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Research Article

A Multiparameter Hardy–Hilbert-Type Inequality Containing Partial Sums as the Terms of Series

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In this study, a multiparameter Hardy–Hilbert-type inequality for double series is established, which contains partial sums as the terms of one of the series. Based on the obtained inequality, we discuss the equivalent statements of the best possible constant factor related to several parameters. Moreover, we illustrate how the inequality obtained can generate some new Hardy–Hilbert-type inequalities.

1. Introduction

In [1], Krnić and Pečarić proposed an interesting result on the extension of Hardy–Hilbert inequality with two parameters, as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p((1-\lambda_1)-1}a_m^p\right]^{(1/p)} \cdot \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right]^{(1/q)}, \tag{1}$$

where $p>1$, $(1/p)+(1/q)=1, a_m,\ b_n\geq 0, 0<\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p<\infty$, $0<\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q<\infty$, $\lambda_i\in(0,2]$ $(i=1,2)$, $\lambda_1+\lambda_2=\lambda\in(0,4]$. The constant factor $B(\lambda_1,\lambda_2)$ in (1) is the best possible, which is represented by the beta function.

$$B(u,v)=\int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad (u,v>0). \tag{2}$$

Obviously, for $\lambda_1=1-(1/p), \lambda_2=1-(1/q)$, inequality (1) reduces to the classical Hardy–Hilbert inequality ([2], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{(1/p)}\left(\sum_{n=1}^{\infty}b_n^q\right)^{(1/q)}. \tag{3}$$

In a special case, when $\lambda_1=\lambda_2=(\lambda/2), p=q=2$, inequality (1) leads to a generalization of Hilbert inequality, i.e.,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}}<B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\sum_{m=1}^{\infty}m^{1-\lambda}a_m^2\right)^{(1/2)}\left(\sum_{n=1}^{\infty}n^{1-\lambda}b_n^2\right)^{(1/2)}. \tag{4}$$

Recently, by introducing more parameters, Yang et al. [3] established a further extension of inequality (1), as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\lambda}}<\left(\frac{1}{\beta}k_{\lambda}(\lambda_2)\right)^{(1/p)}\left(\frac{1}{\alpha}k_{\lambda}(\lambda_1)\right)^{(1/q)} \times\left\{\sum_{m=1}^{\infty}m^{p[1-\alpha(((\lambda-\lambda_2)/p)+(\lambda_1/q))]-1}a_m^p\right\}^{(1/p)} \cdot\left\{\sum_{n=1}^{\infty}n^{q[1-\beta(((\lambda-\lambda_1)/q)+(\lambda_2/p))]-1}b_n^q\right\}^{(1/q)}. \tag{5}$$

Inspired by the inequality (5) above, in this study, we construct and prove a new Hardy–Hilbert-type inequality by replacing the term of the series b_n with $B_n=\sum_{k=1}^{n}b_k\,(n=$

1, 2, ...) in the right-hand side of inequality (5). Our method is mainly based on real analysis techniques and the applications of the Euler–Maclaurin summation formula and Abel's partial summation formula. For details of various clever uses of these techniques, we refer the readers to [4–10].

The rest of the study is organized as follows. We first give some lemmas on the construction of weight function and several identities and inequalities related to the weight function. The results are then applied to derive a multiparameter Hardy–Hilbert-type inequality containing partial sums as the terms of one of the series. Finally, we illustrate the applications of the obtained inequality in discovering new Hardy–Hilbert-type inequalities.

2. Some Lemmas

Let us first state the following specified conditions (C1) that we will use in what follows. We suppose that

(i) **(C1)** $p>1, (1/p)+(1/q)=1\alpha, \beta\in(0,1], \lambda\in(0,5], \lambda_1\in(0,(2/\alpha)]\cap(0,\lambda+1), \lambda_2\in(0,(2/\beta)-1]\cap(0,\lambda+1), \widehat{\lambda}_1:=((\lambda-\lambda_2)/p)+(\lambda_1/q), \widehat{\lambda}_2:=((\lambda-\lambda_1)/q)+(\lambda_2/p), k_{\lambda+1}(\lambda_i):=B(\lambda_i,\lambda+1-\lambda_i)(i=1,2)$. Furthermore, for $a_m, b_n\geq 0$, the partial sum B_n is defined by $B_n:=\sum_{k=1}^{n} b_k (n\in\{1,2,\ldots\})$, such that $B_n=o(e^{tn^\beta})(t>0;n\longrightarrow\infty)$ with

$$0<\sum_{m=1}^{\infty} m^{p\left(1-\alpha\widehat{\lambda}_1\right)-1}a_m^p<\infty,$$
$$0<\sum_{n=1}^{\infty} n^{q\left[1-\beta\left(1+\widehat{\lambda}_2\right)\right]-1}B_n^q<\infty. \tag{6}$$

Lemma 1 (See [11]).

