Research Article

On Some Types of Covering-Based $(\mathcal{J}, \mathcal{T})$ -Fuzzy Rough Sets and Their Applications

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The notions of the fuzzy β -minimal and maximal descriptions were established by Yang et al. (Yang and Hu, 2016 and 2019). Recently, Zhang et al. (Zhang et al. 2019) presented the fuzzy covering via $(\mathcal{J}, \mathcal{T})$ -fuzzy rough set model (FC $\mathcal{J}\mathcal{T}$ FRS), and Jiang et al. (Jiang et al., in 2019) introduced the covering through variable precision $(\mathcal{J}, \mathcal{T})$ -fuzzy rough sets (CVP $\mathcal{J}\mathcal{T}$ FRS). To generalize these models in (Jiang et al., 2019 and Zhang et al. 2019), that is, to improve the lower approximation and reduce the upper approximation, the present paper constructs eight novel models of an FC $\mathcal{J}\mathcal{T}$ FRS based on fuzzy β -minimal (maximal) descriptions. Characterizations of these models are discussed. Further, eight types of CVP $\mathcal{J}\mathcal{T}$ FRS are introduced, and we investigate the related properties. Relationships among these models are also proposed. Finally, we illustrate the above study with a numerical example that also describes its practical application.

1. Introduction

In the 8th decade of the 20th century, the researchers found a new style of sets called rough set theory under the scientist Pawlak [1, 2]. This new theory helped many more authors to generalize and improve the studies in many areas such as engineering, economics, medicine, biology, speech recognition, chemistry, market research, data analysis, banking, materials science, networking, data mining, linguistics, variable precision rough sets, and other areas [3–15].

A widely studied extension of RST is a covering through rough sets (CRSs).

Pomykala [16, 17] generated a dual approximation for two pairs of operators. The definitions of granularity and neighborhood presented further insights in the field of

approximation operators by Yao [18]. Using the assumption of incomplete knowledge, Bonikowski et al. [19] introduced a model of CRS, which depends on the of minimal description concept. Couso and Dubois [20] deliberated both pairs of operators as well. Other CRS models and relationships among them are the subjects of Zhu [21] and of Zhu and Wang [22–24]. The additional models of CRS were developed by Tsang et al. [25] and Xu and Zhang [26]. Moreover, Liu and Sai [27] compared the CRS model of Zhu and the CRS models of Xu and Zhang. Based on the concepts of neighborhood and complementary neighborhood, Ma [28] established some neighborhood-related forms covering rough sets in 2012. On the other direction, Dubois et al. [29] defined the fuzzy rough set (FRS) and rough fuzzy set (RFS) which assist the authors to a built a new way of CRS called fuzzy covering rough sets (FCRSs). There are many scholars

working on this idea such as Atef and El Atik [30] presented some types of covering-based multigranulation fuzzy rough sets, Ma et al. [31] proposed new fuzzy rough coverings models using fuzzy α -neighborhood and presented some of its application, and the multigranulation fuzzy rough sets notion over two universes with its decision-making application studied by [32] and Zhan et al. [33] established the concept of a covering-based multigranulation fuzzy rough sets. In this direction, the meaning of fuzzy β -neighborhood was discovered by Ma [34] to create a fuzzy β -covering approximation space (FCAS). Then, Yang and Hu [35–37] investigated the notions of a fuzzy β -complementary neighborhood, a fuzzy β -minimal description, and a fuzzy β -maximal description to generate new models of FCAS. Deer et al. [38] studied fuzzy neighborhoods based on fuzzy coverings. Further, Atef and Azzam [39] and Jiang et al. [40] introduced covering fuzzy rough sets through variable precision. In [41–43], Hu and Wong have constructed generalized interval-valued fuzzy variable precision rough sets and generalized interval-valued fuzzy rough sets by using the operators of fuzzy logical. Meanwhile, L-fuzzy rough sets have been studied by the constructive method and axiomatic method [44–46]. Yeung et al. [47] have proposed a number of definitions for upper and lower fuzzy sets approximation operators by the means of arbitrary fuzzy relations and studied the relationship among them from the viewpoint of a constructive approach. In an axiomatic approach, they have characterized different classes of generalized upper and lower approximation operators of fuzzy sets by different sets of axioms. After that, Zhang et al. [48] set up the FC $\mathcal{J}$ TFRS paradigms in 2019. In addition, Jiang et al. [49] established CVP $\mathcal{J}$ TFRS with its applications to multiattribute-based decision-making. Furthermore, between 2019 and 2021, Zhan et al. introduced some types of decision-making problems [33, 50]. The later extension of the former model takes advantage of the generality provided by a fuzzy logical implication $\mathcal{J}$ and a t -norm $\mathcal{T}$. Both works provided applications of the models to multiattribute group decision-making.

The emphasis of the FC $\mathcal{J}$ TFRS models is on granularity, which here is understood in a fuzzy sense. Other settings require alternative models. For example, the Takagi–Sugeno fuzzy models [51] show a higher ability to capture nonlinear behavior. While the former insists on approximation operators, be they defined by equivalence relations, (fuzzy) coverings, or otherwise, and the latter relies on fuzzy rules. For this reason, the Takagi–Sugeno fuzzy models have also become popular in many other applied fields [52–56].

Based on these recent developments and to extend and generalize the last studies by [48, 49] (i.e., to raise the lower approximation and lowering the previous works' upper

approximation), in this paper, we contribute to generating eight types of FC $\mathcal{J}$ TFRS models through the descriptions of fuzzy β -minimal and fuzzy β -maximal notions and their complementary.

Contrary to the inspiring cases of Zhan et al. [33, 50] and Ma et al. [57], our attention is restricted to one single source of granularity although we keep the general setting that allows for a t -norm and a fuzzy logical implication. The properties of these models as well as their relationships are introduced. Further, eight types of FC $\mathcal{J}$ TFRSs and their related properties are considered. An application to a real issue clarifies the capacity to assist the practitioner for making the decisions.

The structure and body of the paper are as follows. Our next section contains various technical preliminary concepts. Then, Section 3 describes the eight paradigms of FC $\mathcal{J}$ TFRSs that we propose for investigation. Section 4 introduces the corresponding eight types of CVP $\mathcal{J}$ TFRSs. The relations between these different models are set in Section 5. Section 6 gives the applicable example of this theoretical study. Section 7 puts an end to this paper.

2. Preliminary Concepts

Throughout this section, we introduce several fundamental notions of fuzzy logical operators, CFRSs, FC $\mathcal{J}$ TFRSs, and CVP $\mathcal{J}$ TFRSs. In this paper, $\mathcal{F}(\Omega)$ denotes the set of all fuzzy sets on a crisp set Ω . Also, we use $\mathcal{T}_{\mathcal{M}}(x, y) = x \wedge y$ and $\mathcal{J}_{\mathcal{M}}(x, y) = (1 - x) \vee y$. Further, we say that $\mathcal{N}$ is involutive when $\mathcal{N}(\mathcal{N}(x)) = x$ for every $x \in [0, 1]$. The standard negator operator is defined as $\mathcal{N}(x) = 1 - x$, for any $x \in [0, 1]$. For more data, see [58].

Definition 1 (see [59, 60]). Presume that Ω be a universe set. Then, $\tilde{\Gamma} = \{\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_m\}$, where $\tilde{C}_i \in \mathcal{F}(\Omega)$ ($i = 1, 2, \dots, m$), is called a fuzzy covering (FC) of Ω if $(\cup_{i=1}^m \tilde{C}_i)(x) = 1, \forall x \in \Omega$.

As a generalization of FC, Ma [34] defined a fuzzy β -covering by replacing 1 with a threshold β ($0 < \beta \leq 1$). So, $\tilde{\Gamma}$ is said to be a fuzzy β -covering of Ω if $((\cup_{i=1}^m \tilde{C}_i)(x) \geq \beta, \forall x \in \Omega$. Besides, $(\Omega, \tilde{\Gamma})$ is an FCAS.

FCASs produce a number of theoretical elements for discussion.

Definition 2 (see [34–37]). Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for $\beta \in (0, 1]$. For every $x \in \Omega$, the fuzzy β -neighborhood consecutively, fuzzy complementary β -neighborhood, fuzzy β -minimal description, and fuzzy β -maximal description of x are defined as follows:

$$\begin{aligned}
\widetilde{N}_x^\beta &= \cap \{ \widetilde{C}_i \in \widetilde{\Gamma}: \widetilde{C}_i \geq \beta \}, \\
\widetilde{M}_x^\beta(y) &= \widetilde{N}_y^\beta(x), \\
\widetilde{\mathcal{M}d}_x^\beta &= \{ C \in \widetilde{\Gamma}: (C(x) \geq \beta) \wedge (\forall D \in C \wedge D(x) \geq \beta \wedge D \subseteq C \longrightarrow D = C) \}, \\
\widetilde{\mathcal{M}D}_x^\beta &= \{ C \in \widetilde{\Gamma}: (C(x) \geq \beta) \wedge (\forall D \in C \wedge D(x) \geq \beta \wedge D \supseteq C \longrightarrow D = C) \}.
\end{aligned} \tag{1}$$

Definition 3 (see [48]). Presume that $(\Omega, \widetilde{\Gamma})$ be an FCAS for $\beta \in (0, 1]$. For any $x \in \Omega$ and $\widehat{X} \in \mathcal{F}(\Omega)$, define the first type of $(\mathcal{F}, \mathcal{T})$ -fuzzy lower approximation (for short 1-ITFLA)

(resp., 2-ITFLA, 3-ITFLA, and 4-ITFLA) and the first type of $(\mathcal{F}, \mathcal{T})$ -fuzzy upper approximation (for short 1-ITFUA) (resp., 2-ITFUA, 3-ITFUA, and 4-ITFUA) of x as follows:

$$\begin{aligned}
\widetilde{C}_1^-(\widehat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F}[\widetilde{N}_x^\beta(y), \widehat{X}(y)], \\
\widetilde{C}_1^+(\widehat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{T}[\widetilde{N}_x^\beta(y), \widehat{X}(y)], \\
\widetilde{C}_2^-(\widehat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F}[\widetilde{M}_x^\beta(y), \widehat{X}(y)], \\
\widetilde{C}_2^+(\widehat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{T}[\widetilde{M}_x^\beta(y), \widehat{X}(y)], \\
\widetilde{C}_3^-(\widehat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F}[\widetilde{N}_x^\beta(y) \wedge \widetilde{M}_x^\beta(y), \widehat{X}(y)], \\
\widetilde{C}_3^+(\widehat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{T}[\widetilde{N}_x^\beta(y) \wedge \widetilde{M}_x^\beta(y), \widehat{X}(y)], \\
\widetilde{C}_4^-(\widehat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F}[\widetilde{N}_x^\beta(y) \vee \widetilde{M}_x^\beta(y), \widehat{X}(y)], \\
\widetilde{C}_4^+(\widehat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{T}[\widetilde{N}_x^\beta(y) \vee \widetilde{M}_x^\beta(y), \widehat{X}(y)].
\end{aligned} \tag{2}$$

If $\widetilde{C}_1^-(\widehat{X})$ (resp., $\widetilde{C}_2^-(\widehat{X})$, $\widetilde{C}_3^-(\widehat{X})$, and $\widetilde{C}_4^-(\widehat{X}) \neq \widetilde{C}_1^+(\widehat{X})$ (resp., $\widetilde{C}_2^+(\widehat{X})$, $\widetilde{C}_3^+(\widehat{X})$, and $\widetilde{C}_4^+(\widehat{X})$), then we call $\widehat{X}$ a 1-FC $\mathcal{F}$ TFRS (resp., 2-FC $\mathcal{F}$ TFRS, 3-FC $\mathcal{F}$ TFRS, and 4-FC $\mathcal{F}$ TFRS).

Definition 4 (see [49]). Presume that $(\Omega, \widetilde{\Gamma})$ be an FCAS for $\beta \in (0, 1]$. Then, for every $\widehat{X} \in \mathcal{F}(\Omega)$, a parameter $\gamma \in [0, 1]$.

