_1 - \mu_1) \beta_1 \delta_1 \\ (\mu_1 - \lambda_1) \alpha_1 \gamma_1 & -\lambda_1 \beta_1 \gamma_1 + \mu_1 \alpha_1 \delta_1 \end{pmatrix}. \end{aligned}$$
(45)

Substituting (45) into (44) gives

$$\begin{aligned} \lim_{s \rightarrow -\infty} A(s, 0) &= \frac{(\lambda_1 - \mu_1)(\alpha_1 \delta_1 + \beta_1 \gamma_1)(\beta_1^2 - \alpha_1^2) - 2(\lambda_1 - \mu_1)(\beta_1 \delta_1 - \alpha_1 \gamma_1) \alpha_1 \beta_1}{\det(\Gamma_1)(\alpha_1^2 + \beta_1^2)} \\ &= \frac{(\lambda_1 - \mu_1)(\beta_1 \gamma_1 - \alpha_1 \delta_1)(\alpha_1^2 + \beta_1^2)}{\det(\Gamma_1)(\alpha_1^2 + \beta_1^2)} \\ &= \mu_1 - \lambda_1 > 0. \end{aligned}$$
(46)

As $s > T_0$, we infer from (37) that

$$f(a, b) = \frac{1}{\det(\Gamma_2)} \lambda_2 \delta_2 r e^{\lambda_2(s-t_0)} + \mathcal{O}(r^2),$$

$$g(a, b) = -\frac{1}{\det(\Gamma_2)} \lambda_2 \gamma_2 r e^{\lambda_2(s-t_0)} + \mathcal{O}(r^2).$$
(47)

After similar derivation, we have

$$\lim_{s \rightarrow +\infty} A(s, 0) = \lambda_2 - \mu_2 < 0.$$
(48)

This completes the proof. $\square$

By using Lemma 2, one can easily verify that

$$\lim_{t \rightarrow -\infty} e^{-\int_0^t A(\tau_1, 0) d\tau_1} = \lim_{t \rightarrow +\infty} e^{-\int_0^t A(\tau_1, 0) d\tau_1} = +\infty. \quad (49)$$

Observe that $Z^+(t, t_0)$ and $Z^-(t, t_0)$ should be bounded as $t \rightarrow \pm\infty$, and the initial values in (25) and (26) are claimed to be

$$Z_0^+ = \int_0^{+\infty} U(\tau, 0, \tau + t_0) e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau + \sum_{n+m=1}^{+\infty} \varepsilon^{n+m} \int_0^{+\infty} [A_{n,m}(\tau) Z^+(\tau, t_0) + U_{n,m}(\tau, t_0)] (S^+(\tau, t_0))^n (Z^+(\tau, t_0))^m e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau \quad (50)$$

$$Z_0^- = \int_0^{-\infty} U(\tau, 0, \tau + t_0) e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau + \sum_{n+m=1}^{+\infty} \varepsilon^{n+m} \int_0^{-\infty} [A_{n,m}(\tau) Z^-(\tau, t_0) + U_{n,m}(\tau, t_0)] (S^-(\tau, t_0))^n (Z^-(\tau, t_0))^m e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau. \quad (51)$$

Substituting (50) and (51) into (25) and (26), we have the integral equations of the primary stable solution

$$\begin{aligned} Z^+(t, t_0) &= \int_t^{+\infty} U(\tau, 0, \tau + t_0) e^{\int_t^\tau A(\tau_1, 0) d\tau_1} d\tau \\ &\quad + \sum_{n+m \geq 1} \varepsilon^{n+m} \int_t^{+\infty} [A_{n,m}(\tau) Z^+(\tau, t_0) + U_{n,m}(\tau, t_0)] (S^+(\tau, t_0))^n (Z^+(\tau, t_0))^m d\tau, \\ S^+(t, t_0) &= \int_0^t [V(\tau, 0, \tau + t_0) d\tau + \int_0^\tau B(\tau, 0) Z^+(\tau, t_0)] d\tau \\ &\quad + \sum_{n+m \geq 1} \varepsilon^{n+m} \int_0^t [B_{n,m}(\tau) Z^+(\tau, t_0) + V_{n,m}(\tau, t_0)] (S^+(\tau, t_0))^n (Z^+(\tau, t_0))^m d\tau, \end{aligned} \quad (52)$$

and the primary unstable solution

$$\begin{aligned} Z^-(t, t_0) &= \int_t^{-\infty} U(\tau, 0, \tau + t_0) e^{\int_t^\tau A(\tau_1, 0) d\tau_1} d\tau \\ &\quad + \sum_{n+m \geq 1} \varepsilon^{n+m} \int_t^{-\infty} [A_{n,m}(\tau) Z^-(\tau, t_0) + U_{n,m}(\tau, t_0)] (S^-(\tau, t_0))^n (Z^-(\tau, t_0))^m d\tau, \\ S^-(t, t_0) &= \int_0^t [V(\tau, 0, \tau + t_0) d\tau + \int_0^\tau B(\tau, 0) Z^-(\tau, t_0)] d\tau \\ &\quad + \sum_{n+m \geq 1} \varepsilon^{n+m} \int_0^t [B_{n,m}(\tau) Z^-(\tau, t_0) + V_{n,m}(\tau, t_0)] (S^-(\tau, t_0))^n (Z^-(\tau, t_0))^m d\tau. \end{aligned} \quad (53)$$