(i) Let $(-1)^i(d^i/dt^i)g(t)>0$, $t\in[m,\infty)(m\in N)$ with $g^{(i)}(\infty)=0$ $g^{(i)}(\infty)=0(i=0,1,2,3)$, and let $P_i(t)$ and $B_i(i\in N)$ denote, respectively, the Bernoulli functions and the Bernoulli numbers of i-order. Then,

$$\int_m^{\infty} P_{2q-1}(t)g(t)\mathrm{d}t=-\varepsilon_q\frac{B_{2q}}{2q}g(m),$$
$$\left(0<\varepsilon_q<1;q=1,2,\ldots\right). \tag{7}$$

(ii) In particular, when $q=1$, in view of $B_2=(1/6)$, we have

$$-\frac{1}{12}g(m)<\int_m^{\infty}P_1(t)g(t)\mathrm{d}t<0. \tag{8}$$

(iii) When $q=2$, in view of $B_4=-(1/30)$, we have

$$0<\int_m^{\infty}P_3(t)g(t)\mathrm{d}t<\frac{1}{120}g(m). \tag{9}$$

(iv) (See [11]). If $f(t)(>0)\in C^3[m,\infty), f^{(i)}(\infty)=0(i=0,1,2,3)$, then we have the following Euler–Maclaurin summation formulas:

$$\sum_{k=m}^{\infty} f(k)=\int_m^{\infty}f(t)\mathrm{d}t+\frac{1}{2}f(m)+\int_m^{\infty}P_1(t)f'(t)\mathrm{d}t, \tag{10}$$

$$\int_m^{\infty}P_1(t)f'(t)\mathrm{d}tt=-\frac{1}{12}f'(m)+\frac{1}{6}\int_m^{\infty}P_3(t)f'''(t)\mathrm{d}t. \tag{11}$$

Lemma 2. *For $s\in(0,6]$, $s_2\in(0,(2/\beta)]\cap(0,s), k_s(s_2)=B(s_2,s-s_2)$, we define the weight coefficient as follows:*

$$\varpi_\alpha(s_2,m):=m^{\alpha(s-s_2)}\sum_{n=1}^{\infty}\frac{\beta n^{\beta s_2-1}}{\left(m^\alpha+n^\beta\right)^s},\quad (m\in N). \tag{12}$$

Then, the following inequalities hold:

$$0<k_s(s_2)\left(1-O\left(\frac{1}{m^{\alpha s_2}}\right)\right)<\varpi_\alpha(s_2,m)<k_s(s_2),\quad (m\in N), \tag{13}$$

where $O(1/m^{\alpha s_2}):=(1/k_s(s_2))\int_0^{(1/m^\alpha)}(u^{s_2-1}/(1+u)^s)du>0$.

Proof. For fixed $m\in N$, we define the function $g(m,t)$ by

$$g(m,t):=\frac{\beta t^{\beta s_2-1}}{\left(m^\alpha+t^\beta\right)^s},\quad (t>0). \tag{14}$$

By virtue of (10), we have

$$\sum_{n=1}^{\infty}g(m,n)=\int_1^{\infty}g(m,t)\mathrm{d}t+\frac{1}{2}g(m,1)+\int_1^{\infty}P_1(t)g'(m,t)\mathrm{d}t$$
$$=\int_0^{\infty}g(m,t)\mathrm{d}t-h(m),$$
$$h(m):=\int_0^1 g(m,t)\mathrm{d}t-\frac{1}{2}g(m,1)-\int_1^{\infty}P_1(t)g'(m,t)\mathrm{d}t. \tag{15}$$

Note that $-(1/2)g(m,1)=(-\beta/2(m^\alpha+1)^s)$. Integration by parts, we find

$$\begin{aligned}
\int_0^1 g(m,t)\mathrm{d}t &= \beta\int_0^1 \frac{t^{\beta s_2-1}}{\left(m^\alpha+t^\beta\right)^s}\mathrm{d}t \overset{u=t^\beta}{=} \int_0^1 \frac{u^{s_2-1}}{\left(m^\alpha+u\right)^s}\mathrm{d}u \\
&= \frac{1}{s_2}\int_0^1 \frac{\mathrm{d}u^{s_2}}{\left(m^\alpha+u\right)^s} = \frac{1}{s_2}\frac{u^{s_2}}{\left(m^\alpha+u\right)^s}\Big|_0^1 + \frac{s}{s_2}\int_0^1 \frac{u^{s_2}}{\left(m^\alpha+u\right)^{s+1}}\mathrm{d}u \\
&= \frac{1}{s_2}\frac{1}{\left(m^\alpha+1\right)^s} + \frac{s}{s_2\left(s_2+1\right)}\int_0^1 \frac{\mathrm{d}u^{s_2+1}}{\left(m^\alpha+u\right)^{s+1}} \\
&> \frac{1}{s_2}\frac{1}{\left(m^\alpha+1\right)^s} + \frac{s}{s_2\left(s_2+1\right)}\left[\frac{u^{s_2+1}}{\left(m^\alpha+u\right)^{s+1}}\right]_0^1 + \frac{s(s+1)}{s_2\left(s_2+1\right)\left(m^\alpha+1\right)^{s+2}}\int_0^1 u^{s_2+1}\mathrm{d}u \qquad (16)\\
&= \frac{1}{s_2}\frac{1}{\left(m^\alpha+1\right)^s} + \frac{\lambda}{s_2\left(s_2+1\right)}\frac{1}{\left(m^\alpha+1\right)^{s+1}} + \frac{s(s+1)}{s_2\left(s_2+1\right)\left(s_2+2\right)^{s+2}}\frac{1}{\left(m^\alpha+1\right)^{s+1}}, \\
-g'(m,t) &= -\frac{\beta\left(\beta s_2-1\right)t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^s} + \frac{\beta^2 s t^{\beta+\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^{s+1}} \\
&= -\frac{\beta\left(\beta s_2-1\right)t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^s} + \frac{\beta^2 s\left(m^\alpha+t^\beta-m^\alpha\right)t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^{s+1}} = \frac{\beta\left(\beta s-\beta s_2+1\right)t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^s} - \frac{\beta^2 s m^\alpha t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^{s+1}},
\end{aligned}$$

then from $0<s_2\le(2/\beta), 0<\beta\le 1, s_2<s\le 6$, it follows that

$$\begin{aligned}
&(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[\frac{t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^s}\right]>0, \\
&(-1)^i\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left[\frac{t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^{s+1}}\right]>0, \quad (i=0,1,2,3).
\end{aligned} \tag{17}$$