For any $x \in \Omega$, define the first type of the variable precision $(\mathcal{F}, \mathcal{T})$ -fuzzy lower approximation (for short I-VPITFLA) (resp., II-VPITFLA, III-VPITFLA, and IV-VPITFLA) and the first type of the variable precision $(\mathcal{F}, \mathcal{T})$ -fuzzy upper approximation (for short I-VPITFUA) (resp., II-VPITFUA, III-VPITFUA, and IV-VPITFUA) as follows:

$$\begin{aligned}
{}_I\mathcal{L}_{vp}^\mathcal{F}(\widehat{X})(x) &= \left(\bigwedge_{\widehat{X}(y) \leq \gamma} \mathcal{F}(\widetilde{N}_x^\beta(y), \gamma) \right) \wedge \left(\bigwedge_{\widehat{X}(y) > \gamma} \mathcal{F}(\widetilde{N}_x^\beta(y), \widehat{X}(y)) \right), \\
{}_I\mathcal{U}_{vp}^\mathcal{T}(\widehat{X})(x) &= \left(\bigvee_{\widehat{X}(y) \geq 1-\gamma} \mathcal{T}(\widetilde{N}_x^\beta(y), 1-\gamma) \right) \vee \left(\bigvee_{\widehat{X}(y) < 1-\gamma} \mathcal{T}(\widetilde{N}_x^\beta(y), \widehat{X}(y)) \right), \\
{}_{II}\mathcal{L}_{vp}^\mathcal{F}(\widehat{X})(x) &= \left(\bigwedge_{\widehat{X}(y) \leq \gamma} \mathcal{F}(\widetilde{M}_x^\beta(y), \gamma) \right) \wedge \left(\bigwedge_{\widehat{X}(y) > \gamma} \mathcal{F}(\widetilde{M}_x^\beta(y), \widehat{X}(y)) \right), \\
{}_{II}\mathcal{U}_{vp}^\mathcal{T}(\widehat{X})(x) &= \left(\bigvee_{\widehat{X}(y) \geq 1-\gamma} \mathcal{T}(\widetilde{M}_x^\beta(y), 1-\gamma) \right) \vee \left(\bigvee_{\widehat{X}(y) < 1-\gamma} \mathcal{T}(\widetilde{M}_x^\beta(y), \widehat{X}(y)) \right),
\end{aligned}$$

$$\begin{aligned}
\text{III } \mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}([\tilde{N}_x^\beta(y) \wedge \tilde{M}_x^\beta(y)], \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}([\tilde{N}_x^\beta(y) \wedge \tilde{M}_x^\beta(y)], \hat{X}(y)) \right), \\
\text{III } \mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{F}([\tilde{N}_x^\beta(y) \wedge \tilde{M}_x^\beta(y)], 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{F}([\tilde{N}_x^\beta(y) \wedge \tilde{M}_x^\beta(y)], \hat{X}(y)) \right), \\
\text{IV } \mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}([\tilde{N}_x^\beta(y) \vee \tilde{M}_x^\beta(y)], \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}([\tilde{N}_x^\beta(y) \vee \tilde{M}_x^\beta(y)], \hat{X}(y)) \right), \\
\text{IV } \mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{F}([\tilde{N}_x^\beta(y) \vee \tilde{M}_x^\beta(y)], 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{F}([\tilde{N}_x^\beta(y) \vee \tilde{M}_x^\beta(y)], \hat{X}(y)) \right).
\end{aligned} \tag{3}$$

If ${}_1\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$ (resp., ${}_2\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_3\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$, and ${}_4\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \neq {}_1\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$ (resp., ${}_2\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_3\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$, and ${}_4\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$), then $\hat{X}$ is called a I-CVP $\mathcal{F}$ TFRS (resp., II-CVP $\mathcal{F}$ TFRS, III-CVP $\mathcal{F}$ TFRS, and IV-CVP $\mathcal{F}$ TFRS); otherwise, it is variable precision fuzzy definable.

3. Eight Types of Covering-Based ($\mathcal{F}, \mathcal{T}$)-Fuzzy Rough Sets

This section contains the definitions of eight types of FC $\mathcal{F}$ TFRS models. Also, the related characteristics are discussed. The study by Zhan et al. [50] is a direct inspiration by their utilization of a fuzzy logical implication and a

t -norm in a similar setting. However, we refrain from using various sources of granularity, which opens the door for additional modelizations.

3.1. Four Kinds of FC $\mathcal{F}$ TFRS via Fuzzy β -Minimal Description

Definition 5. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. For any $x \in \Omega$, the ${}^1\text{ITFLA}$ (resp., ${}^2\text{ITFLA}$, ${}^3\text{ITFLA}$, and ${}^4\text{ITFLA}$) and the ${}^1\text{ITFUA}$ (resp., ${}^2\text{ITFUA}$, ${}^3\text{ITFUA}$, and ${}^4\text{ITFUA}$) are defined as in the following way:

$$\begin{aligned}
{}_1\mathcal{L}^{\mathcal{F}}(\hat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F}\{\sqrt{\mathcal{M}}d_x^\beta(y), \hat{X}(y)\}, \\
{}_1\mathcal{U}^{\mathcal{F}}(\hat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{F}\{\sqrt{\mathcal{M}}d_x^\beta(y), \hat{X}(y)\}, \\
{}_2\mathcal{L}^{\mathcal{F}}(\hat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F}\{d\sqrt{\mathcal{M}}_x^\beta(y), \hat{X}(y)\}, \\
{}_2\mathcal{U}^{\mathcal{F}}(\hat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{F}\{d\sqrt{\mathcal{M}}_x^\beta(y), \hat{X}(y)\}, \\
{}_3\mathcal{L}^{\mathcal{F}}(\hat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F}\{\sqrt{\mathcal{M}}d_x^\beta(y) \cap d\sqrt{\mathcal{M}}_x^\beta(y), \hat{X}(y)\}, \\
{}_3\mathcal{U}^{\mathcal{F}}(\hat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{F}\{\sqrt{\mathcal{M}}d_x^\beta(y) \cap d\sqrt{\mathcal{M}}_x^\beta(y), \hat{X}(y)\}, \\
{}_4\mathcal{L}^{\mathcal{F}}(\hat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F}\{\sqrt{\mathcal{M}}d_x^\beta(y) \cup d\sqrt{\mathcal{M}}_x^\beta(y), \hat{X}(y)\}, \\
{}_4\mathcal{U}^{\mathcal{F}}(\hat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{F}\{\sqrt{\mathcal{M}}d_x^\beta(y) \cup d\sqrt{\mathcal{M}}_x^\beta(y), \hat{X}(y)\}.
\end{aligned} \tag{4}$$

If ${}_1\mathcal{L}^{\mathcal{F}}(\hat{X})$ (resp., ${}_2\mathcal{L}^{\mathcal{F}}(\hat{X})$, ${}_3\mathcal{L}^{\mathcal{F}}(\hat{X})$, and ${}_4\mathcal{L}^{\mathcal{F}}(\hat{X}) \neq {}_1\mathcal{U}^{\mathcal{F}}(\hat{X})$ (resp., ${}_2\mathcal{U}^{\mathcal{F}}(\hat{X})$, ${}_3\mathcal{U}^{\mathcal{F}}(\hat{X})$, and ${}_4\mathcal{U}^{\mathcal{F}}(\hat{X})$), then $\hat{X}$ is a ${}^1\text{FC}\mathcal{F}$ TFRS (resp., ${}^2\text{FC}\mathcal{F}$ TFRS, ${}^3\text{FC}\mathcal{F}$ TFRS, and ${}^4\text{FC}\mathcal{F}$ TFRS). In other cases, we say that it is definable.

Example 1. Let $(\Omega, \tilde{\Gamma})$ be an FCAS, where $\beta = 0.5$ and $\Omega = \{x_1, \dots, x_6\}$ as in Table 1.

Let us fix $\mathcal{F}_m$ and $\mathcal{T}_m$. Then, we have the following. Then, we can calculate the $\sqrt{\mathcal{M}}d_{x_i}^\beta$ and $\mathcal{M}D_{x_i}^\beta$ ($i = 1, 2, 3, 4, 5, 6$), respectively. See Table 2 for the results of the corresponding computations.

Now, if $\hat{X} = (0.6/x_1) + (0.8/x_2) + (0.5/x_3) + (0.7/x_4) + (0.8/x_5) + (0.9/x_6)$ then we introduce fuzzy β -neighborhood and the union of the fuzzy β -minimal description for $\tilde{\Gamma} = \{\tilde{C}_1\}$ as in Tables 3 and 4.

TABLE 1: A fuzzy β -covering approximation space.

	x_1	x_2	x_3	x_4	x_5	x_6
$\tilde{\Gamma}_1$	0.6	0.1	0.2	0.5	0.4	0.6
$\tilde{\Gamma}_2$	0.5	0.1	0.8	0.6	0.3	0.7
$\tilde{\Gamma}_3$	0.1	0.8	0.3	0.4	0.2	0.1
$\tilde{\Gamma}_4$	0.3	0.6	0.4	0.8	0.8	0.3
$\tilde{\Gamma}_5$	0.5	0.3	0.8	0.9	0.5	0.7

TABLE 2: $\tilde{\mathcal{N}}_C^{0.5}(x_i)$, $\widetilde{\mathcal{M}d}_{x_i}^\beta$, and $\widetilde{\mathcal{M}D}_{x_i}^\beta$ for $i = 1, 2, 3, 4, 5, 6$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\tilde{\mathcal{N}}_C^{0.5}(x_i)$	$\{\Gamma_1, \Gamma_2, \Gamma_5\}$	$\{\Gamma_3, \Gamma_4\}$	$\{\Gamma_2, \Gamma_5\}$	$\{\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5\}$	$\{\Gamma_4, \Gamma_5\}$	$\{\Gamma_1, \Gamma_2, \Gamma_5\}$
$\widetilde{\mathcal{M}d}_{x_i}^{0.5}$	$\{\tilde{\Gamma}_1, \tilde{\Gamma}_2\}$	$\{\tilde{\Gamma}_3, \tilde{\Gamma}_4\}$	$\{\tilde{\Gamma}_2\}$	$\{\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_4, \tilde{\Gamma}_5\}$	$\{\tilde{\Gamma}_4, \tilde{\Gamma}_5\}$	$\{\tilde{\Gamma}_1, \tilde{\Gamma}_2\}$
$\widetilde{\mathcal{M}D}_{x_i}^{0.5}$	$\{\tilde{\Gamma}_1, \tilde{\Gamma}_5\}$	$\{\tilde{\Gamma}_3, \tilde{\Gamma}_4\}$	$\{\tilde{\Gamma}_5\}$	$\{\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_4, \tilde{\Gamma}_5\}$	$\{\tilde{\Gamma}_4, \tilde{\Gamma}_5\}$	$\{\tilde{\Gamma}_1, \tilde{\Gamma}_5\}$

TABLE 3: Fuzzy β -neighborhood of $\tilde{\Gamma}$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\tilde{\mathcal{N}}_{x_1}^{0.5}$	0.5	0.1	0.2	0.5	0.3	0.6
$\tilde{\mathcal{N}}_{x_2}^{0.5}$	0.1	0.6	0.3	0.4	0.2	0.1
$\tilde{\mathcal{N}}_{x_3}^{0.5}$	0.5	0.1	0.8	0.6	0.3	0.7
$\tilde{\mathcal{N}}_{x_4}^{0.5}$	0.3	0.1	0.2	0.5	0.3	0.3
$\tilde{\mathcal{N}}_{x_5}^{0.5}$	0.3	0.3	0.4	0.8	0.5	0.3
$\tilde{\mathcal{N}}_{x_6}^{0.5}$	0.5	0.1	0.2	0.5	0.3	0.6

TABLE 4: Values of $\vee \widetilde{\mathcal{M}d}_x^\beta(y)$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\vee \widetilde{\mathcal{M}d}_{x_1}^{0.5}$	0.6	0.1	0.8	0.6	0.4	0.7
$\vee \widetilde{\mathcal{M}d}_{x_2}^{0.5}$	0.3	0.8	0.4	0.8	0.8	0.3
$\vee \widetilde{\mathcal{M}d}_{x_3}^{0.5}$	0.5	0.1	0.8	0.6	0.3	0.7
$\vee \widetilde{\mathcal{M}d}_{x_4}^{0.5}$	0.6	0.6	0.8	0.9	0.8	0.7
$\vee \widetilde{\mathcal{M}d}_{x_5}^{0.5}$	0.5	0.6	0.8	0.9	0.8	0.7
$\vee \widetilde{\mathcal{M}d}_{x_6}^{0.5}$	0.6	0.1	0.8	0.9	0.4	0.7

So, we have the following results:

$$\begin{aligned} {}_1\mathcal{L}^\mathcal{F}(\hat{X}) &= \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.5}{x_5} + \frac{0.5}{x_6}, \\ {}_1\mathcal{U}^\mathcal{F}(\hat{X}) &= \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.8}{x_5} + \frac{0.7}{x_6}. \end{aligned} \quad (5)$$

We proceed to calculate the fuzzy β -complementary for $\vee \widetilde{\mathcal{M}d}_{x_i}^\beta$ which is denoted by $\widetilde{\mathcal{M}d}_{x_i}^\beta$ and is set in Table 5.