3.3. Derivation of High-Order Melnikov Functions. The properties of $(Z^\pm(t, t_0), S^\pm(t, t_0))$ here are coincident with the ones in [13]. We have the following power series:

$$\begin{aligned} Z^\pm(t, t_0) &= Z_0^\pm(t, t_0) + \sum_{k=1}^{\infty} \varepsilon^k Z_k^\pm(t, t_0), \\ S^\pm(t, t_0) &= S_0^\pm(t, t_0) + \sum_{k=1}^{\infty} \varepsilon^k S_k^\pm(t, t_0). \end{aligned} \quad (54)$$

In what follows, we substitute (54) into (52) and (53) to determine $Z_k^\pm(t, t_0)$ ($k = 0, 1, \dots$) and $S_k^\pm(t, t_0)$ ($k = 0, 1, \dots$) inductively. They are

$$\begin{aligned} Z_0^\pm(t, t_0) &= \int_0^{\pm\infty} U(\tau, 0, \tau + t_0) E(0, \tau) d\tau, \\ S_0^\pm(t, t_0) &= \int_0^{\pm t} [V(\tau, 0) + B(\tau, 0) Z_0^\pm(\tau)] d\tau, \end{aligned} \quad (55)$$

$$\begin{aligned} Z_k^\pm(t, t_0) &= \sum_{n+m+i+\rho=k} \int_t^{\pm\infty} [A_{n,m}(\tau) Z_i^\pm(\tau, t_0) + U_{n,m}(\tau, t_0)] C_{\rho,\pm}^{n,m}(\tau, t_0) e^{\int_t^\tau A(\tau_1, 0) d\tau_1} d\tau, \\ S_k^\pm(t, t_0) &= \sum_{n+m+i+\rho=k} \int_0^t [B_{n,m}(\tau) Z_i^\pm(\tau, t_0) + V_{n,m}(\tau, t_0)] C_{\rho,\pm}^{n,m}(\tau, t_0) d\tau, \end{aligned} \quad (56)$$

for $k = 1, 2, \dots$. Here, n, m, i , and ρ are nonnegative integers, and $n + m \geq 1$, $i \geq 0$, and $\rho \geq 0$. $C_{\rho,+}^{n,m}(t, t_0)$ and

$C_{\rho,-}^{n,m}(t, t_0)$ are coefficient functions of ε^ρ , which have the form of

$$C_{\rho,\pm}^{n,m}(t, t_0) = \sum_{\substack{\ell_0+\dots+\ell_\rho=n, l_0+\dots+l_\rho=m, \\ (\ell_1+l_1)+\dots+(\ell_\rho+l_\rho)=\rho}} K \prod_{j=0}^{\rho} (S_j^\pm(t, t_0))^{\ell_j} \cdot (Z_j^\pm(t, t_0))^{l_j}. \quad (57)$$

Here, K is a constant depending on the specific term. In the next section, these notations contribute to the proof of Theorem 2.

Recall the definition of $D(\varepsilon, t_0)$, and we have

$$D(\varepsilon, t_0) = \varepsilon(Z^+(0, t_0) - Z^-(0, t_0)) = \sum_{k=0}^{+\infty} \varepsilon^{k+1} (Z_k^+(0, t_0) - Z_k^-(0, t_0)). \quad (58)$$

Then,

$$M_0(t_0) = \int_{-\infty}^{+\infty} U(t, 0, t + t_0) e^{\int_0^t A(\omega, 0) d\omega} dt, \quad (59)$$

and for $k = 1, 2, \dots$, the $(k + 1)$ -th order Melnikov function is

$$M_k(t_0) = Z_k^+(0, t_0) - Z_k^-(0, t_0), \quad (60)$$

where $Z_k^+(0, t_0)$ and $Z_k^-(0, t_0)$ are the values by letting $t = 0$ in (56). Particularly, we have

$$\begin{aligned} Z_1^+(0, t_0) &= \sum_{n+m+i+\rho=1} \int_0^{+\infty} [A_{n,m}(\tau) Z_i^+(\tau, t_0) + U_{n,m}(\tau, t_0)] C_{\rho,+}^{n,m}(\tau, t_0) e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau, \\ Z_1^-(0, t_0) &= \sum_{n+m+i+\rho=1} \int_0^{-\infty} [A_{n,m}(\tau) Z_i^-(\tau, t_0) + U_{n,m}(\tau, t_0)] C_{\rho,-}^{n,m}(\tau, t_0) e^{\int_0^\tau A(\tau_1, 0) d\tau_1} d\tau. \end{aligned} \quad (61)$$

Up to now, the formula of $M_1(t_0)$ can be obtained.