By (8), (9), (10), and (11), we obtain

$$\begin{aligned}
\beta\left(\beta s-\beta s_2+1\right)\int_1^\infty P_1(t)\frac{t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^s}\mathrm{d}t &> -\frac{\beta\left(\beta s-\beta s_2+1\right)}{12\left(m^\alpha+1\right)^s}, \\
-\beta^2 m^\alpha s\int_1^\infty P_1(t)\frac{t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^{s+1}}\mathrm{d}t \\
&= \frac{\beta^2 m^\alpha s}{12\left(m^\alpha+1\right)^{s+1}} - \frac{\beta^2 m^\alpha s}{6}\int_1^\infty P_3(t)\left[\frac{t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^{s+1}}\right]''\mathrm{d}t \\
&> \frac{\beta^2 m^\alpha s}{12\left(m^\alpha+1\right)^{s+1}} - \frac{\beta^2 m^\alpha s}{720}\left[\frac{t^{\beta s_2-2}}{\left(m^\alpha+t^\beta\right)^{s+1}}\right]''_{t=1}
\end{aligned}$$

$$
\begin{aligned}
&> \frac{\beta^2 (m^\alpha + 1 - 1)s}{12 (m^\alpha + 1)^{s+1}} - \frac{\beta^2 (m^\alpha + 1)s}{720} \left[\frac{(s+1)(s+2)\beta^2}{(m^\alpha + 1)^{s+3}} + \frac{\beta (s+1)(5 - \beta - 2\beta s_2)}{(m^\alpha + 1)^{s+2}} \right. \\
&\quad \left. + \frac{(2 - \beta s_2)(3 - \beta s_2)}{(m^\alpha + 1)^{s+1}} \right] \\
&= \frac{\beta^2 s}{12 (m^\alpha + 1)^{s}} - \frac{\beta^2 s}{12 (m^\alpha + 1)^{s+1}} \\
&\quad - \frac{\beta^2 s}{720} \left[\frac{(s+1)(s+2)\beta^2}{(m^\alpha + 1)^{s+2}} + \frac{\beta (s+1)(5 - \beta - 2\beta s_2)}{(m^\alpha + 1)^{s+1}} + \frac{(2 - \beta s_2)(3 - \beta s_2)}{(m^\alpha + 1)^{s}} \right].
\end{aligned} \tag{18}
$$

Thus, we have

$$
h(m) > \frac{1}{(m^\alpha + 1)^s} h_1 + \frac{\lambda}{(m^\alpha + 1)^{s+1}} h_2 + \frac{s(s+1)}{(m^\alpha + 1)^{s+2}} h_3, \tag{19}
$$

where

$$
\begin{aligned}
h_1 &:= \frac{1}{s_2} - \frac{\beta}{2} - \frac{\beta - \beta^2 s_2}{12} - \frac{\beta^2 s (2 - \beta s_2)(3 - \beta s_2)}{720}, \\
h_2 &:= \frac{1}{s_2 (s_2 + 1)} - \frac{\beta^2}{12} - \frac{\beta^3 (s+1)(5 - \beta - 2\beta s_2)}{720}, \\
h_3 &:= \frac{1}{s_2 (s_2 + 1)(s_2 + 2)} - \frac{\beta^4 (s+2)}{720}.
\end{aligned} \tag{20}
$$

It is easy to observe that $h_1 \geq (1/s_2) - (\beta/2) - ((\beta - \beta^2 s_2)/12) - (s\beta^2 (2 - \beta s_2)(3 - \beta s_2)/720) = (g(s_2)/720 s_2)$, where the function $g(\sigma)\,(\sigma \in (0, (2/\beta)])$ is defined by

$$
g(\sigma) := 720 - \left(420\beta + 6s\beta^2\right)\sigma + \left(60\beta^2 + 5s\beta^3\right)\sigma^2 - s\beta^4 \sigma^3. \tag{21}
$$

Therefore, we deduce that, for $\beta \in (0, 1], s \in (0, 6]$,

$$
\begin{aligned}
g'(\sigma) &= -\left(420\beta + 6s\beta^2\right) + 2\left(60\beta^2 + 5s\beta^3\right)\sigma - 3\beta^4 \sigma^2 \\
&\leq -420\beta - 6s\beta^2 + 2\left(60\beta^2 + 5s\beta^3\right)\frac{2}{\beta} \\
&= (14 s\beta - 180)\beta < 0,
\end{aligned} \tag{22}
$$

thus, it follows that $h_1 \geq (g(s_2)/720 s_2) \geq (g(2/\beta)/720 s_2) = (1/6 s_2) > 0$.