So, we obtain the following values:

$$\begin{aligned} {}_2\mathcal{L}^\mathcal{F}(\hat{X}) &= \frac{0.5}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.5}{x_6}, \\ {}_2\mathcal{U}^\mathcal{F}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.8}{x_3} + \frac{0.9}{x_4} + \frac{0.8}{x_5} + \frac{0.7}{x_6}. \end{aligned} \quad (6)$$

Then, we calculate $\vee \widetilde{\mathcal{M}d}_x^\beta \cap_d \widetilde{\mathcal{M}d}_x^\beta$ as shown in Table 6.

Thus, we have

$$\begin{aligned} {}_3\mathcal{L}^\mathcal{F}(\hat{X}) &= \frac{0.5}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.5}{x_6}, \\ {}_3\mathcal{U}^\mathcal{F}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.8}{x_5} + \frac{0.7}{x_6}. \end{aligned} \quad (7)$$

Let us now calculate $\vee \widetilde{\mathcal{M}d}_x^\beta \cup_d \widetilde{\mathcal{M}d}_x^\beta$, which produces Table 7.

Then, we get the following outcomes:

$$\begin{aligned} {}_4\mathcal{L}^\mathcal{F}(\hat{X}) &= \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.5}{x_5} + \frac{0.5}{x_6}, \\ {}_4\mathcal{U}^\mathcal{F}(\hat{X}) &= \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.8}{x_3} + \frac{0.9}{x_4} + \frac{0.8}{x_5} + \frac{0.7}{x_6}. \end{aligned} \quad (8)$$

Theorem 1 satisfies all kinds of operators in Definition 5. Here, we prove it for the first kind only and the other kinds similarly.

TABLE 5: Values of $\widetilde{\mathcal{M}}_x^\beta(y)$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\widetilde{\mathcal{M}}_{x_1}^{0.5}$	0.6	0.3	0.5	0.6	0.5	0.6
$\widetilde{\mathcal{M}}_{x_2}^{0.5}$	0.1	0.8	0.1	0.6	0.6	0.1
$\widetilde{\mathcal{M}}_{x_3}^{0.5}$	0.8	0.4	0.8	0.8	0.8	0.8
$\widetilde{\mathcal{M}}_{x_4}^{0.5}$	0.6	0.8	0.6	0.9	0.9	0.9
$\widetilde{\mathcal{M}}_{x_5}^{0.5}$	0.4	0.8	0.3	0.8	0.8	0.4
$\widetilde{\mathcal{M}}_{x_6}^{0.5}$	0.7	0.3	0.7	0.7	0.7	0.7

TABLE 6: The result for $\vee \widetilde{\mathcal{M}} d_x^\beta \cap \widetilde{\mathcal{M}}_x^\beta$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\vee \widetilde{\mathcal{M}} d_{x_1}^{0.5} \cap \widetilde{\mathcal{M}}_{x_1}^{0.5}$	0.6	0.1	0.5	0.6	0.4	0.7
$\vee \widetilde{\mathcal{M}} d_{x_2}^{0.5} \cap \widetilde{\mathcal{M}}_{x_2}^{0.5}$	0.1	0.8	0.1	0.6	0.6	0.1
$\vee \widetilde{\mathcal{M}} d_{x_3}^{0.5} \cap \widetilde{\mathcal{M}}_{x_3}^{0.5}$	0.5	0.1	0.8	0.6	0.3	0.7
$\vee \widetilde{\mathcal{M}} d_{x_4}^{0.5} \cap \widetilde{\mathcal{M}}_{x_4}^{0.5}$	0.6	0.6	0.6	0.9	0.8	0.7
$\vee \widetilde{\mathcal{M}} d_{x_5}^{0.5} \cap \widetilde{\mathcal{M}}_{x_5}^{0.5}$	0.4	0.6	0.3	0.8	0.8	0.4
$\vee \widetilde{\mathcal{M}} d_{x_6}^{0.5} \cap \widetilde{\mathcal{M}}_{x_6}^{0.5}$	0.6	0.1	0.7	0.7	0.4	0.7

TABLE 7: The result for $\vee \widetilde{\mathcal{M}} d_x^\beta \cup \widetilde{\mathcal{M}}_x^\beta$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\vee \widetilde{\mathcal{M}} d_{x_1}^{0.5} \cup \widetilde{\mathcal{M}}_{x_1}^{0.5}$	0.6	0.3	0.8	0.6	0.5	0.7
$\vee \widetilde{\mathcal{M}} d_{x_2}^{0.5} \cup \widetilde{\mathcal{M}}_{x_2}^{0.5}$	0.3	0.8	0.4	0.8	0.8	0.3
$\vee \widetilde{\mathcal{M}} d_{x_3}^{0.5} \cup \widetilde{\mathcal{M}}_{x_3}^{0.5}$	0.8	0.4	0.8	0.8	0.8	0.8
$\vee \widetilde{\mathcal{M}} d_{x_4}^{0.5} \cup \widetilde{\mathcal{M}}_{x_4}^{0.5}$	0.6	0.8	0.8	0.9	0.9	0.9
$\vee \widetilde{\mathcal{M}} d_{x_5}^{0.5} \cup \widetilde{\mathcal{M}}_{x_5}^{0.5}$	0.5	0.8	0.8	0.9	0.8	0.7
$\vee \widetilde{\mathcal{M}} d_{x_6}^{0.5} \cup \widetilde{\mathcal{M}}_{x_6}^{0.5}$	0.7	0.3	0.8	0.9	0.7	0.7

Theorem 1. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a given $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$ and $\forall x \in \Omega$. Then, the following affirmations hold true:

- (1) ${}_1\mathcal{L}^\mathcal{F}(\hat{X}) = ({}_1\mathcal{U}^\mathcal{F}(\hat{X}^{c_N}))^{c_N}$ and ${}_1\mathcal{U}^\mathcal{F}(\hat{X}) = ({}_1\mathcal{L}^\mathcal{F}(\hat{X}^{c_N}))^{c_N}$ when $\mathcal{F}$ is an $\mathcal{S}$ -implicator
- (2) ${}_1\mathcal{L}^\mathcal{F}(\Omega) = \Omega$ and ${}_1\mathcal{U}^\mathcal{F}(\emptyset) = \emptyset$ when $\mathcal{F}$ is a left monotonicity
- (3) If $\hat{X} \subseteq \hat{Y}$, then ${}_1\mathcal{L}^\mathcal{F}(\hat{X}) \subseteq {}_1\mathcal{L}^\mathcal{F}(\hat{Y})$ and ${}_1\mathcal{U}^\mathcal{F}(\hat{X}) \subseteq {}_1\mathcal{U}^\mathcal{F}(\hat{Y})$ when $\mathcal{F}$ is a right monotonicity
- (4) ${}_1\mathcal{L}^\mathcal{F}(\hat{X} \cap \hat{Y}) = {}_1\mathcal{L}^\mathcal{F}(\hat{X}) \cap {}_1\mathcal{L}^\mathcal{F}(\hat{Y})$ and ${}_1\mathcal{U}^\mathcal{F}(\hat{X} \cap \hat{Y}) \subseteq {}_1\mathcal{U}^\mathcal{F}(\hat{X}) \cap {}_1\mathcal{U}^\mathcal{F}(\hat{Y})$ when $\mathcal{F}$ is a right monotonicity

- (5) ${}_1\mathcal{L}^\mathcal{F}(\hat{X} \cup \hat{Y}) \supseteq {}_1\mathcal{L}^\mathcal{F}(\hat{X}) \cup {}_1\mathcal{L}^\mathcal{F}(\hat{Y})$ and ${}_1\mathcal{U}^\mathcal{F}(\hat{X} \cup \hat{Y}) = {}_1\mathcal{U}^\mathcal{F}(\hat{X}) \cup {}_1\mathcal{U}^\mathcal{F}(\hat{Y})$ when $\mathcal{F}$ is a right monotonicity

Proof

- (1) $({}_1\mathcal{U}^\mathcal{F}(\hat{X}^{c_N}))^{c_N} = \mathcal{N}(\vee_{y \in \Omega} \mathcal{F}\{\vee \widetilde{\mathcal{M}} d_x^\beta(y), \mathcal{N}(\hat{X}(y))\}) = \mathcal{N}(\vee_{y \in \Omega} \mathcal{N}(\mathcal{F}\{\vee \widetilde{\mathcal{M}} d_x^\beta(y), \hat{X}(y)\})) = \mathcal{N}(\mathcal{N}(\wedge_{y \in \Omega} (\mathcal{F}\{\vee \widetilde{\mathcal{M}} d_x^\beta(y), \hat{X}(y)\}))) = \wedge_{y \in \Omega} \mathcal{F}\{\vee \widetilde{\mathcal{M}} d_x^\beta(y), \hat{X}(y)\} = {}_1\mathcal{L}^\mathcal{F}(\hat{X})$.

By the same manner, we also obtain ${}_1\mathcal{U}^\mathcal{F}(\hat{X}) = ({}_1\mathcal{L}^\mathcal{F}(\hat{X}^{c_N}))^{c_N}$.

- (2) As $\mathcal{F}$ is a left monotonic, then we have

$${}_1\mathcal{L}^\mathcal{F} = \bigwedge_{y \in \Omega} \mathcal{F}\{\vee \widetilde{\mathcal{M}} d_x^\beta(y), \Omega(y)\} = \bigwedge_{y \in \Omega} \mathcal{F}\{\vee \widetilde{\mathcal{M}} d_x^\beta(y), 1\} = 1 = \Omega(x) \quad (9)$$

Also, we have

$${}_1\mathcal{U}^{\mathcal{F}}(\emptyset) = \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), \emptyset(y) \right\} = \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), 0 \right\} = 0 = \emptyset(x). \quad (10)$$

(3) As $\mathcal{F}$ is a right monotonic, if $\widehat{X} \subseteq \widehat{Y}$, then we have

$${}_1\mathcal{L}^{\mathcal{F}}(\widehat{X}) = \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), \widehat{X}(y) \right\} \leq \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), \widehat{Y}(y) \right\} = {}_1\mathcal{L}^{\mathcal{F}}(\widehat{Y}). \quad (11)$$

Thus, ${}_1\mathcal{L}^{\mathcal{F}}(\widehat{X}) \subseteq {}_1\mathcal{L}^{\mathcal{F}}(\widehat{Y})$ holds, and similarly ${}_1\mathcal{U}^{\mathcal{F}}(\widehat{X}) \subseteq {}_1\mathcal{U}^{\mathcal{F}}(\widehat{Y})$ holds.

(4) As $\mathcal{F}$ is a right monotonic, then we have

$$\begin{aligned} {}_1\mathcal{L}^{\mathcal{F}}(\widehat{X} \cap \widehat{Y})(x) &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), (\widehat{X} \cap \widehat{Y})(y) \right\} \\ &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), \widehat{X}(y) \right\} \\ &\quad \wedge \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), \widehat{Y}(y) \right\} \\ &= {}_1\mathcal{L}^{\mathcal{F}}(\widehat{X})(x) \wedge {}_1\mathcal{L}^{\mathcal{F}}(\widehat{Y})(x) \\ &= ({}_1\mathcal{L}^{\mathcal{F}}(\widehat{X}) \cap {}_1\mathcal{L}^{\mathcal{F}}(\widehat{Y}))(x). \end{aligned} \quad (12)$$

Because $\widehat{X} \cap \widehat{Y} \subseteq \widehat{X}$ and $\widehat{X} \cap \widehat{Y} \subseteq \widehat{Y}$, we deduce from (3), ${}_1\mathcal{U}^{\mathcal{F}}(\widehat{X} \cap \widehat{Y}) \subseteq {}_1\mathcal{U}^{\mathcal{F}}(\widehat{X})$ and ${}_1\mathcal{U}^{\mathcal{F}}(\widehat{X} \cap \widehat{Y}) \subseteq {}_1\mathcal{U}^{\mathcal{F}}(\widehat{Y})$. Thus, ${}_1\mathcal{U}^{\mathcal{F}}(\widehat{X} \cap \widehat{Y}) \subseteq {}_1\mathcal{U}^{\mathcal{F}}(\widehat{X}) \cap {}_1\mathcal{U}^{\mathcal{F}}(\widehat{Y})$.