4. Heteroclinic Persisting Perturbations

This section serves as the proof of Theorem 2. For autonomous perturbations, all the functions above having the variable t_0 are now dropped off. Before starting our proof, we need some lemmas.

Lemma 2. *Under the conditions of Theorem 2, we have the following:*

- (1) *If n is odd, then $A_{n,m}(t)$ and $U_{n,m}(t)$ are even in t , while $B_{n,m}(t)$ and $V_{n,m}(t)$ are odd in t*
- (2) *If n is even, then $A_{n,m}(t)$ and $U_{n,m}(t)$ are odd in t , while $B_{n,m}(t)$ and $V_{n,m}(t)$ are even in t .*

Proof. We will show the parity of $V_{n,m}(t)$, and the other three can be verified similarly. Recall that

$$V(s, z) = \frac{\dot{a}P(a + z\dot{b}, b - z\dot{a}) + \dot{b}Q(a + z\dot{b}, b - z\dot{a})}{\dot{a}^2 + \dot{b}^2 + z(\dot{a}\ddot{b} - \ddot{a}\dot{b})}. \quad (62)$$

Since $a(s)$ is an odd function while $b(s)$ is even, then $\dot{a}(s)$ and $\dot{b}(s)$ are even and $\ddot{a}(s)$ and $\ddot{b}(s)$ are odd. It follows that

$$V(-s, z) = V(s, z). \quad (63)$$

Denote $V_{n,m}(s, z) = \partial^{n+m} V(s, z) / (\partial s)^n (\partial z)^m$. We have

$$V_{1,0}(-s, z) = -\frac{\partial V(-s, z)}{\partial s} = -\frac{\partial V(s, z)}{\partial s} = -V_{1,0}(s, z). \quad (64)$$

Moreover, the parity in s will not change by taking a derivative with respect to z . Inductively, we obtain

$$V_{n,m}(s, z) = \begin{cases} -V_{n,m}(-s, z), & n \text{ is odd,} \\ V_{n,m}(-s, z), & n \text{ is even.} \end{cases} \quad (65)$$

Since $V_{n,m}(t)$ differs from $V_{n,m}(t, 0)$ by a constant factor, the proof is now completed by letting $(s, z) = (t, 0)$. $\square$

At the end of this section, let us prove Theorem 2 with the help of the preceding preparations.

Proof of Theorem 2. To start with, we prove, for all $t \geq 0$,

$$\begin{aligned} Z_k^+(t) &= Z_k^+(-t), \\ S_k^+(t) &= -S_k^+(-t), \quad k \in \mathbb{N}. \end{aligned} \quad (66)$$

By using mathematical induction, we first consider the case $k = 0$. Recall that

$$\begin{aligned} Z_0^-(t) &= \int_{-t}^{-\infty} U(\tau, 0) e^{\int_t^\tau A(\tau_1, 0) d\tau_1} d\tau, \\ S_0^-(t) &= \int_0^{-t} [V(\tau, 0) + B(\tau, 0)Z_0^-(\tau)] d\tau, \end{aligned} \quad (67)$$

where $U(\tau, 0)$ is an odd function in τ and $B(\tau, 0)$ and $V(\tau, 0)$ are even functions in τ . Moreover, we have

$$e^{\int_{-t}^{-\tau} A(\tau_1, 0) d\tau_1} = e^{\int_t^\tau A(-\tau_1, 0) d(-\tau_1)} = e^{\int_t^\tau A(\tau_1, 0) d\tau_1}. \quad (68)$$

By means of variable substitution, we have

$$\begin{aligned} Z_0^-(t) &= \int_t^{+\infty} U(-\tau, 0) e^{\int_t^{-\tau} A(\tau_1, 0) d\tau_1} d(-\tau) = \int_t^{+\infty} U(\tau, 0) e^{\int_t^\tau A(\tau_1, 0) d\tau_1} d\tau = Z_0^+(t), \\ S_0^-(t) &= \int_0^t [V(-\tau, 0) + B(-\tau, 0)Z_0^-(\tau)] d(-\tau) = -\int_0^t [V(\tau, 0) + B(\tau, 0)Z_0^+(\tau)] d\tau = -S_0^+(t). \end{aligned} \quad (69)$$