We find that for $s_2 \in (0, (2/\beta)]$, $h_2 > (\beta^2/6) - (\beta^2/12) - (5(s+1)\beta^2/720) = ((1/12) - ((s+1)/140))\beta^2 > 0$ and

$$
h_3 \geq \left(\frac{1}{24} - \frac{s+2}{720}\right)\beta^3 > 0, \quad (0 < s \leq 6). \tag{23}
$$

Hence, we have $h(m) > 0$, and then setting $t = m^{\alpha/\beta} u^{1/\beta}$, it follows that

$$
\begin{aligned}
\varpi_\alpha (s_2, m) &= m^{\alpha (s - s_2)} \sum_{n=1}^{\infty} g(m, n) < m^{\alpha (s - s_2)} \int_0^\infty g(m, t) \mathrm{d}t \\
&= \beta m^{\alpha (s - s_2)} \int_0^\infty \frac{t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^s} \mathrm{d}t = \int_0^\infty \frac{u^{s_2 - 1}}{(1 + u)^s} \mathrm{d}u = k_s (s_2).
\end{aligned} \tag{24}
$$

On the other hand, by (10), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} g(m, n) = \int_1^\infty g(m, t) \mathrm{d}t + \frac{1}{2} g(m, 1) + \int_1^\infty P_1 (t) g'(m, t) \mathrm{d}t = \int_1^\infty g(m, t) \mathrm{d}t + H(m), \\
&H(m) := \frac{1}{2} g(m, 1) + \int_1^\infty P_1 (t) g'(m, t) \mathrm{d}t.
\end{aligned} \tag{25}
$$

We have obtained that $(1/2) g(m, 1) = (\beta / 2 (m^\alpha + 1)^s)$ and $g'(m, t) = -(\beta (\beta s - \beta s_2 + 1) t^{\beta s_2 - 2} / (m^\alpha + t^\beta)^s) + (\beta^2 s m^\alpha t^{\beta s_2 - 2} / (m^\alpha + t^\beta)^{s+1})$.

For $s_2 \in (0, (2/\beta)] \cap (0, s), 0 < s \leq 6$, by (8), we find $-\beta (\beta s - \beta s_2 + 1) \int_1^\infty P_1 (t) (t^{\beta s_2 - 2} / (m^\alpha + t^\beta)^s) \mathrm{d}t > 0$ and

$$
\begin{aligned}
&\beta^2 m^\alpha s \int_1^\infty P_1(t) \frac{t^{\beta s_2 - 2}}{\left(m^\alpha + t^\beta\right)^{s+1}} \mathrm{d}t \\
&> -\frac{\beta^2 m^\alpha s}{12\left(m^\alpha + 1\right)^{s+1}} > -\frac{\beta^2 s}{12\left(m^\alpha + 1\right)^s}.
\end{aligned} \tag{26}
$$

Hence, we have

$$
\begin{aligned}
H(m) &> \frac{\beta}{2\left(m^\alpha + 1\right)^s} - \frac{\beta^2 s}{12\left(m^\alpha + 1\right)^s} \\
&\geq \frac{\beta}{2\left(m^\alpha + 1\right)^s} - \frac{6\beta}{12\left(m^\alpha + 1\right)^s} = 0,
\end{aligned} \tag{27}
$$

then, we obtain

$$
\begin{aligned}
\varpi_\alpha(\lambda_2, m) &= m^{\alpha(s-s_2)} \sum_{n=1}^\infty g(m,n) \\
&> m^{\alpha(s-s_2)} \int_1^\infty g(m,t)\mathrm{d}t \\
&= m^{\alpha(s-s_2)} \int_0^\infty g(m,t)\mathrm{d}t - m^{\alpha(s-s_2)} \int_0^1 g(m,t)\mathrm{d}t \\
&= k_s(s_2)\left[1 - \frac{1}{k_s(s_2)} \int_0^{(1/m^\alpha)} \frac{u^{s_2-1}}{(1+u)^s} \mathrm{d}u\right] > 0,
\end{aligned} \tag{28}
$$

where we set $O(1/m^{\alpha s_2}) = (1/k_s(s_2)) \int_0^{(1/m^\alpha)} (u^{s_2-1}/(1+u)^s)\mathrm{d}u$ satisfying

$$
0 < \int_0^{(1/m^\alpha)} \frac{u^{s_2-1}}{(1+u)^s} \mathrm{d}u < \int_0^{(1/m^\alpha)} u^{s_2-1} \mathrm{d}u = \frac{1}{s_2 m^{\alpha s_2}}. \tag{29}
$$

Therefore, we obtain the required inequalities (13). This completes the proof of Lemma 2. □

Lemma 3. *For* $s \in (0,6]$, $s_1 \in (0,(2/\alpha)] \cap (0,s)$, $s_2 \in (0,(2/\beta)] \cap (0,s)$, *we have the following Hardy–Hilbert-type inequality with the internal variables:*

$$
\begin{aligned}
I = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\left(m^\alpha + n^\beta\right)^s} &\leq \left(\frac{1}{\beta} k_s(s_2)\right)^{(1/p)} \left(\frac{1}{\alpha} k_s(s_1)\right)^{(1/q)} \\
&\times \left\{\sum_{m=1}^\infty m^{p[1-\alpha(((s-s_2)/p)+t(s_1/q))]-1} a_m^p\right\}^{(1/p)} \\
&\cdot \left\{\sum_{n=1}^\infty n^{q[1-\beta(((s-s_1)/q)+t(s_2/p))]-1} b_n^q\right\}^{(1/q)}.
\end{aligned} \tag{30}
$$