(5) As $\mathcal{F}$ is a right monotonic, $\widehat{X} \subseteq \widehat{X} \cup \widehat{Y}$ and $\widehat{Y} \subseteq \widehat{X} \cup \widehat{Y}$; from (3), we have ${}_1\mathcal{L}^{\mathcal{F}}(\widehat{X}) \subseteq {}_1\mathcal{L}^{\mathcal{F}}(\widehat{X} \cup \widehat{Y})$ and ${}_1\mathcal{L}^{\mathcal{F}}(\widehat{Y}) \subseteq {}_1\mathcal{L}^{\mathcal{F}}(\widehat{X} \cup \widehat{Y})$. Thus, ${}_1\mathcal{L}^{\mathcal{F}}(\widehat{X} \cup \widehat{Y}) \supseteq {}_1\mathcal{L}^{\mathcal{F}}(\widehat{X}) \cup {}_1\mathcal{L}^{\mathcal{F}}(\widehat{Y})$.

Also, we obtain

$$\begin{aligned} {}_1\mathcal{U}^{\mathcal{F}}(\widehat{X} \cup \widehat{Y})(x) &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), (\widehat{X} \cup \widehat{Y})(y) \right\} \\ &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), \widehat{X}(y) \right\} \\ &\quad \vee \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigvee \mathcal{M} d_x^{\beta}(y), \widehat{Y}(y) \right\} \\ &= {}_1\mathcal{U}^{\mathcal{F}}(\widehat{X})(x) \vee {}_1\mathcal{U}^{\mathcal{F}}(\widehat{Y})(x) \\ &= ({}_1\mathcal{U}^{\mathcal{F}}(\widehat{X}) \cup {}_1\mathcal{U}^{\mathcal{F}}(\widehat{Y}))(x). \end{aligned} \quad (13)$$

3.2. Another Four FC- $\mathcal{F}$ TFRS Models through Fuzzy β -Maximal Description

Definition 6. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\widehat{X} \in \mathcal{F}(\Omega)$. For each $x \in \Omega$, define ${}^5\text{ITFLA}$ (resp., ${}^6\text{ITFLA}$, ${}^7\text{ITFLA}$, and ${}^8\text{ITFLA}$) and ${}^5\text{ITFUA}$ (resp., ${}^6\text{ITFUA}$, ${}^7\text{ITFUA}$, and ${}^8\text{ITFUA}$) as in the following way:

$$\begin{aligned} {}^5\mathcal{L}^{\mathcal{F}}(\widehat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} \mathcal{D}_x^{\beta}(y), \widehat{X}(y) \right\}, \\ {}^5\mathcal{U}^{\mathcal{F}}(\widehat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} \mathcal{D}_x^{\beta}(y), \widehat{X}(y) \right\}, \\ {}^6\mathcal{L}^{\mathcal{F}}(\widehat{X})(x) &= \bigwedge_{i=1}^n \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M}_x^{\beta}(y), \widehat{X}(y) \right\}, \\ {}^6\mathcal{U}^{\mathcal{F}}(\widehat{X})(x) &= \bigwedge_{i=1}^n \bigvee_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M}_x^{\beta}(y), \widehat{X}(y) \right\}, \\ {}^7\mathcal{L}^{\mathcal{F}}(\widehat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} \mathcal{D}_x^{\beta}(y) \cap \bigwedge \mathcal{M}_x^{\beta}(y), \widehat{X}(y) \right\}, \\ {}^7\mathcal{U}^{\mathcal{F}}(\widehat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} \mathcal{D}_x^{\beta}(y) \cap \bigwedge \mathcal{M}_x^{\beta}(y), \widehat{X}(y) \right\}, \\ {}^8\mathcal{L}^{\mathcal{F}}(\widehat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} \mathcal{D}_x^{\beta}(y) \cup \bigwedge \mathcal{M}_x^{\beta}(y), \widehat{X}(y) \right\}, \\ {}^8\mathcal{U}^{\mathcal{F}}(\widehat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} \mathcal{D}_x^{\beta}(y) \cup \bigwedge \mathcal{M}_x^{\beta}(y), \widehat{X}(y) \right\}. \end{aligned} \quad (14)$$

If ${}_5\mathcal{L}^\mathcal{F}(\hat{X})$ (resp., ${}_6\mathcal{L}^\mathcal{F}(\hat{X})$, ${}_7\mathcal{L}^\mathcal{F}(\hat{X})$ and ${}_8\mathcal{L}^\mathcal{F}(\hat{X}) \neq {}_5\mathcal{U}^\mathcal{F}(\hat{X})$ (resp., ${}_6\mathcal{U}^\mathcal{F}(\hat{X})$, ${}_7\mathcal{U}^\mathcal{F}(\hat{X})$ and ${}_8\mathcal{U}^\mathcal{F}(\hat{X})$), then $\hat{X}$ is a ${}^5\text{FC}\mathcal{F}\text{TFRS}$ (resp., ${}^6\text{FC}\mathcal{F}\text{TFRS}$, ${}^7\text{FC}\mathcal{F}\text{TFRS}$ and ${}^8\text{FC}\mathcal{F}\text{TFRS}$); otherwise, it is definable.

Example 2 (in continuation of Example 1).

If $\hat{X} = (0.6/x_1) + (0.8/x_2) + (0.5/x_3) + (0.7/x_4) + (0.8/x_5) + (0.9/x_6)$, then we build $\wedge \mathcal{M}d_{x_i}^\beta$ ($\forall i = 1, 2, 3, 4, 5, 6$) in Table 8.

Hence, we have

$$\begin{aligned} {}_5\mathcal{L}^\mathcal{F}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\ {}_5\mathcal{U}^\mathcal{F}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{0.5}{x_4} + \frac{0.7}{x_5} + \frac{0.6}{x_6}. \end{aligned} \quad (15)$$

Then, we compute the fuzzy β -complementary for the intersection of the fuzzy β -maximal description, which is denoted by $\widetilde{\mathcal{M}d_{x_i}^\beta}$ and is shown in Table 9.

Thus, we obtain the following:

$$\begin{aligned} {}_6\mathcal{L}^\mathcal{F}(\hat{X}) &= \frac{0.5}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.5}{x_5} + \frac{0.5}{x_6}, \\ {}_6\mathcal{U}^\mathcal{F}(\hat{X}) &= \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.8}{x_4} + \frac{0.5}{x_5} + \frac{0.6}{x_6}. \end{aligned} \quad (16)$$

Then, we calculate $\wedge \mathcal{M}d_x^\beta \cap \widetilde{\mathcal{M}d_x^\beta}$. The results are listed in Table 10.

So, the following results are obtained:

$$\begin{aligned} {}_7\mathcal{L}^\mathcal{F}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\ {}_7\mathcal{U}^\mathcal{F}(\hat{X}) &= \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.5}{x_5} + \frac{0.6}{x_6}. \end{aligned} \quad (17)$$

Then, we calculate $\wedge \mathcal{M}d_x^\beta \cup \widetilde{\mathcal{M}d_x^\beta}$ listed in Table 11. Therefore, we get the following outcomes:

$$\begin{aligned} {}_8\mathcal{L}^\mathcal{F}(\hat{X}) &= \frac{0.5}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} + \frac{0.5}{x_5} + \frac{0.5}{x_6}, \\ {}_8\mathcal{U}^\mathcal{F}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{0.8}{x_4} + \frac{0.7}{x_5} + \frac{0.6}{x_6}. \end{aligned} \quad (18)$$

Theorem 2 satisfies all types of operators in Definition 6. Next, we prove it for the fifth model only and the other models similarly.

Theorem 2. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a given $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$ and $\forall x \in \Omega$. Then, the following affirmations hold true:

- (1) ${}_5\mathcal{L}^\mathcal{F}(\hat{X}) = ({}_5\mathcal{U}^\mathcal{F}(\hat{X}^{c_{\mathcal{N}}}))^{c_{\mathcal{N}}}$ and ${}_5\mathcal{U}^\mathcal{F}(\hat{X}) = ({}_5\mathcal{L}^\mathcal{F}(\hat{X}^{c_{\mathcal{N}}}))^{c_{\mathcal{N}}}$ when $\mathcal{F}$ is an $\mathcal{S}$ -implicator
- (2) ${}_5\mathcal{L}^\mathcal{F}(\Omega) = \Omega$ and ${}_5\mathcal{U}^\mathcal{F}(\emptyset) = \emptyset$ when $\mathcal{F}$ is a left monotonicity
- (3) If $\hat{X} \subseteq \hat{Y}$, then ${}_5\mathcal{L}^\mathcal{F}(\hat{X}) \subseteq {}_5\mathcal{L}^\mathcal{F}(\hat{Y})$ and ${}_5\mathcal{U}^\mathcal{F}(\hat{X}) \subseteq {}_5\mathcal{U}^\mathcal{F}(\hat{Y})$ when $\mathcal{F}$ is a right monotonicity
- (4) ${}_5\mathcal{L}^\mathcal{F}(\hat{X} \cap \hat{Y}) = {}_5\mathcal{L}^\mathcal{F}(\hat{X}) \cap {}_5\mathcal{L}^\mathcal{F}(\hat{Y})$ and ${}_5\mathcal{U}^\mathcal{F}(\hat{X} \cap \hat{Y}) \subseteq {}_5\mathcal{U}^\mathcal{F}(\hat{X}) \cap {}_5\mathcal{U}^\mathcal{F}(\hat{Y})$ when $\mathcal{F}$ is a right monotonicity
- (5) ${}_5\mathcal{L}^\mathcal{F}(\hat{X} \cup \hat{Y}) \supseteq {}_5\mathcal{L}^\mathcal{F}(\hat{X}) \cup {}_5\mathcal{L}^\mathcal{F}(\hat{Y})$ and ${}_5\mathcal{U}^\mathcal{F}(\hat{X} \cup \hat{Y}) = {}_5\mathcal{U}^\mathcal{F}(\hat{X}) \cup {}_5\mathcal{U}^\mathcal{F}(\hat{Y})$ when $\mathcal{F}$ is a right monotonicity

Proof

$$\begin{aligned} (1) \quad ({}_5\mathcal{U}^\mathcal{F}(\hat{X}^{c_{\mathcal{N}}}))^{c_{\mathcal{N}}} &= \mathcal{N}(\vee_{y \in \Omega} \mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), \mathcal{N}(\hat{X}(y))\}) \\ &= \mathcal{N}(\mathcal{N}(\wedge_{y \in \Omega} (\mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), \hat{X}(y)\}))) = \wedge_{y \in \Omega} \mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), \hat{X}(y)\} \\ &= {}_5\mathcal{L}^\mathcal{F}(\hat{X}). \end{aligned}$$

By the same manner, we also obtain ${}_5\mathcal{U}^\mathcal{F}(\hat{X}) = ({}_5\mathcal{L}^\mathcal{F}(\hat{X}^{c_{\mathcal{N}}}))^{c_{\mathcal{N}}}$.