Now, we let L be a positive integer, and suppose that (66) holds for all nonnegative integer $k < L$. According to (56), we have

$$\begin{aligned} Z_L^-(-t) &= \sum_{n+m+i+\rho=L} \int_{-t}^{-\infty} [A_{n,m}(\tau)Z_i^-(\tau) + U_{n,m}(\tau)]C_{\rho,-}^{n,m}(\tau)e^{\int_{-t}^{\tau} A(\tau_1,0)d\tau_1}d\tau, \\ S_L^-(-t) &= \sum_{n+m+i+\rho=L} \int_0^{-t} [B_{n,m}(\tau)Z_i^-(\tau) + V_{n,m}(\tau)]C_{\rho,-}^{n,m}(\tau)d\tau. \end{aligned} \quad (70)$$

Notice that $C_{\rho,-}^{n,m}(\tau)$ is the counterpart of $C_{\rho,+}^{n,m}(\tau)$, changing “ $Z_j^+(\tau)$ ” and “ $S_j^+(\tau)$ ” to “ $Z_j^-(\tau)$ ” and “ $S_j^-(\tau)$,” respectively. Since $\rho < L$, we obtain

$$\begin{aligned} C_{\rho,-}^{n,m}(-\tau) &= \sum_{\substack{\ell_0+\dots+\ell_p=n, \ell_0+\dots+\ell_p=m, \\ (\ell_1+l_1)+\dots+(\ell_p+l_p)=\rho}} K \prod_{j=0}^{\rho} (S_j^-(\tau))^{\ell_j} \cdot (Z_j^-(\tau))^{l_j} \\ &= \sum_{\substack{\ell_0+\dots+\ell_p=n, \ell_0+\dots+\ell_p=m, \\ (\ell_1+l_1)+\dots+(\ell_p+l_p)=\rho}} K \prod_{j=0}^{\rho} (-S_j^+(\tau))^{\ell_j} \cdot (Z_j^+(\tau))^{l_j} \\ &= \sum_{\substack{\ell_0+\dots+\ell_p=n, \ell_0+\dots+\ell_p=m, \\ (\ell_1+l_1)+\dots+(\ell_p+l_p)=\rho}} (-1)^n K \prod_{j=0}^{\rho} (S_j^+(\tau))^{\ell_j} \cdot (Z_j^+(\tau))^{l_j} \\ &= (-1)^n C_{\rho,+}^{n,m}(\tau). \end{aligned} \quad (71)$$

That is,

$$C_{\rho,-}^{n,m}(-\tau) = \begin{cases} -C_{\rho,+}^{n,m}(\tau), & n \text{ is odd,} \\ C_{\rho,+}^{n,m}(\tau), & n \text{ is even.} \end{cases} \quad (72)$$

From Lemma 2, we have

$$\begin{aligned} &[A_{n,m}(-\tau)Z_i^-(\tau) + U_{n,m}(-\tau)]C_{\rho,-}^{n,m}(-\tau)e^{\int_{-t}^{-\tau} A(\tau_1,0)d\tau_1} \\ &= -[A_{n,m}(\tau)Z_i^+(\tau) + U_{n,m}(\tau)]C_{\rho,+}^{n,m}(\tau)e^{\int_t^{\tau} A(\tau_1,0)d\tau_1}, \\ &[B_{n,m}(-\tau)Z_i^-(\tau) + V_{n,m}(-\tau)]C_{\rho,-}^{n,m}(-\tau) = [B_{n,m}(\tau)Z_i^+(\tau) + V_{n,m}(\tau)]C_{\rho,+}^{n,m}(\tau). \end{aligned} \quad (73)$$

Finally, we obtain

$$\begin{aligned} Z_L^-(-t) &= Z_L^+(t), \\ S_L^-(-t) &= -S_L^+(t). \end{aligned} \quad (74)$$

The proof of (66) is completed.

Recall (60), and for autonomous perturbation, we have

$$M_k(P, Q) = Z_k^+(0) - Z_k^-(0) = 0, \quad k = 0, 1, 2, \dots \quad (75)$$

Due to (14), we thus arrive the conclusion that $D(\varepsilon, P, Q) = 0$ for all small ε . This means the stable manifold and unstable manifold coincide. As a result, there exists a heteroclinic orbit ℓ_ε which satisfies

$$\lim_{\varepsilon \rightarrow 0} \ell_\varepsilon = \ell. \quad (76)$$

Therefore, (P, Q) is the heteroclinic persisting perturbation of ℓ . $\square$

5. An Concrete Example: Van der Pol System

Let us consider the Van der Pol system on a torus:

$$\begin{aligned} \frac{dx}{dt} &= \sin y, \\ \frac{dy}{dt} &= -\sin x, \end{aligned} \quad (77)$$