Proof. In the same way of proving inequality (13), we can prove the following inequalities for the weight coefficient $\omega_\beta(s_1,n)$:

$$
\begin{aligned}
k_s(s_1)\left(1 - O\left(\frac{1}{n^{\beta s_1}}\right)\right) < \omega_\beta(s_1,n) &:= n^{\beta(s-s_1)} \sum_{m=1}^\infty \frac{\alpha m^{\alpha s_1 - 1}}{\left(m^\alpha + n^\beta\right)^s} \\
&< k_s(s_1), \quad (n \in N).
\end{aligned} \tag{31}
$$

By Hölder's inequality ([12]), we obtain

$$
\begin{aligned}
I &= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{\left(m^\alpha + n^\beta\right)^s} \left[\frac{m^{\alpha(1-s_1)/q}\left(\beta n^{\beta-1}\right)^{(1/p)}}{n^{\beta(1-s_2)/p}\left(\alpha m^{\alpha-1}\right)^{(1/q)}} a_m\right] \left[\frac{n^{\beta(1-s_2)/p}\left(\alpha m^{\alpha-1}\right)^{(1/q)}}{m^{\alpha(1-s_1)/q}\left(\beta n^{\beta-1}\right)^{(1/p)}} b_n\right] \\
&\leq \left[\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\beta}{(m^\alpha + n^\beta)^s} \frac{m^{\alpha(1-s_1)(p-1)} n^{\beta-1} a_m^p}{n^{\beta(1-s_2)} (\alpha m^{\alpha-1})^{p-1}}\right]^{(1/p)} \left[\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\alpha}{(m^\alpha + n^\beta)^s} \frac{n^{\beta(1-s_2)((-1)} m^{\alpha-1} b_n^q}{m^{\alpha(1-s_1)} (\beta n^{\beta-1})^{q-1}}\right]^{(1/q)} \\
&= \frac{1}{\alpha^{(1/q)} \beta^{(1/p)}} \left\{\sum_{m=1}^\infty \varpi_\alpha(s_2,m) m^{p[1-\alpha(((s-s_2)/p)+t(s_1/q))]-1} a_m^p\right\}^{(1/p)} \\
&\times \left\{\sum_{n=1}^\infty \omega_\beta(s_1,n) n^{q[1-\beta(((s-s_1)/q)+t(s_2/p))]-1} b_n^q\right\}^{(1/q)}.
\end{aligned} \tag{32}
$$

Furthermore, by using (13) and (31), we get inequality (30). Lemma 3 is proved. □

Remark 1. In particular, for

$$s=\lambda+1\in(1,6],\quad \lambda\in(0,5],$$
$$s_1=\lambda_1\in\left(0,\frac{2}{\alpha}\right]\cap(0,\lambda+1), \tag{33}$$
$$s_2=\lambda_2+1\in\left(1,\frac{2}{\beta}\right],\quad \lambda_2\in\left(0,\frac{2}{\beta}-1\right]\cap(0,\lambda+1),$$

in (30), replacing b_n by B_n, in view of (6), one has

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mB_n}{\left(m^{\alpha}+n^{\beta}\right)^{\lambda+1}}<\left(\frac{1}{\beta}k_{\lambda+1}(\lambda_2+1)\right)^{(1/p)}\left(\frac{1}{\alpha}k_{\lambda+1}(\lambda_1)\right)^{(1/q)}\times\left\{\sum_{m=1}^{\infty}m^{p\left(1-\alpha\widehat{\lambda}_1\right)-1}a_m^p\right\}^{(1/p)}\left\{\sum_{n=1}^{\infty}n^{q\left[1-\beta\left(1+\widehat{\lambda}_2\right)\right]-1}B_n^q\right\}^{(1/q)}. \tag{34}$$

Lemma 4. *For $t>0$, the following inequality holds:*

$$\sum_{n=1}^{\infty}e^{-tn^{\beta}}b_n\le t\sum_{n=1}^{\infty}e^{-tn^{\beta}}B_n. \tag{35}$$

Proof. In view of $B_ne^{-tn^{\beta}}=o(1)\,(t>0;n\longrightarrow\infty)$ and applying Abel's summation by parts formula, we find

$$\begin{aligned}\sum_{n=1}^{\infty}e^{-tn^{\beta}}b_n&=\lim_{n\longrightarrow\infty}B_ne^{-tn^{\beta}}+\sum_{n=1}^{\infty}B_n\left[e^{-tn^{\beta}}-e^{-t(n+1)^{\beta}}\right]\\&=\sum_{n=1}^{\infty}B_n\left[e^{-tn^{\beta}}-e^{-t(n+1)^{\beta}}\right].\end{aligned} \tag{36}$$