- (2) As $\mathcal{F}$ is a left monotonic, then we have

$${}_5\mathcal{L}^\mathcal{F}(\Omega) = \bigwedge_{y \in \Omega} \mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), \Omega(y)\} = \bigwedge_{y \in \Omega} \mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), 1\} = 1 = \Omega(x). \quad (19)$$

Also, we have

$${}_5\mathcal{L}^\mathcal{F}(\emptyset) = \bigwedge_{y \in \Omega} \mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), \emptyset(y)\} = \bigwedge_{y \in \Omega} \mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), 0\} = 0 = \emptyset(x). \quad (20)$$

- (3) As $\mathcal{F}$ is a right monotonic, if $\hat{X} \subseteq \hat{Y}$, then we have

$${}_5\mathcal{L}^\mathcal{F}(\hat{X}) = \bigwedge_{y \in \Omega} \mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), \hat{X}(y)\} \leq \bigwedge_{y \in \Omega} \mathcal{F}\{\wedge \mathcal{M}d_x^\beta(y), \hat{Y}(y)\} = {}_5\mathcal{L}^\mathcal{F}(\hat{Y}). \quad (21)$$

Thus, ${}_5\mathcal{L}^\mathcal{F}(\hat{X}) \subseteq {}_5\mathcal{L}^\mathcal{F}(\hat{Y})$ holds, and similarly ${}_5\mathcal{U}^\mathcal{F}(\hat{X}) \subseteq {}_5\mathcal{U}^\mathcal{F}(\hat{Y})$ holds.

- (4) As $\mathcal{F}$ is a right monotonic. Then, we have

TABLE 8: Values of $\wedge \widetilde{\mathcal{MD}}_x^\beta$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\wedge \widetilde{\mathcal{MD}}_{x_1}^{0.5}$	0.5	0.1	0.2	0.5	0.4	0.6
$\wedge \widetilde{\mathcal{MD}}_{x_2}^{0.5}$	0.1	0.6	0.3	0.4	0.2	0.1
$\wedge \widetilde{\mathcal{MD}}_{x_3}^{0.5}$	0.5	0.3	0.8	0.9	0.5	0.7
$\wedge \widetilde{\mathcal{MD}}_{x_4}^{0.5}$	0.3	0.1	0.2	0.5	0.3	0.3
$\wedge \widetilde{\mathcal{MD}}_{x_5}^{0.5}$	0.3	0.3	0.4	0.8	0.5	0.3
$\wedge \widetilde{\mathcal{MD}}_{x_6}^{0.5}$	0.5	0.1	0.2	0.5	0.4	0.6

TABLE 9: Values of $\widetilde{\mathcal{M}}_x^\beta(y)$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\widetilde{\mathcal{M}}_{x_1}^{0.5}$	0.5	0.1	0.5	0.3	0.3	0.5
$\widetilde{\mathcal{M}}_{x_2}^{0.5}$	0.1	0.6	0.3	0.1	0.3	0.1
$\widetilde{\mathcal{M}}_{x_3}^{0.5}$	0.2	0.3	0.8	0.2	0.4	0.2
$\widetilde{\mathcal{M}}_{x_4}^{0.5}$	0.5	0.4	0.9	0.5	0.8	0.5
$\widetilde{\mathcal{M}}_{x_5}^{0.5}$	0.4	0.2	0.5	0.3	0.5	0.4
$\widetilde{\mathcal{M}}_{x_6}^{0.5}$	0.6	0.1	0.7	0.3	0.3	0.6

TABLE 10: The result for $\wedge \widetilde{\mathcal{MD}}_x^\beta \cap \widetilde{\mathcal{M}}_x^\beta$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\wedge \widetilde{\mathcal{MD}}_{x_1}^{0.5} \cap \widetilde{\mathcal{M}}_{x_1}^{0.5}$	0.5	0.1	0.2	0.3	0.3	0.5
$\wedge \widetilde{\mathcal{MD}}_{x_2}^{0.5} \cap \widetilde{\mathcal{M}}_{x_2}^{0.5}$	0.1	0.6	0.3	0.1	0.2	0.1
$\wedge \widetilde{\mathcal{MD}}_{x_3}^{0.5} \cap \widetilde{\mathcal{M}}_{x_3}^{0.5}$	0.2	0.3	0.8	0.2	0.4	0.2
$\wedge \widetilde{\mathcal{MD}}_{x_4}^{0.5} \cap \widetilde{\mathcal{M}}_{x_4}^{0.5}$	0.3	0.1	0.2	0.5	0.3	0.3
$\wedge \widetilde{\mathcal{MD}}_{x_5}^{0.5} \cap \widetilde{\mathcal{M}}_{x_5}^{0.5}$	0.3	0.2	0.4	0.3	0.5	0.3
$\wedge \widetilde{\mathcal{MD}}_{x_6}^{0.5} \cap \widetilde{\mathcal{M}}_{x_6}^{0.5}$	0.5	0.1	0.2	0.3	0.3	0.6

TABLE 11: The result for $\wedge \widetilde{\mathcal{MD}}_x^\beta \cup \widetilde{\mathcal{M}}_x^\beta$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\wedge \widetilde{\mathcal{MD}}_{x_1}^{0.5} \cup \widetilde{\mathcal{M}}_{x_1}^{0.5}$	0.5	0.1	0.5	0.5	0.4	0.6
$\wedge \widetilde{\mathcal{MD}}_{x_2}^{0.5} \cup \widetilde{\mathcal{M}}_{x_2}^{0.5}$	0.1	0.6	0.3	0.4	0.3	0.1
$\wedge \widetilde{\mathcal{MD}}_{x_3}^{0.5} \cup \widetilde{\mathcal{M}}_{x_3}^{0.5}$	0.5	0.3	0.8	0.9	0.5	0.7
$\wedge \widetilde{\mathcal{MD}}_{x_4}^{0.5} \cup \widetilde{\mathcal{M}}_{x_4}^{0.5}$	0.5	0.4	0.9	0.5	0.8	0.5
$\wedge \widetilde{\mathcal{MD}}_{x_5}^{0.5} \cup \widetilde{\mathcal{M}}_{x_5}^{0.5}$	0.4	0.3	0.5	0.8	0.5	0.4
$\wedge \widetilde{\mathcal{MD}}_{x_6}^{0.5} \cup \widetilde{\mathcal{M}}_{x_6}^{0.5}$	0.6	0.1	0.7	0.5	0.4	0.6

$$\begin{aligned}
{}_5\mathcal{L}^{\mathcal{F}}(\hat{X} \cap \hat{Y})(x) &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \wedge \widetilde{\mathcal{MD}}_x^\beta(y), (\hat{X} \cap \hat{Y})(y) \right\} \\
&= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \wedge \widetilde{\mathcal{MD}}_x^\beta(y), \hat{X}(y) \right\} \wedge \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \wedge \widetilde{\mathcal{MD}}_x^\beta(y), \hat{Y}(y) \right\} \\
&= {}_5\mathcal{L}^{\mathcal{F}}(\hat{X})(x) \wedge {}_5\mathcal{L}^{\mathcal{F}}(\hat{Y})(x) = ({}_5\mathcal{L}^{\mathcal{F}}(\hat{X}) \cap {}_5\mathcal{L}^{\mathcal{F}}(\hat{Y}))(x).
\end{aligned} \tag{22}$$

As $\hat{X} \cap \hat{Y} \subseteq \hat{X}$ and $\hat{X} \cap \hat{Y} \subseteq \hat{Y}$, from (3), we obtain ${}_5\mathcal{U}^{\mathcal{F}}(\hat{X} \cap \hat{Y}) \subseteq {}_5\mathcal{U}^{\mathcal{F}}(\hat{X})$ and ${}_5\mathcal{U}^{\mathcal{F}}(\hat{X} \cap \hat{Y}) \subseteq {}_5\mathcal{U}^{\mathcal{F}}(\hat{Y})$. Thus, ${}_5\mathcal{U}^{\mathcal{F}}(\hat{X} \cap \hat{Y}) \subseteq {}_5\mathcal{U}^{\mathcal{F}}(\hat{X}) \cap {}_5\mathcal{U}^{\mathcal{F}}(\hat{Y})$.

- (5) As $\mathcal{F}$ is a right monotonic, $\hat{X} \subseteq \hat{X} \cup \hat{Y}$ and $\hat{Y} \subseteq \hat{X} \cup \hat{Y}$, from (3), we have ${}_5\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_5\mathcal{L}^{\mathcal{F}}(\hat{X} \cup \hat{Y})$ and

$${}_5\mathcal{L}^{\mathcal{F}}(\hat{Y}) \subseteq {}_5\mathcal{L}^{\mathcal{F}}(\hat{X} \cup \hat{Y}).$$

$${}_5\mathcal{L}^{\mathcal{F}}(\hat{X} \cup \hat{Y}) \supseteq {}_5\mathcal{L}^{\mathcal{F}}(\hat{X}) \cup {}_5\mathcal{L}^{\mathcal{F}}(\hat{Y}).$$

Thus,

Also, we obtain

$$\begin{aligned} {}_5\mathcal{U}^{\mathcal{F}}(\hat{X} \cup \hat{Y})(x) &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} d_x^{\beta}(y), (\hat{X} \cup \hat{Y})(y) \right\} \\ &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} d_x^{\beta}(y), \hat{X}(y) \right\} \vee \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M} d_x^{\beta}(y), \hat{Y}(y) \right\} \\ &= {}_5\mathcal{U}^{\mathcal{F}}(\hat{X})(x) \vee {}_5\mathcal{U}^{\mathcal{F}}(\hat{Y})(x) \\ &= ({}_5\mathcal{U}^{\mathcal{F}}(\hat{X}) \cup {}_5\mathcal{U}^{\mathcal{F}}(\hat{Y}))(x). \end{aligned} \quad (23)$$

□

4. Eight Types of Covering-Based Variable Precision $(\mathcal{F}, \mathcal{T})$ -Fuzzy Rough Sets

Based on Section 3, this section establishes eight new CVP $\mathcal{F}$ TFRS models. We discuss some of their features too.

Definition 7. Let $(\Omega, \tilde{\Gamma})$ be an FCAS and for a given $\beta \in (0, 1]$. Let us fix a parameter $\gamma \in [0, 1]$. Define 1-VPITFLA (resp., 2-VPITFLA, 3-VPITFLA, 4-VPITFLA, 5-VPITFLA, 6-VPITFLA, 7-VPITFLA, and 8-VPITFLA) and 1-VPITFUA (resp., 2-VPITFUA, 3-VPITFUA, 4-VPITFUA, 5-VPITFUA, 6-VPITFUA, 7-VPITFUA, and 8-VPITFUA), for every $\hat{X} \in \mathcal{F}(\Omega)$ and $x \in \Omega$ as follows:

$$\begin{aligned} {}_1\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}(\bigwedge \mathcal{M} d_x^{\beta}(y), \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}(\bigwedge \mathcal{M} d_x^{\beta}(y), \hat{X}(y)) \right), \\ {}_1\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{T}(\bigvee \mathcal{M} d_x^{\beta}(y), 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{T}(\bigvee \mathcal{M} d_x^{\beta}(y), \hat{X}(y)) \right), \\ {}_2\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}(\bigwedge \mathcal{M} d_x^{\beta}(y), \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}(\bigwedge \mathcal{M} d_x^{\beta}(y), \hat{X}(y)) \right), \\ {}_2\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{T}(\bigvee \mathcal{M} d_x^{\beta}(y), 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{T}(\bigvee \mathcal{M} d_x^{\beta}(y), \hat{X}(y)) \right), \\ {}_3\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}([\bigvee \mathcal{M} d_x^{\beta}(y) \cap \bigwedge \mathcal{M} d_x^{\beta}(y)], \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}([\bigvee \mathcal{M} d_x^{\beta}(y) \cap \bigwedge \mathcal{M} d_x^{\beta}(y)], \hat{X}(y)) \right), \\ {}_3\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{T}([\bigvee \mathcal{M} d_x^{\beta}(y) \cap \bigwedge \mathcal{M} d_x^{\beta}(y)], 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{T}([\bigvee \mathcal{M} d_x^{\beta}(y) \cap \bigwedge \mathcal{M} d_x^{\beta}(y)], \hat{X}(y)) \right), \\ {}_4\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}([\bigvee \mathcal{M} d_x^{\beta}(y) \cup \bigwedge \mathcal{M} d_x^{\beta}(y)], \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}([\bigvee \mathcal{M} d_x^{\beta}(y) \cup \bigwedge \mathcal{M} d_x^{\beta}(y)], \hat{X}(y)) \right), \\ {}_4\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{T}([\bigvee \mathcal{M} d_x^{\beta}(y) \cup \bigwedge \mathcal{M} d_x^{\beta}(y)], 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{T}([\bigvee \mathcal{M} d_x^{\beta}(y) \cup \bigwedge \mathcal{M} d_x^{\beta}(y)], \hat{X}(y)) \right), \\ {}_5\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}(\bigwedge \mathcal{M} \mathcal{D}_x^{\beta}(y), \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}(\bigwedge \mathcal{M} \mathcal{D}_x^{\beta}(y), \hat{X}(y)) \right), \\ {}_5\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{T}(\bigvee \mathcal{M} \mathcal{D}_x^{\beta}(y), 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{T}(\bigvee \mathcal{M} \mathcal{D}_x^{\beta}(y), \hat{X}(y)) \right), \end{aligned}$$