where $(x, y) \in \mathbb{R}^2$. Without loss of generality, we fix a periodic region of the vector field in which we have four hyperbolic saddle fixed points:

$$O_1(-\pi, 0), O_2(0, \pi), O_3(\pi, 0), O_4(0, -\pi). \quad (78)$$

We write $f(x, y) = \sin y$ and $g(x, y) = -\sin x$, and the heteroclinic solutions are

$$\begin{aligned} \ell_{12}: & \left\{ \left(-2\arctan(e^{-t}), \pi - 2\arctan(e^{-t}) \right), \quad t \in \mathbb{R} \right\}, \\ \ell_{23}: & \left\{ \left(2\arctan(e^t), \pi - 2\arctan(e^t) \right), \quad t \in \mathbb{R} \right\}, \\ \ell_{34}: & \left\{ \left(2\arctan(e^{-t}), 2\arctan(e^{-t}) - \pi \right), \quad t \in \mathbb{R} \right\}, \\ \ell_{41}: & \left\{ \left(-2\arctan(e^t), 2\arctan(e^t) - \pi \right), \quad t \in \mathbb{R} \right\}. \end{aligned} \quad (79)$$

The vector field of (77) is displayed in Figure 1.

In order to obtain the heteroclinic persisting perturbations of these heteroclinic orbits, we need some rotation transformations of coordinates to satisfy the preconditions in Theorem 2.

5.1. Persistence of ℓ_{12} and ℓ_{34} . Introducing the change of variables

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (80)$$

we obtain

$$\begin{aligned} \frac{du_1}{dt} &= \frac{\sqrt{2}}{2} \left[\sin\left(\frac{u_1 + v_1}{\sqrt{2}}\right) - \sin\left(\frac{u_1 - v_1}{\sqrt{2}}\right) \right], \\ \frac{dv_1}{dt} &= -\frac{\sqrt{2}}{2} \left[\sin\left(\frac{u_1 + v_1}{\sqrt{2}}\right) + \sin\left(\frac{u_1 - v_1}{\sqrt{2}}\right) \right]. \end{aligned} \quad (81)$$

In such a case, the saddle points are transformed to

$$\begin{aligned} O'_1 & \left(-\frac{\sqrt{2}}{2}\pi, -\frac{\sqrt{2}}{2}\pi \right), \\ O'_2 & \left(-\frac{\sqrt{2}}{2}\pi, \frac{\sqrt{2}}{2}\pi \right), \\ O'_3 & \left(\frac{\sqrt{2}}{2}\pi, \frac{\sqrt{2}}{2}\pi \right), \\ O'_4 & \left(\frac{\sqrt{2}}{2}\pi, -\frac{\sqrt{2}}{2}\pi \right). \end{aligned} \quad (82)$$

Denote

FIGURE 1: Vector field of (77).

$$\begin{aligned} f_1(u_1, v_1) &= \frac{\sqrt{2}}{2} \left[\sin\left(\frac{u_1 + v_1}{\sqrt{2}}\right) - \sin\left(\frac{u_1 - v_1}{\sqrt{2}}\right) \right], \\ g_1(u_1, v_1) &= -\frac{\sqrt{2}}{2} \left[\sin\left(\frac{u_1 + v_1}{\sqrt{2}}\right) + \sin\left(\frac{u_1 - v_1}{\sqrt{2}}\right) \right]. \end{aligned} \quad (83)$$

The new vector field and heteroclinic orbits are plotted in Figure 2.

We note that the image of ℓ_{12} is

$$\ell'_{12}: \left\{ \left(u_1^{12}(t), v_1^{12}(t) \right) = \left(\frac{\sqrt{2}}{2} (4 \arctan(e^t) - \pi), \frac{\sqrt{2}}{2} \pi \right), \quad t \in \mathbb{R} \right\}. \quad (84)$$

One can easily verify that $u_1^{12}(t)$ is odd in t while $v_1^{12}(t)$ is even and $f_1(u_1, v_1)$ is even in u_1 while $g_1(u_1, v_1)$ is odd. For the convenience of derivation, we denote

$$\Lambda(\ell'_{12}) = \{(P_1, Q_1): P_1(u_1, v_1) \text{ is even in } u_1, Q_1(u_1, v_1) \text{ is odd in } u_1\}. \quad (85)$$