Since $1-e^{-t}<t\,(t>0)$ and the fact that for $\beta\in(0,1]$,

$$\begin{aligned}&e^{-t(n+1)^{\beta}}\ge e^{-t\left(n^{\beta}+1\right)}\Leftrightarrow e^{t\left[(n+1)^{\beta}-n^{\beta}-1\right]}\le 1\\&\Leftrightarrow(n+1)^{\beta}-n^{\beta}-1=\beta\left(n+\theta_n\right)^{\beta-1}-1\le 0,\\&\left(\theta_n\in(0,1)\right),\end{aligned} \tag{37}$$

by (36), we have

$$\begin{aligned}\sum_{n=1}^{\infty}e^{-tn^{\beta}}b_n&\le\sum_{n=1}^{\infty}B_n\left[e^{-tn^{\beta}}-e^{-t\left(n^{\beta}+1\right)}\right]\\&=\left(1-e^{-t}\right)\sum_{n=1}^{\infty}B_ne^{-tn^{\beta}}\le t\sum_{n=1}^{\infty}B_ne^{-tn^{\beta}}.\end{aligned} \tag{38}$$

Hence, we derive the inequality (35). The proof of Lemma 4 is complete. □

3. Main Results

Theorem 1. *Under the assumptions described in (C1), we have the following inequality containing partial sums as the terms of series:*

$$I:=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{\left(m^{\alpha}+n^{\beta}\right)^{\lambda}}<\lambda\left(\frac{1}{\beta}k_{\lambda+1}(\lambda_2+1)\right)^{(1/p)}\left(\frac{1}{\alpha}k_{\lambda+1}(\lambda_1)\right)^{(1/q)}\times\left\{\sum_{m=1}^{\infty}m^{p\left(1-\alpha\widehat{\lambda}_1\right)-1}a_m^p\right\}^{(1/p)}\left\{\sum_{n=1}^{\infty}n^{q\left[1-\beta\left(1+\widehat{\lambda}_2\right)\right]-1}B_n^q\right\}^{(1/q)}. \tag{39}$$

In particular, for $\lambda_1+\lambda_2=\lambda$ with

$$\begin{aligned}&\lambda_1\in\left(0,\frac{2}{\alpha}\right]\cap(0,\lambda),\\&\lambda_2\in\left(0,\frac{2}{\beta}-1\right]\cap(0,\lambda),\quad(\lambda\in(0,5])\\&0<\sum_{m=1}^{\infty}m^{p(1-\alpha\lambda_1)-1}a_m^p<\infty,\\&0<\sum_{n=1}^{\infty}n^{q[1-\beta(1+\lambda_2)]-1}B_n^q<\infty,\end{aligned} \tag{40}$$

we have the following inequality:

$$I=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{\left(m^{\alpha}+n^{\beta}\right)^{\lambda}}<\frac{\lambda_2}{\beta^{(1/p)}\alpha^{(1/q)}}B(\lambda_1,\lambda_2)\times\left[\sum_{m=1}^{\infty}m^{p(1-\alpha\lambda_1)-1}a_m^p\right]^{(1/p)}\left\{\sum_{n=1}^{\infty}n^{q[1-\beta(1+\lambda_2)]-1}B_n^q\right\}^{(1/q)}. \tag{41}$$

Proof. By virtue of the fact that

$$\frac{1}{\left(m^{\alpha}+n^{\beta}\right)^{\lambda}}=\frac{1}{\Gamma(\lambda)}\int_0^{\infty}t^{\lambda-1}e^{-\left(m^{\alpha}+n^{\beta}\right)t}\mathrm{d}t, \tag{42}$$

from (35), we obtain

$$\begin{aligned}I&=\frac{1}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_mb_n\int_0^{\infty}t^{\lambda-1}e^{-\left(m^{\alpha}+n^{\beta}\right)t}\mathrm{d}t\\&=\frac{1}{\Gamma(\lambda)}\int_0^{\infty}t^{\lambda-1}\sum_{m=1}^{\infty}e^{-m^{\alpha}t}a_m\sum_{n=1}^{\infty}e^{-n^{\beta}t}b_n\mathrm{d}t\\&\le\frac{1}{\Gamma(\lambda)}\int_0^{\infty}t^{\lambda-1}\sum_{m=1}^{\infty}e^{-m^{\alpha}t}a_m\left(t\sum_{n=1}^{\infty}e^{-n^{\beta}t}B_n\mathrm{d}t\right)\\&=\frac{1}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_mB_n\int_0^{\infty}t^{(\lambda+1)-1}e^{-\left(m^{\alpha}+n^{\beta}\right)t}\mathrm{d}t\\&=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mB_n}{\left(m^{\alpha}+n^{\beta}\right)^{\lambda+1}}\end{aligned} \tag{43}$$

Then, by (34), we derive inequality (39). Theorem 1 is proved. □

Remark 2. Putting $\beta = 1$ in (39), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^\alpha + n)^\lambda} < \frac{\lambda}{\alpha^{(1/q)}}(k_{\lambda+1}(\lambda_2+1))^{(1/p)}(k_{\lambda+1}(\lambda_1))^{(1/q)} \times \left[\sum_{m=1}^{\infty} m^{p(1-\alpha\widehat{\lambda}_1)-1} a_m^p\right]^{(1/p)}\left(\sum_{n=1}^{\infty} n^{-q\widehat{\lambda}_2-1} B_n^q\right)^{(1/q)}. \tag{44}$$

Theorem 2. *If $\lambda_1 + \lambda_2 = \lambda (\in (0,5])$, then the constant factor*

$$\frac{\lambda}{\alpha^{(1/q)}}(k_{\lambda+1}(\lambda_2+1))^{(1/p)}(k_{\lambda+1}(\lambda_1))^{(1/q)}, \tag{45}$$

in (44) is the best possible. Moreover, if

$$\lambda_1, \lambda - \lambda_2 \in \left(0, \frac{2}{\alpha}\right] \cap (0,\lambda), \quad \lambda_2, \lambda - \lambda_1 \in (0,1] \cap (0,\lambda), \tag{46}$$

and the constant factor $(\lambda/\alpha^{(1/q)})(k_{\lambda+1}(\lambda_2+1))^{(1/p)}(k_{\lambda+1}(\lambda_1))^{(1/q)}$ in (44) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$..