$$\begin{aligned}
{}_6\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}(\overline{\mathcal{M}}_x^{\beta}(y), \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}(\overline{\mathcal{M}}_x^{\beta}(y), \hat{X}(y)) \right), \\
{}_6\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{F}(\overline{\mathcal{M}}_x^{\beta}(y), 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{F}(\overline{\mathcal{M}}_x^{\beta}(y), \hat{X}(y)) \right), \\
{}_7\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}([\vee \overline{\mathcal{M}}_x^{\beta}(y) \cap \overline{\mathcal{M}}_x^{\beta}(y)], \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}([\vee \overline{\mathcal{M}}_x^{\beta}(y) \cap \overline{\mathcal{M}}_x^{\beta}(y)], \hat{X}(y)) \right), \\
{}_7\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{F}([\vee \overline{\mathcal{M}}_x^{\beta}(y) \cap \overline{\mathcal{M}}_x^{\beta}(y)], 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{F}([\vee \overline{\mathcal{M}}_x^{\beta}(y) \cap \overline{\mathcal{M}}_x^{\beta}(y)], \hat{X}(y)) \right), \\
{}_8\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigwedge_{\hat{X}(y) \leq \gamma} \mathcal{F}([\vee \overline{\mathcal{M}}_x^{\beta}(y) \cup \overline{\mathcal{M}}_x^{\beta}(y)], \gamma) \right) \wedge \left(\bigwedge_{\hat{X}(y) > \gamma} \mathcal{F}([\vee \overline{\mathcal{M}}_x^{\beta}(y) \cup \overline{\mathcal{M}}_x^{\beta}(y)], \hat{X}(y)) \right), \\
{}_8\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})(x) &= \left(\bigvee_{\hat{X}(y) \geq 1-\gamma} \mathcal{F}([\vee \overline{\mathcal{M}}_x^{\beta}(y) \cup \overline{\mathcal{M}}_x^{\beta}(y)], 1-\gamma) \right) \vee \left(\bigvee_{\hat{X}(y) < 1-\gamma} \mathcal{F}([\vee \overline{\mathcal{M}}_x^{\beta}(y) \cup \overline{\mathcal{M}}_x^{\beta}(y)], \hat{X}(y)) \right).
\end{aligned} \tag{24}$$

If ${}_1\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$ (resp., ${}_2\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_3\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_4\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_5\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_6\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_7\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X})$, and ${}_8\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \neq {}_1\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$ (resp., ${}_2\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_3\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_4\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_5\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_6\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$, ${}_7\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$, and ${}_8\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X})$), then $\hat{X}$ is a 1-VP $\mathcal{F}$ TFRS (resp., 2-VP $\mathcal{F}$ TFRS, 3-VP $\mathcal{F}$ TFRS, 4-VP $\mathcal{F}$ TFRS, 5-VP $\mathcal{F}$ TFRS, 6-VP $\mathcal{F}$ TFRS, 7-VP $\mathcal{F}$ TFRS, and 8-VP $\mathcal{F}$ TFRS). In other cases, it is definable.

Example 3 (in continuation of Examples 1 and 2). Let $\gamma = 0.6$. We now have

$$\begin{aligned}
{}_1\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\
{}_1\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.1}{x_1} + \frac{0.3}{x_2} + \frac{0.1}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.1}{x_6}, \\
{}_2\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\
{}_2\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5} + \frac{0.3}{x_6}, \\
{}_3\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\
{}_3\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5} + \frac{0.1}{x_6}, \\
{}_4\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\
{}_4\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} + \frac{0.3}{x_6}, \\
{}_5\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.7}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6},
\end{aligned}$$

$$\begin{aligned}
{}_5\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.3}{x_5} + \frac{0.1}{x_6}, \\
{}_6\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\
{}_6\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.4}{x_4} + \frac{0.2}{x_5} + \frac{0.1}{x_6}, \\
{}_7\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.7}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\
{}_7\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.1}{x_4} + \frac{0.2}{x_5} + \frac{0.1}{x_6}, \\
{}_8\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.6}{x_1} + \frac{0.7}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} + \frac{0.6}{x_6}, \\
{}_8\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.3}{x_3} + \frac{0.4}{x_4} + \frac{0.3}{x_5} + \frac{0.1}{x_6}.
\end{aligned} \tag{25}$$

Next, we have the relevant properties of Definition 7. The proof of each point in Theorem 3 is easily obtained, so we omit it. Let us select $j \in \{1, 2, 3, \dots, 8\}$.

Theorem 3. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a given $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$ and $\forall x \in \Omega$. Then, the following affirmations hold true:

- (1) ${}_j\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) = ({}_j\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}^{c,\mathcal{N}}))^{c,\mathcal{N}}$ and ${}_j\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) = ({}_j\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}^{*\mathcal{P}}))^{c,\mathcal{N}}$ when $\mathcal{F}$ is an $\mathcal{S}$ -implicator
- (2) ${}_j\mathcal{L}_{vp}^{\mathcal{F}}(\Omega) = \Omega$ and ${}_j\mathcal{U}_{vp}^{\mathcal{F}}(\emptyset) = \emptyset$ when $\mathcal{F}$ is a left monotonicity
- (3) If $\hat{X} \subseteq \hat{Y}$, then ${}_j\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_j\mathcal{L}_{vp}^{\mathcal{F}}(\hat{Y})$ and ${}_j\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_j\mathcal{U}_{vp}^{\mathcal{F}}(\hat{Y})$ when $\mathcal{F}$ is a right monotonicity
- (4) ${}_j\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X} \cap \hat{Y}) = {}_j\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \cap {}_j\mathcal{L}_{vp}^{\mathcal{F}}(\hat{Y})$ and ${}_j\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X} \cap \hat{Y}) = {}_j\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \cap {}_j\mathcal{U}_{vp}^{\mathcal{F}}(\hat{Y})$ when $\mathcal{F}$ is a right monotonicity

$$(5) {}_j\mathcal{L}^{\mathcal{F}}(\hat{X} \cup \hat{Y}) \supseteq {}_j\mathcal{L}^{\mathcal{F}}(\hat{X}) \cup {}_j\mathcal{L}^{\mathcal{F}}(\hat{Y}) \quad \text{and} \quad {}_j\mathcal{U}^{\mathcal{F}}(\hat{X} \cup \hat{Y}) = {}_j\mathcal{U}^{\mathcal{F}}(\hat{X}) \cup {}_j\mathcal{U}^{\mathcal{F}}(\hat{Y}) \quad \text{when } \mathcal{F} \text{ is a right monotonicity}$$

5. The Relationships among Our Models

Now, we explain some relationships among the paradigms proposed previously.

Remark 1. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. Then, the following affirmation holds true:

$$\begin{aligned} (1) \quad & \wedge \mathcal{M} \mathcal{D}_x^\beta = \tilde{N}_x^\beta \\ (2) \quad & \vee \mathcal{M} \mathcal{D}_x^\beta = \vee \{\tilde{C}_i \in \tilde{\Gamma}: \tilde{C}_i \geq \beta\} \\ (3) \quad & (\vee \mathcal{M} \mathcal{D}_x^\beta \cap \vee \mathcal{M} \mathcal{D}_x^\beta) \leq \vee \mathcal{M} \mathcal{D}_x^\beta, (\vee \mathcal{M} \mathcal{D}_x^\beta \cap \vee \mathcal{M} \mathcal{D}_x^\beta) \\ & \leq \vee \mathcal{M} \mathcal{D}_x^\beta \\ (4) \quad & (\vee \mathcal{M} \mathcal{D}_x^\beta \cup \vee \mathcal{M} \mathcal{D}_x^\beta) \geq \vee \mathcal{M} \mathcal{D}_x^\beta, (\vee \mathcal{M} \mathcal{D}_x^\beta \cup \vee \mathcal{M} \mathcal{D}_x^\beta) \\ & \geq \vee \mathcal{M} \mathcal{D}_x^\beta \\ (5) \quad & (\vee \mathcal{M} \mathcal{D}_x^\beta \cap \vee \mathcal{M} \mathcal{D}_x^\beta) \leq \vee \mathcal{M} \mathcal{D}_x^\beta, (\vee \mathcal{M} \mathcal{D}_x^\beta \cap \vee \mathcal{M} \mathcal{D}_x^\beta) \leq \vee \mathcal{M} \mathcal{D}_x^\beta \\ (6) \quad & (\vee \mathcal{M} \mathcal{D}_x^\beta \cup \vee \mathcal{M} \mathcal{D}_x^\beta) \geq \vee \mathcal{M} \mathcal{D}_x^\beta, (\vee \mathcal{M} \mathcal{D}_x^\beta \cup \vee \mathcal{M} \mathcal{D}_x^\beta) \geq \vee \mathcal{M} \mathcal{D}_x^\beta \end{aligned}$$

Proposition 1. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. Then, the following affirmation holds true:

$$\begin{aligned} (1) \quad & {}_4\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_1\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_3\mathcal{L}^{\mathcal{F}}(\hat{X}) \\ (2) \quad & {}_4\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_2\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_3\mathcal{L}^{\mathcal{F}}(\hat{X}) \\ (3) \quad & {}_3\mathcal{U}^{\mathcal{F}}(\hat{X}) \subseteq {}_1\mathcal{U}^{\mathcal{F}}(\hat{X}) \subseteq {}_4\mathcal{U}^{\mathcal{F}}(\hat{X}) \\ (4) \quad & {}_3\mathcal{U}^{\mathcal{F}}(\hat{X}) \subseteq {}_2\mathcal{U}^{\mathcal{F}}(\hat{X}) \subseteq {}_4\mathcal{U}^{\mathcal{F}}(\hat{X}) \\ (5) \quad & {}_3\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_1\mathcal{L}^{\mathcal{F}}(\hat{X}) \cup {}_2\mathcal{L}^{\mathcal{F}}(\hat{X}) \\ (6) \quad & {}_4\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_1\mathcal{L}^{\mathcal{F}}(\hat{X}) \cap {}_2\mathcal{L}^{\mathcal{F}}(\hat{X}) \\ (7) \quad & {}_3\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_1\mathcal{U}^{\mathcal{F}}(\hat{X}) \cap {}_2\mathcal{U}^{\mathcal{F}}(\hat{X}) \\ (8) \quad & {}_4\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_1\mathcal{U}^{\mathcal{F}}(\hat{X}) \cup {}_2\mathcal{U}^{\mathcal{F}}(\hat{X}) \end{aligned}$$

Proof. It is clear by using Definition 5. $\square$

Proposition 2. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. Then, the following affirmation holds true:

$$\begin{aligned} (1) \quad & {}_8\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_5\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_7\mathcal{L}^{\mathcal{F}}(\hat{X}) \\ (2) \quad & {}_8\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_6\mathcal{L}^{\mathcal{F}}(\hat{X}) \subseteq {}_7\mathcal{L}^{\mathcal{F}}(\hat{X}) \\ (3) \quad & {}_7\mathcal{U}^{\mathcal{F}}(\hat{X}) \subseteq {}_5\mathcal{U}^{\mathcal{F}}(\hat{X}) \subseteq {}_8\mathcal{U}^{\mathcal{F}}(\hat{X}) \\ (4) \quad & {}_7\mathcal{U}^{\mathcal{F}}(\hat{X}) \subseteq {}_6\mathcal{U}^{\mathcal{F}}(\hat{X}) \subseteq {}_8\mathcal{U}^{\mathcal{F}}(\hat{X}) \\ (5) \quad & {}_7\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_5\mathcal{L}^{\mathcal{F}}(\hat{X}) \cup {}_6\mathcal{L}^{\mathcal{F}}(\hat{X}) \\ (6) \quad & {}_8\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_5\mathcal{L}^{\mathcal{F}}(\hat{X}) \cap {}_6\mathcal{L}^{\mathcal{F}}(\hat{X}) \\ (7) \quad & {}_7\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_5\mathcal{U}^{\mathcal{F}}(\hat{X}) \cap {}_6\mathcal{U}^{\mathcal{F}}(\hat{X}) \\ (8) \quad & {}_8\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_5\mathcal{U}^{\mathcal{F}}(\hat{X}) \cup {}_6\mathcal{U}^{\mathcal{F}}(\hat{X}) \end{aligned}$$