Applying Theorem 2, we have $\Lambda(\ell'_{12}) \subset HP(\ell'_{12})$. Now, we consider the following equations:

$$\begin{aligned} \frac{du_1}{dt} &= \frac{\sqrt{2}}{2} \left[\sin\left(\frac{u_1 + v_1}{\sqrt{2}}\right) - \sin\left(\frac{u_1 - v_1}{\sqrt{2}}\right) \right] + \varepsilon P_1(u_1, v_1), \\ \frac{dv_1}{dt} &= -\frac{\sqrt{2}}{2} \left[\sin\left(\frac{u_1 + v_1}{\sqrt{2}}\right) + \sin\left(\frac{u_1 - v_1}{\sqrt{2}}\right) \right] + \varepsilon Q_1(u_1, v_1). \end{aligned} \quad (86)$$

where $(P_1(u_1, v_1), Q_1(u_1, v_1)) \in \Lambda(\ell'_{12})$. Let us go back to the (x, y) -plane and replace $\varepsilon/\sqrt{2}$ by ε for easy computation. It follows that

FIGURE 2: Vector field of (81).

$$\begin{aligned}\frac{dx}{dt} &= \sin y + \varepsilon \left[P_1 \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right) + Q_1 \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right) \right], \\ \frac{dy}{dt} &= -\sin x + \varepsilon \left[-P_1 \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right) + Q_1 \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right) \right].\end{aligned}\quad (87)$$

We introduce $\Lambda(\ell_{12})$ to be the set obtained by $\Lambda(\ell'_{12})$ through the transformation of coordinates (80), which means given $(P(x, y), Q(x, y)) \in \Lambda(\ell_{12})$, there exists a $(P_1(u_1, v_1), Q_1(u_1, v_1)) \in \Lambda(\ell'_{12})$ such that

$$\begin{aligned}P(x, y) &= P_1 \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right) + Q_1 \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right), \\ Q(x, y) &= -P_1 \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right) + Q_1 \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right).\end{aligned}\quad (88)$$

Thus, we have

$$\Lambda(\ell_{12}) = \{(P, Q) : \exists (P_1, Q_1) \in \Lambda(\ell'_{12}) \text{ s.t. (88) holds.}\}.\quad (89)$$

Obviously, $HP(\ell_{12}) = HP(\ell_{34})$. We also have $\Lambda(\ell_{12}) \subset HP(\ell_{34})$. In particular, we set

$$P_1(u_1, v_1) = \sin \left(\frac{u_1 + v_1}{\sqrt{2}} \right) \sin \left(\frac{u_1 - v_1}{\sqrt{2}} \right),\quad (90)$$

$$Q_1(u_1, v_1) = 0.$$

This implies

$$\begin{aligned}P(x, y) &= -\sin x \sin y, \\ Q(x, y) &= \sin x \sin y.\end{aligned}\quad (91)$$

We consider the following system:

$$\begin{aligned}\frac{dx}{dt} &= \sin y - \varepsilon \sin x \sin y, \\ \frac{dy}{dt} &= -\sin x + \varepsilon \sin x \sin y.\end{aligned}\quad (92)$$

It follows that there exist two heteroclinic solutions which satisfy equality (10) in Definition 1. The perturbed vector field and the two heteroclinic orbits ℓ_{12}^ε and ℓ_{34}^ε are plotted in Figure 3. Figure 4 shows the other two are broken.

5.2. Persistence of ℓ_{23} and ℓ_{41} . For the heteroclinic orbits ℓ_{23} and ℓ_{41} , we introduce another transformation of coordinates:

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.\quad (93)$$

It follows that

$$\frac{du_2}{dt} = \frac{\sqrt{2}}{2} \left[\sin \left(\frac{u_2 + v_2}{\sqrt{2}} \right) - \sin \left(\frac{u_2 - v_2}{\sqrt{2}} \right) \right],\quad (94)$$

$$\frac{dv_2}{dt} = -\frac{\sqrt{2}}{2} \left[\sin \left(\frac{u_2 + v_2}{\sqrt{2}} \right) + \sin \left(\frac{u_2 - v_2}{\sqrt{2}} \right) \right].$$

Denote

$$f_2(u_2, v_2) = \frac{\sqrt{2}}{2} \left[\sin \left(\frac{u_2 + v_2}{\sqrt{2}} \right) - \sin \left(\frac{u_2 - v_2}{\sqrt{2}} \right) \right],\quad (95)$$

$$g_2(u_2, v_2) = -\frac{\sqrt{2}}{2} \left[\sin \left(\frac{u_2 + v_2}{\sqrt{2}} \right) + \sin \left(\frac{u_2 - v_2}{\sqrt{2}} \right) \right].$$

Similarly, the image of ℓ_{23} is transformed to

$$\ell_{23}'' : \left\{ (u_2^{23}(t), v_2^{23}(t)) = \left(\frac{\sqrt{2}}{2} (\pi - 4 \arctan(e^{-t})), \frac{\sqrt{2}}{2} \pi \right), \quad t \in \mathbb{R} \right\}.\quad (96)$$