Proof. If $\lambda_1 + \lambda_2 = \lambda (\in (0,5])$, then we find $\lambda_1 \in (0,(2/\alpha)] \cap (0,\lambda), \lambda_2 \in (0,1] \cap (0,\lambda)$,

$$k_{\lambda+1}(\lambda_2+1) = k_{\lambda+1}(\lambda_1) = B(\lambda_1,\lambda_2+1) = \frac{\lambda_2}{\lambda}B(\lambda_1,\lambda_2), \tag{47}$$

and then inequality (44) reduces to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^\alpha + n)^\lambda} < \frac{\lambda}{\alpha^{(1/q)}}B(\lambda_1,\lambda_2) \cdot \left(\sum_{m=1}^{\infty} m^{p(1-\alpha\lambda_1)-1}a_m^p\right)^{(1/p)} \cdot \left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q\right)^{(1/q)}. \tag{48}$$

For any $0 < \varepsilon < q\lambda_2$, we set $\widetilde{a}_m := m^{\alpha(\lambda_1-(\varepsilon/p))-1}$, $\widetilde{b}_n := n^{\lambda_2-(\varepsilon/q)-1} (m,n \in N)$. Then, we have

$$\widetilde{B}_n := \sum_{k=1}^{n}\widetilde{b}_k = \sum_{k=1}^{n} k^{\lambda_2-(\varepsilon/p)-1} < \int_0^n t^{\lambda_2-(\varepsilon/q)-1}\mathrm{d}t = \frac{1}{\lambda_2-(\varepsilon/q)}n^{\lambda_2-(\varepsilon/q)}, \quad (n \in N), \tag{49}$$

$$\widetilde{B}_n = o\left(e^{tn}\right), \quad (t > 0; n \longrightarrow \infty).$$

If there exists a positive constant $M \le (\lambda_2/\alpha^{(1/q)})B(\lambda_1,\lambda_2)$, (48) is valid when we replace $(\lambda_2/\alpha^{(1/q)})B(\lambda_1,\lambda_2)$ by M. Then, we have

$$\widetilde{I} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha + n)^\lambda}\widetilde{a}_m\widetilde{b}_n < M\left\{\sum_{m=1}^{\infty} m^{p(1-\alpha\lambda_1)-1}\widetilde{a}_m^p\right\}^{(1/p)} \cdot \left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1}\widetilde{B}_n^q\right)^{(1/q)}. \tag{50}$$

By (50) and the decreasing property of series, we obtain

$$\begin{aligned}
\widetilde{I} &< M\left(\sum_{m=1}^{\infty} m^{p(1-\alpha\lambda_1)-1}m^{p\alpha\lambda_1-\alpha\varepsilon-p}\right)^{(1/p)} \cdot \frac{1}{\lambda_2-(\varepsilon/q)}\left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1}n^{q\lambda_2-\varepsilon}\right)^{(1/q)} \\
&= \frac{M}{\lambda_2-(\varepsilon/q)}\left(1+\sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1}\right)^{(1/p)}\left(1+\sum_{n=2}^{\infty} n^{-\varepsilon-1}\right)^{(1/q)} \\
&< \frac{M}{\lambda_2-(\varepsilon/q)}\left(1+\int_1^{\infty} x^{-\alpha\varepsilon-1}\mathrm{d}x\right)^{(1/q)}\left(1+\int_1^{\infty} y^{-\varepsilon-1}\mathrm{d}y\right)^{(1/q)} \\
&= \frac{1}{\varepsilon\alpha^{(1/p)}}\frac{M}{\lambda_2-(\varepsilon/q)}(\alpha\varepsilon+1)^{(1/p)}(\varepsilon+1)^{(1/q)}.
\end{aligned} \tag{51}$$

By employing (31) (for $\beta = 1, s = \lambda, s_1 = \lambda_1$) and setting

$$\widetilde{\lambda}_1 := \lambda_1 - \frac{\varepsilon}{p} \in \left(0,\frac{2}{\alpha}\right) \cap (0,\lambda), \quad \left(0 < \widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} = \lambda - \widetilde{\lambda}_1 < \lambda\right), \tag{52}$$