Proof. It is clear by using Definition 6. $\square$

Proposition 3. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. Then, $\vee \mathcal{M} \mathcal{D}_x^\beta = \mathcal{M} \mathcal{D}_x^\beta$ for every

$$x \in \Omega \Leftrightarrow \text{for each } \hat{X} \in \mathcal{F}(\Omega), \text{ either } {}_1\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_2\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_3\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_4\mathcal{L}^{\mathcal{F}}(\hat{X}) \text{ or } {}_1\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_2\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_3\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_4\mathcal{U}^{\mathcal{F}}(\hat{X}).$$

Proof. The proof is obtained by using Definition 5 and Proposition 1. $\square$

Proposition 4. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. Then, $\wedge \mathcal{M} \mathcal{D}_x^\beta = \mathcal{M} \mathcal{D}_x^\beta$ for every $x \in \Omega \Leftrightarrow \text{for each } \hat{X} \in \mathcal{F}(\Omega), \text{ either } {}_5\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_6\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_7\mathcal{L}^{\mathcal{F}}(\hat{X}) = {}_8\mathcal{L}^{\mathcal{F}}(\hat{X}) \text{ or } {}_5\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_6\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_7\mathcal{U}^{\mathcal{F}}(\hat{X}) = {}_8\mathcal{U}^{\mathcal{F}}(\hat{X}).$

Proof. The proof is obtained by using Definition 6 and Proposition 2. $\square$

Proposition 5. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. Then, $\vee \mathcal{M} \mathcal{D}_x^\beta = \tilde{N}_x^\beta$, $\forall x \in \Omega \Leftrightarrow$

$$\begin{aligned} (1) \quad & {}_1\mathcal{L}^{\mathcal{F}}(\hat{X}) = \tilde{C}_1^-(\hat{X}) \text{ or } {}_1\mathcal{U}^{\mathcal{F}}(\hat{X}) = \tilde{C}_1^+(\hat{X}) \\ (2) \quad & {}_2\mathcal{L}^{\mathcal{F}}(\hat{X}) = \tilde{C}_2^-(\hat{X}) \text{ or } {}_2\mathcal{U}^{\mathcal{F}}(\hat{X}) = \tilde{C}_2^+(\hat{X}) \\ (3) \quad & {}_3\mathcal{L}^{\mathcal{F}}(\hat{X}) = \tilde{C}_3^-(\hat{X}) \text{ or } {}_3\mathcal{U}^{\mathcal{F}}(\hat{X}) = \tilde{C}_3^+(\hat{X}) \\ (4) \quad & {}_4\mathcal{L}^{\mathcal{F}}(\hat{X}) = \tilde{C}_4^-(\hat{X}) \text{ or } {}_4\mathcal{U}^{\mathcal{F}}(\hat{X}) = \tilde{C}_4^+(\hat{X}) \end{aligned}$$

Proof. The proof is obtained by using Definitions 3 and 5. $\square$

Proposition 6. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. $\wedge \mathcal{M} \mathcal{D}_x^\beta = \tilde{N}_x^\beta$, $\forall x \in \Omega \Leftrightarrow$

$$\begin{aligned} (1) \quad & {}_5\mathcal{L}^{\mathcal{F}}(\hat{X}) = \tilde{C}_1^-(\hat{X}) \text{ or } {}_5\mathcal{U}^{\mathcal{F}}(\hat{X}) = \tilde{C}_1^+(\hat{X}) \\ (2) \quad & {}_6\mathcal{L}^{\mathcal{F}}(\hat{X}) = \tilde{C}_2^-(\hat{X}) \text{ or } {}_6\mathcal{U}^{\mathcal{F}}(\hat{X}) = \tilde{C}_2^+(\hat{X}) \\ (3) \quad & {}_7\mathcal{L}^{\mathcal{F}}(\hat{X}) = \tilde{C}_3^-(\hat{X}) \text{ or } {}_7\mathcal{U}^{\mathcal{F}}(\hat{X}) = \tilde{C}_3^+(\hat{X}) \\ (4) \quad & {}_8\mathcal{L}^{\mathcal{F}}(\hat{X}) = \tilde{C}_4^-(\hat{X}) \text{ or } {}_8\mathcal{U}^{\mathcal{F}}(\hat{X}) = \tilde{C}_4^+(\hat{X}) \end{aligned}$$

Proof. The proof is obtained by using Definitions 3 and 6. $\square$

Proposition 7. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. Then, the following affirmation holds true:

$$\begin{aligned} (1) \quad & {}_4\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_1\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_3\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \\ (2) \quad & {}_4\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_2\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_3\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \\ (3) \quad & {}_3\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_1\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_4\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \\ (4) \quad & {}_3\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_2\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_4\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \end{aligned}$$

Proof. It is obvious from Definition 7. $\square$

Proposition 8. Presume that $(\Omega, \tilde{\Gamma})$ be an FCAS for a fixed $\beta \in (0, 1]$. Select $\hat{X} \in \mathcal{F}(\Omega)$. Then, the following affirmation holds true:

$$\begin{aligned} (1) \quad & {}_8\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_5\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_7\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \\ (2) \quad & {}_8\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_6\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_7\mathcal{L}_{vp}^{\mathcal{F}}(\hat{X}) \\ (3) \quad & {}_7\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_5\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_8\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \\ (4) \quad & {}_7\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_6\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \subseteq {}_8\mathcal{U}_{vp}^{\mathcal{F}}(\hat{X}) \end{aligned}$$

Proof. It is obvious from Definition 7. $\square$

6. Proposed Decision-Making Approach

Now, we use the presented study to obtain a decision on a realistic issue.

6.1. Description and Process. Suppose that $\Omega = \{u_1, u_2, \dots, u_n\}$ be n candidates and $\tilde{\Gamma} = \{\tilde{\Gamma}_1, \tilde{\Gamma}_2, \dots, \tilde{\Gamma}_m\}$ be a set of features. Thus, $\tilde{\Gamma}(u_j)$ denotes the experts rating value related to the candidate u_j to the attribute $\tilde{\Gamma}$, and we assume that for a fixed $\beta \in (0, 1]$, $(\Omega, \tilde{\Gamma})$ is an FCAS. Based on the explained covering methods, we submit an algorithm that seeks to obtain the best choice as in the following procedures:

Step 1. Produce a matrix $\mathcal{D}_{\mathcal{M}}$ as

$$\mathcal{D}_{\mathcal{M}} = \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1m} \\ e_{21} & e_{22} & \cdots & e_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & \cdots & e_{nm} \end{pmatrix}, \quad (26)$$

where e_{ij} is the experts rating value of alternative u_i .
Step 2. Count the favorable ideal $I^{\oplus}$ and the unfavorable ideal $I^{\ominus}$ fuzzy sets with the help of the following equations:

$$\begin{aligned} I^{\oplus} &= \left\{ \left(u_1, \bigvee_{1 \leq j \leq m} e_{1j} \right), \left(u_2, \bigvee_{1 \leq j \leq m} e_{2j} \right), \dots, \left(u_n, \bigvee_{1 \leq j \leq m} e_{nj} \right) \right\}, \\ I^{\ominus} &= \left\{ \left(u_1, \bigwedge_{1 \leq j \leq m} e_{1j} \right), \left(u_2, \bigwedge_{1 \leq j \leq m} e_{2j} \right), \dots, \left(u_n, \bigwedge_{1 \leq j \leq m} e_{nj} \right) \right\}, \end{aligned} \quad (27)$$

where $\vee$ and $\wedge$ symbolize to “maximum” and “minimum,” respectively.

Set the integrated ideal fuzzy set $\mathcal{N}_{\mathcal{J}}$ as follows:

$$\mathcal{N}_{\mathcal{J}} = \left\{ \left(u_1, \sum_{1 \leq j \leq m} e_{1j} w_j \right), \left(u_2, \sum_{1 \leq j \leq m} e_{2j} w_j \right), \dots, \left(u_n, \sum_{1 \leq j \leq m} e_{nj} w_j \right) \right\}, \quad (28)$$

where $\mathcal{W}_{\mathcal{J}} = (w_1, w_2, \dots, w_m)$ is the weight vector, $\sum_{r=1}^m w_r = 1$, and $w_j \in [0, 1]$ for every $j \in \{1, 2, \dots, n\}$.

Step 3. Count the lower and upper approximations of $I^{\oplus}$, $I^{\ominus}$, and $\mathcal{N}_{\mathcal{J}}$ as defined by the 7-FCFCSTFRS model.

Step 4. Count the respective distances $\overline{\mathcal{D}}$ and $\underline{\mathcal{D}}$ between the lower approximation of $\mathcal{N}_{\mathcal{J}}$ and the lower approximation of $I^{\oplus}$ and also the distance among their parallel upper approximations as follows:

$$\begin{aligned} \overline{\mathcal{D}}_i &= \nu(\mathcal{D}(\tilde{\Gamma}^{\ominus}(I^{\oplus})(u_i), \tilde{\Gamma}^{\ominus}(\mathcal{N}_{\mathcal{J}})(u_i))) + (1 - \nu)(\mathcal{D}(\tilde{\Gamma}^{\oplus}(I^{\oplus})(u_i), \tilde{\Gamma}^{\oplus}(\mathcal{N}_{\mathcal{J}})(u_i))), \\ \underline{\mathcal{D}}_i &= \nu(\mathcal{D}(\tilde{\Gamma}^{\ominus}(I^{\ominus})(u_i), \tilde{\Gamma}^{\ominus}(\mathcal{N}_{\mathcal{J}})(u_i))) + (1 - \nu)(\mathcal{D}(\tilde{\Gamma}^{\oplus}(I^{\ominus})(u_i), \tilde{\Gamma}^{\oplus}(\mathcal{N}_{\mathcal{J}})(u_i))), \end{aligned} \quad (29)$$

where ν is a controlling result which is determined by the experts and $\mathcal{D}(x, y) = |x - y|$.

Step 5. Count the closeness coefficient grade by

$$R_I(u_i) = \frac{\underline{\mathcal{D}}(u_i)}{\underline{\mathcal{D}}(u_i) + \overline{\mathcal{D}}(u_i)}, \quad (30)$$

and hence order the candidates.

Based on these procedures, we establish an algorithm to fix the decision-making issues according to 7-FCSTFRS paradigm. The steps corresponding to it are summarized in Algorithm 1.

6.2. Illustrative Example. The above procedures have been clarified with a test example as follows.

Example 4 (see [48]). Nominee engineers form a set $\Omega = \{u_1, u_2, \dots, u_6\}$, and the related characteristics are combined via attribute set $\tilde{\Gamma} = \{\text{passionate firmness } (\Gamma_1), \text{verbal connection expertise } (\Gamma_2), \text{individuality } (\Gamma_3), \text{former practice } (\Gamma_4), \text{aplomb } (\Gamma_5)\}$. Here, we proceed with the above-declared algorithm as in the following steps.

Step 1. The evaluation for each nominee through the decision-makers is summarized in Table 12.

Input: Fuzzy information systems.

Output: Decision-making.

- (1) Get $\hat{X}$, β , $\tilde{\Gamma}$, and Ω
- (2) Establish $\mathcal{D}_M$
- (3) By Definition 2, count $\bigwedge \mathcal{M}_u^\beta$
- (4) By Step (3), count $\bigwedge \mathcal{M}_u^\beta$
- (5) By Steps (3) and (4), count $\bigwedge \mathcal{M}_u^\beta \cap \bigwedge \mathcal{M}_u^\beta$
- (6) Count $\mathcal{I}^\oplus$ and $\mathcal{I}^\ominus$
- (7) Count $\mathcal{N}_{\mathcal{F}}$
- (8) Count the lower and upper approximations of $\mathcal{I}^\oplus, \mathcal{I}^\ominus$, and $\mathcal{N}_{\mathcal{F}}$, according to Definition 6
- (9) Count $\overline{\mathcal{D}}_i$ and $\underline{\mathcal{D}}_i$
- (10) Count $\mathcal{R}_l$
- (11) Get the judgment

ALGORITHM 1: Algorithm for MADM.