The new saddle points of (94) are

$$\begin{aligned}O_1'' &\left(-\frac{\sqrt{2}}{2} \pi, \frac{\sqrt{2}}{2} \pi \right), \\ O_2'' &\left(\frac{\sqrt{2}}{2} \pi, \frac{\sqrt{2}}{2} \pi \right), \\ O_3'' &\left(\frac{\sqrt{2}}{2} \pi, -\frac{\sqrt{2}}{2} \pi \right), \\ O_4'' &\left(-\frac{\sqrt{2}}{2} \pi, -\frac{\sqrt{2}}{2} \pi \right).\end{aligned}\quad (97)$$

One can easily verify that $u_1^{23}(t)$ is odd in t while $v_1^{23}(t)$ is even and $f_2(u_2, v_2)$ is even in u_2 while $g_2(u_2, v_2)$ is odd. Denote

FIGURE 3: Vector field of $(92)_{\varepsilon=0.5}$ and ℓ_{12}^ε and ℓ_{34}^ε . FIGURE 4: Vector field of $(92)_{\varepsilon=0.5}$ and ℓ_{23}^ε and ℓ_{41}^ε .

$$\Lambda(\ell_{23}'') = \{(P_2, Q_2): P_2(u_2, v_2) \text{ is even in } u_2, Q_2(u_2, v_2) \text{ is odd in } u_2\} \quad (98)$$

It then follows from Theorem 2 that $\Lambda(\ell_{23}'') \subset HP(\ell_{23}'')$. We take $(P_2(u, v), Q_2(u, v)) \in \Lambda(\ell_{23}'')$, and consider the following perturbed system:

$$\begin{aligned} \frac{du_2}{dt} &= \frac{\sqrt{2}}{2} \left[\sin\left(\frac{u_2 + v_2}{\sqrt{2}}\right) - \sin\left(\frac{u_2 - v_2}{\sqrt{2}}\right) \right] + \varepsilon P_2(u_2, v_2), \\ \frac{dv_2}{dt} &= -\frac{\sqrt{2}}{2} \left[\sin\left(\frac{u_2 + v_2}{\sqrt{2}}\right) + \sin\left(\frac{u_2 - v_2}{\sqrt{2}}\right) \right] + \varepsilon Q_2(u_2, v_2). \end{aligned} \quad (99)$$

Back to (x, y) -plane, we have

$$\begin{aligned} \frac{dx}{dt} &= \sin y + \varepsilon \left[P_2\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) - Q_2\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) \right], \\ \frac{dy}{dt} &= -\sin x + \varepsilon \left[P_2\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) + Q_2\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) \right]. \end{aligned} \quad (100)$$

Similarly, we obtain the set $\Lambda(\ell_{23})$ through transformation (93). Then, for any $(P(x, y), Q(x, y)) \in \Lambda(\ell_{23})$, we obtain

$$\begin{aligned} P(x, y) &= P_2\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) - Q_2\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right), \\ Q(x, y) &= P_2\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) + Q_2\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right). \end{aligned} \quad (101)$$

Consequently, we have

$$\Lambda(\ell_{23}) = \{(P, Q): \exists (P_2, Q_2) \in CP(\ell_{23}'') \text{ s.t. (101) holds.}\}. \quad (102)$$

Since $HP(\ell_{23}) = HP(\ell_{41})$, we also have $\Lambda(\ell_{23}) \subset HP(\ell_{41})$.

We take

$$P_2(u_2, v_2) = \sin\left(\frac{u_2 + v_2}{\sqrt{2}}\right) \sin\left(\frac{u_2 - v_2}{\sqrt{2}}\right), \quad (103)$$

$$Q_2(u_2, v_2) = 0.$$

It follows that

$$\begin{aligned} P(x, y) &= \sin x \sin y, \\ Q(x, y) &= \sin x \sin y. \end{aligned} \quad (104)$$

Hence, we get the following perturbed system:

$$\begin{aligned} \frac{dx}{dt} &= \sin y + \varepsilon \sin x \sin y, \\ \frac{dy}{dt} &= -\sin x + \varepsilon \sin x \sin y. \end{aligned} \quad (105)$$

Figures 5 and 6 are the plots of the perturbed vector field of (105) and the two heteroclinic orbits ℓ_{23}^ε and ℓ_{41}^ε . The other two are broken.

FIGURE 5: Vector field of $(105)_{\varepsilon=0.5}$ and ℓ_{23}^e and ℓ_{41}^e . FIGURE 6: Vector field of $(105)_{\varepsilon=0.5}$ and ℓ_{12}^e and ℓ_{34}^e .

5.3. Persistence of the Heteroclinic 4-Cycle. We note that in unperturbed system (77), $\ell_{12} \cup \ell_{23} \cup \ell_{34} \cup \ell_{41} \cup \{O_1, O_2, O_3, O_4\}$ forms a heteroclinic 4-cycle. It is natural to ask when the heteroclinic 4-cycle remains unbroken. We rewrite $\Lambda(\ell_{12})$ and $\Lambda(\ell_{23})$ as

$$\begin{aligned}\Lambda(\ell_{12}) &= \{(P, Q): P(x, y) = P_1(u_1, v_1) + Q_1(u_1, v_1), \\ Q(x, y) &= -P_1(u_1, v_1) + Q_1(u_1, v_1), \quad (P_1, Q_1) \in \Lambda(\ell'_{12})\}, \\ \Lambda(\ell_{23}) &= \{(P, Q): P(x, y) = P_2(u_2, v_2) - Q_2(u_2, v_2), \\ Q(x, y) &= P_2(u_2, v_2) + Q_2(u_2, v_2), \quad (P_2, Q_2) \in \Lambda(\ell'_{23})\}.\end{aligned}\quad (106)$$

From (80) and (93), we have the relation between phase variables (u_1, v_1) and (u_2, v_2) as

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ u_2 \end{pmatrix}.\quad (107)$$

Thus, we have

$$\begin{aligned}\Lambda(\ell_{12}) &= \{(P, Q): P(x, y) = P_1(-v_2, u_2) + Q_1(-v_2, u_2), \\ Q(x, y) &= -P_1(-v_2, u_2) + Q_1(-v_2, u_2), \quad (P_1, Q_1) \in \Lambda(\ell'_{12})\} \\ &= \{(P, Q): P(x, y) = P_1(v_2, u_2) - Q_1(v_2, u_2), \\ Q(x, y) &= -P_1(v_2, u_2) - Q_1(v_2, u_2), \quad (P_1, Q_1) \in \Lambda(\ell'_{12})\}.\end{aligned}\quad (108)$$

Our goal is to figure out the elements in $CP(\ell_{12}) \cap CP(\ell_{23})$ and so we let

$$\begin{cases} P_1(v_2, u_2) - Q_1(v_2, u_2) = P_2(u_2, v_2) - Q_2(u_2, v_2), \\ -P_1(v_2, u_2) - Q_1(v_2, u_2) = P_2(u_2, v_2) + Q_2(u_2, v_2). \end{cases}\quad (109)$$

It then follows that

$$\begin{cases} P_1(v_2, u_2) = -Q_2(u_2, v_2), \\ Q_1(v_2, u_2) = -P_2(u_2, v_2). \end{cases}\quad (110)$$

Depending on the parities of P_1, P_2, Q_1 , and Q_2 with respect to the first variable, we can use (110) to obtain their parities with respect to the second variable. Let

$$\begin{aligned}\Omega &= \{(P_2, Q_2): P_2(u_2, v_2) \text{ is even in } u_2 \text{ and odd in } v_2, \\ Q_2(u_2, v_2) &\text{ is odd in } u_2 \text{ and even in } v_2\}.\end{aligned}\quad (111)$$

We finally get

$$\begin{aligned}\Lambda(\ell_{12}) \cap \Lambda(\ell_{23}) &= \{(P, Q): P(x, y) = P_2(u_2, v_2) - Q_2(u_2, v_2), \\ Q(x, y) &= P_2(u_2, v_2) + Q_2(u_2, v_2), \quad (P_2, Q_2) \in \Omega\},\end{aligned}\quad (112)$$

in which $u_2 = \sqrt{2}/2(x + y)$ and $v_2 = -\sqrt{2}/2(x - y)$.

In order to visualize this result, we consider the following perturbations:

FIGURE 7: Vector field of $(105)_{\varepsilon=0.5}$.

$$P(x, y) = \sin x \sin y \left[\sin\left(\frac{x-y}{2}\right) + \sin\left(\frac{x+y}{2}\right) \right], \quad (113)$$

$$Q(x, y) = \sin x \sin y \left[\sin\left(\frac{x-y}{2}\right) - \sin\left(\frac{x+y}{2}\right) \right].$$

Adding them to system (77) gives

$$\begin{aligned} \frac{dx}{dt} &= \sin y + \varepsilon \sin x \sin y \left[\sin\left(\frac{x-y}{2}\right) + \sin\left(\frac{x+y}{2}\right) \right], \\ \frac{dy}{dt} &= -\sin x + \varepsilon \sin x \sin y \left[\sin\left(\frac{x-y}{2}\right) - \sin\left(\frac{x+y}{2}\right) \right]. \end{aligned} \quad (114)$$

Since $(P(x, y), Q(x, y)) \in \Lambda(\ell_{12}) \cap \Lambda(\ell_{23})$, all the four heteroclinic orbits of (114) remain unbroken (see Figure 7).

Data Availability

All data included in this study are available upon request to the author.

Conflicts of Interest

The author declares no conflicts of interest.

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