Step 2. Based on the significance of these five features, we have $\mathcal{W}_{\mathcal{F}} = \{0.1, 0.3, 0.25, 0.15, 0.2\}$, we calculate $\mathcal{I}^\oplus, \mathcal{I}^\ominus$, and $\mathcal{N}_{\mathcal{F}}$, and outcomes are listed in Table 13.

Step 3. Using $\mathcal{F}_m$ and $\mathcal{T}_M$, through 7-FCSTFRS, we obtain the the following values:

$$\begin{aligned} {}_7\mathcal{L}^{\mathcal{F}}(\hat{X})(x) &= \bigwedge_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M}_x^\beta(y) \cap \bigwedge \mathcal{M}_x^\beta(y), \hat{X}(y) \right\}, \\ {}_7\mathcal{U}^{\mathcal{F}}(\hat{X})(x) &= \bigvee_{y \in \Omega} \mathcal{F} \left\{ \bigwedge \mathcal{M}_x^\beta(y) \cap \bigwedge \mathcal{M}_x^\beta(y), \hat{X}(y) \right\}. \end{aligned} \quad (31)$$

So, we evaluate $\bigwedge \mathcal{M}_x^\beta$ and $\bigwedge \mathcal{M}_x^\beta$, and the counted outcomes are shown in Tables 14 and 15, and hence we compute $\bigwedge \mathcal{M}_u^\beta \cap \bigwedge \mathcal{M}_u^\beta$ whose result is listed in Table 16.

Based on these information, we get

$$\begin{aligned} \tilde{\Gamma}^\ominus(\mathcal{F}^\oplus) &= \frac{0.7}{u_1} + \frac{0.7}{u_2} + \frac{0.7}{u_3} + \frac{0.7}{u_4} + \frac{0.8}{u_5} + \frac{0.7}{u_6}, \\ \tilde{\Gamma}^\oplus(\mathcal{F}^\oplus) &= \frac{0.5}{u_1} + \frac{0.6}{u_2} + \frac{0.5}{u_3} + \frac{0.5}{u_4} + \frac{0.2}{u_5} + \frac{0.5}{u_6}, \\ \tilde{\Gamma}^\ominus(\mathcal{F}^\ominus) &= \frac{0.5}{u_1} + \frac{0.4}{u_2} + \frac{0.5}{u_3} + \frac{0.5}{u_4} + \frac{0.8}{u_5} + \frac{0.5}{u_6}, \\ \tilde{\Gamma}^\oplus(\mathcal{F}^\ominus) &= \frac{0.3}{u_1} + \frac{0.3}{u_2} + \frac{0.3}{u_3} + \frac{0.2}{u_4} + \frac{0.1}{u_5} + \frac{0.2}{u_6}, \\ \tilde{\Gamma}^\ominus(\mathcal{N}_{\mathcal{F}}) &= \frac{0.5}{u_1} + \frac{0.4}{u_2} + \frac{0.5}{u_3} + \frac{0.535}{u_4} + \frac{0.8}{u_5} + \frac{0.515}{u_6}, \\ \tilde{\Gamma}^\oplus(\mathcal{N}_{\mathcal{F}}) &= \frac{0.405}{u_1} + \frac{0.39}{u_2} + \frac{0.475}{u_3} + \frac{0.5}{u_4} + \frac{0.2}{u_5} + \frac{0.5}{u_6}. \end{aligned} \quad (32)$$

Step 4. Count $\overline{\mathcal{D}}_i$ and $\underline{\mathcal{D}}_i$ indicated as follows:

$$\begin{aligned} \overline{\mathcal{D}}_i &= \frac{0.1475}{u_1} + \frac{0.255}{u_2} + \frac{0.1125}{u_3} + \frac{0.0825}{u_4} + \frac{0}{u_5} + \frac{0.0925}{u_6}, \\ \underline{\mathcal{D}}_i &= \frac{0.0525}{u_1} + \frac{0.045}{u_2} + \frac{0.0875}{u_3} + \frac{0.1675}{u_4} + \frac{0.05}{u_5} + \frac{0.1575}{u_6}. \end{aligned} \quad (33)$$

Step 5. According to the results on Step 4, we count $\mathcal{R}_l$ as follows:

$$\mathcal{R}_l = \frac{0.2625}{u_1} + \frac{0.15}{u_2} + \frac{0.4375}{u_3} + \frac{0.67}{u_4} + \frac{1}{u_5} + \frac{0.63}{u_6}, \quad (34)$$

and hence get the ranking order as

TABLE 12: Values of $\mathcal{D}_{\mathcal{M}}$.

	u_1	u_2	u_3	u_4	u_5	u_6
$\bar{\Gamma}_1$	0.7	0.6	0.4	0.5	0.1	0.6
$\bar{\Gamma}_2$	0.5	0.3	0.3	0.7	0.4	0.8
$\bar{\Gamma}_3$	0.4	0.3	0.5	0.5	0.2	0.4
$\bar{\Gamma}_4$	0.3	0.7	0.8	0.2	0.6	0.1
$\bar{\Gamma}_5$	0.2	0.3	0.5	0.6	0.1	0.5

TABLE 13: Values of $\mathcal{I}^\oplus$, $\mathcal{I}^\ominus$, and $\mathcal{N}_{\mathcal{J}}$.

	u_1	u_2	u_3	u_4	u_5	u_6
$\mathcal{J}^\oplus$	0.7	0.7	0.8	0.7	0.6	0.8
$\mathcal{J}^\ominus$	0.2	0.3	0.3	0.2	0.1	0.1
$\mathcal{N}_{\mathcal{J}}$	0.405	0.39	0.475	0.535	0.29	0.515

TABLE 14: Values of $\wedge \widetilde{\mathcal{MD}}_x^\beta$.

	u_1	u_2	u_3	u_4	u_5	u_6
$\wedge \widetilde{\mathcal{MD}}_{u_1}^{0.5}$	0.5	0.3	0.3	0.5	0.1	0.6
$\wedge \widetilde{\mathcal{MD}}_{u_2}^{0.5}$	0.3	0.6	0.4	0.2	0.1	0.1
$\wedge \widetilde{\mathcal{MD}}_{u_3}^{0.5}$	0.2	0.3	0.5	0.2	0.1	0.1
$\wedge \widetilde{\mathcal{MD}}_{u_4}^{0.5}$	0.2	0.3	0.3	0.5	0.1	0.4
$\wedge \widetilde{\mathcal{MD}}_{u_5}^{0.5}$	0.3	0.7	0.8	0.2	0.6	0.1
$\wedge \widetilde{\mathcal{MD}}_{u_6}^{0.5}$	0.2	0.3	0.3	0.5	0.1	0.5

TABLE 15: Values of $\widetilde{\mathcal{MD}}_x^\beta(y)$.

	u_1	u_2	u_3	u_4	u_5	u_6
$\wedge_{\mathcal{D}} \widetilde{\mathcal{M}}_{u_1}^{0.5}$	0.5	0.3	0.2	0.2	0.3	0.2
$\wedge_{\mathcal{D}} \widetilde{\mathcal{M}}_{u_2}^{0.5}$	0.3	0.6	0.3	0.3	0.7	0.3
$\wedge_{\mathcal{D}} \widetilde{\mathcal{M}}_{u_3}^{0.5}$	0.3	0.4	0.5	0.3	0.8	0.3
$\wedge_{\mathcal{D}} \widetilde{\mathcal{M}}_{u_4}^{0.5}$	0.5	0.2	0.2	0.5	0.2	0.5
$\wedge_{\mathcal{D}} \widetilde{\mathcal{M}}_{u_5}^{0.5}$	0.1	0.1	0.1	0.1	0.6	0.1
$\wedge_{\mathcal{D}} \widetilde{\mathcal{M}}_{u_6}^{0.5}$	0.6	0.1	0.1	0.4	0.1	0.5

$$u_5 \geq u_4 \geq u_6 \geq u_3 \geq u_1 \geq u_2. \quad (35)$$

By the above procedures and counting, you can say that the 5th engineer is the best nominee among the others.

6.3. Comparative Analysis. This section aims to explain the merit of the proposed method by comparing it to the last study in the area which was given by Zhang et al. [48]. The purpose here is it is able to raise the lower approximation and down the upper approximation of Zhang et al.'s models as you see in Example 4. We established Tables 17 and 18 to explain the differences between Zhang et al.'s model and our models.

To view another way to illustrate this comparison between Zhang et al.'s [48] and our's, we present Figures 1 and 2.

Figure 1 displays the relations among Zhang-lower positive ideal, Zhang-lower negative ideal, and Zhang-lower integrated ideal and our-lower positive ideal, our-lower

negative ideal, and our-lower integrated ideal. From this point of view, you can see that our-lower (positive, negative, and integrated ideal) is superior to Zhang-lower (positive, negative, and integrated ideal). This means that our-lower is better than Zhang-lower.

Figure 2 discusses the links among Zhang-upper positive ideal, Zhang-upper negative ideal, and Zhang-upper integrated ideal and our-upper positive ideal, our-upper negative ideal, and our-upper integrated ideal. Thus, you can see that our-upper (positive, negative, and integrated ideal) is down than Zhang-upper (positive, negative, and integrated ideal). This means that our-upper is lower than Zhang-upper.

Through Tables 17 and 18 and Figures 1 and 2, it is easy to say that our work is considered as an improvement of Zhang et al.'s [48] work via raising the lower approximation and down the upper approximation (for instance, see Example 4). This means that our method is effective and reliable.

TABLE 16: The result for $\wedge \mathcal{M}_u^\beta \cap \mathcal{M}_u^\beta$.

	u_1	u_2	u_3	u_4	u_5	u_6
$\wedge \mathcal{M}_{u_1}^{0.5} \cap \mathcal{M}_{u_1}^{0.5}$	0.5	0.3	0.2	0.2	0.1	0.2
$\wedge \mathcal{M}_{u_2}^{0.5} \cap \mathcal{M}_{u_2}^{0.5}$	0.3	0.6	0.3	0.2	0.1	0.1
$\wedge \mathcal{M}_{u_3}^{0.5} \cap \mathcal{M}_{u_3}^{0.5}$	0.2	0.3	0.5	0.2	0.1	0.1
$\wedge \mathcal{M}_{u_4}^{0.5} \cap \mathcal{M}_{u_4}^{0.5}$	0.2	0.2	0.2	0.5	0.1	0.4
$\wedge \mathcal{M}_{u_5}^{0.5} \cap \mathcal{M}_{u_5}^{0.5}$	0.1	0.1	0.1	0.1	0.2	0.1
$\wedge \mathcal{M}_{u_6}^{0.5} \cap \mathcal{M}_{u_6}^{0.5}$	0.2	0.1	0.1	0.4	0.1	0.5

TABLE 17: Values by FCITFRS.

Different processes	Obtain a value					
	u_1	u_2	u_3	u_4	u_5	u_6
Zhang's model [48]	0.2625	0.15	0.4375	0.67	0.2714	0.63
Our model	0.2625	0.15	0.4375	0.67	1	0.63

TABLE 18: Values by FCITFRS.

Different processes	Obtain a decision
Zhang's process [48]	$u_4 \geq u_6 \geq u_3 \geq u_5 \geq u_1 \geq u_2$
Our process	$u_5 \geq u_4 \geq u_6 \geq u_1 \geq u_2$

FIGURE 1: The lower approximation using our model and Zhang et al.'s [48] model.

FIGURE 2: The upper approximation using our model and Zhang et al.'s [48] model.

7. Conclusion

The major purpose of the presented work is to improve Zhang et al.'s model [48] and Jiang et al.'s model [49]. In this paper, we considered the issue of the neighborhood-related β covering the classification of approximation spaces. Eight new paradigms of FC $\mathcal{FTFRS}$ s models can be viewed as an improvement of Zhang et al. [48] model through concepts of fuzzy β -minimal description and fuzzy β -maximal description. Also, we introduce eight kinds CVP $\mathcal{FTFRS}$ s as generalizations of Jiang et al. [49] model. Further, we study the relations between our models and the previous models in [48, 49]. After that, we place the present study on a real issue to show its effectiveness and reliability.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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