we deduce that

$$\begin{aligned}
\widetilde{I} &= \frac{1}{\alpha}\sum_{n=1}^{\infty}\left[n^{\widetilde{\lambda}_2}\sum_{m=1}^{\infty}\frac{\alpha}{(m+n)^\lambda}m^{\alpha\widetilde{\lambda}_1-1}\right]n^{-\varepsilon-1} \\
&= \frac{1}{\alpha}\sum_{n=1}^{\infty}\omega_1\left(\widetilde{\lambda}_1,n\right)n^{-\varepsilon-1} > \frac{1}{\alpha}k_\lambda\left(\widetilde{\lambda}_1\right)\sum_{n=1}^{\infty}\left(1-O\left(\frac{1}{n^{\widetilde{\lambda}_1}}\right)\right)n^{-\varepsilon-1} \\
&= \frac{1}{\alpha}B\left(\widetilde{\lambda}_1,\widetilde{\lambda}_2\right)\left(\sum_{n=1}^{\infty} n^{-\varepsilon-1} - \sum_{n=1}^{\infty} O\left(\frac{1}{n^{\lambda_1-(\varepsilon/p)+1}}\right)\right) \\
&> \frac{1}{\alpha}B\left(\widetilde{\lambda}_1,\widetilde{\lambda}_2\right)\left(\int_1^{\infty} y^{-\varepsilon-1}\mathrm{d}y - O(1)\right) \\
&= \frac{1}{\varepsilon\alpha}B\left(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\right)(1-\varepsilon O(1)).
\end{aligned} \tag{53}$$

Thus, we have

$$\frac{1}{\alpha}B\left(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\right)(1-\varepsilon O(1))$$
$$<\varepsilon\widetilde{I}<\frac{1}{\alpha^{(1/p)}}\frac{M}{\lambda_2-(\varepsilon/q)}(\alpha\varepsilon+1)^{(1/p)}(\varepsilon+1)^{(1/q)}. \tag{54}$$

For $\varepsilon\longrightarrow 0^+$, in view of the continuity of the beta function, we find $(\lambda_2/\alpha^{(1/p)})B(\lambda_1,\lambda_2)\le M$. Hence, $M=(\lambda_2/\alpha^{(1/p)})B(\lambda_1,\lambda_2)$ is the best possible constant factor of (48).

On the other hand, since $\lambda_1,\lambda-\lambda_2\in(0,(2/\alpha)]\cap(0,\lambda),\lambda_2,\lambda-\lambda_1\in(0,1]\cap(0,\lambda)$, we find

$$\widehat{\lambda}_1+\widehat{\lambda}_2=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_1}{p}=\lambda,$$
$$0<\widehat{\lambda}_1,\widehat{\lambda}_2<\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda,\widehat{\lambda}_1\le\frac{2/\alpha}{p}+\frac{2/\alpha}{1}=\frac{2}{\alpha},\quad \widehat{\lambda}_2\le 1, \tag{55}$$

and $\widehat{\lambda}_2k_\lambda(\widehat{\lambda}_1)=\widehat{\lambda}_2B(\widehat{\lambda}_1,\widehat{\lambda}_2)\in R_+=(0,\infty)$.

By using Hölder's inequality ([12]), we obtain

$$\begin{aligned}k_{\lambda+1}\left(\widehat{\lambda}_1\right)&=k_{\lambda+1}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)\\
&=\int_0^\infty\frac{1}{(1+u)^{\lambda+1}}u^{((\lambda-\lambda_2)/p)+(\lambda_1/q)-1}\mathrm{d}u=\int_0^\infty\frac{1}{(1+u)^{\lambda+1}}\left(u^{((\lambda-\lambda_2-1)/p)}\right)\left(u^{(\lambda_1-1/q)}\right)\mathrm{d}u\\
&\le\left[\int_0^\infty\frac{1}{(1+u)^{\lambda+1}}u^{\lambda-\lambda_2-1}\mathrm{d}u\right]^{(1/p)}\left[\int_0^\infty\frac{1}{(1+u)^{\lambda+1}}u^{\lambda_1-1}du\right]^{(1/q)}\\
&=\left[\int_0^\infty\frac{1}{(1+v)^{\lambda+1}}v^{(\lambda_2+1)-1}\mathrm{d}v\right]^{(1/p)}\left[\int_0^\infty\frac{1}{(1+u)^{\lambda+1}}u^{\lambda_1-1}\mathrm{d}u\right]^{(1/q)}\\
&=\left(k_{\lambda+1}(\lambda_2+1)\right)^{(1/p)}\left(k_{\lambda+1}(\lambda_1)\right)^{(1/q)}.\end{aligned} \tag{56}$$

If the constant factor $(\lambda/\alpha^{(1/q)})(k_{\lambda+1}(\lambda_2+1))^{(1/p)}(k_{\lambda+1}(\lambda_1))^{(1/q)}$ in (44) is the best possible, then by (48) (for $\lambda_i=\widehat{\lambda}_i(i=1,2)$), we have the following inequality:

$$\frac{\lambda}{\alpha^{(1/q)}}(k_{\lambda+1}(\lambda_2+1))^{(1/p)}(k_{\lambda+1}(\lambda_1))^{(1/q)}\le\frac{\widehat{\lambda}_2}{\alpha^{(1/q)}}B\left(\widehat{\lambda}_1,\widehat{\lambda}_2\right)$$
$$=\frac{\lambda}{\alpha^{(1/q)}}k_{\lambda+1}\left(\widehat{\lambda}_1\right), \tag{57}$$

namely, $k_{\lambda+1}(\widehat{\lambda}_1)\ge(k_{\lambda+1}(\lambda_2+1))^{(1/p)}(k_{\lambda+1}(\lambda_1))^{(1/q)}$. Hence, (56) keeps the form of equality.

We observe that (56) keeps the form of equality if and only if there exist constants A and B, such that they are not all zero satisfying ([12]) $Au^{\lambda-\lambda_2-1}=Bu^{\lambda_1-1}$ a.e. in R_+.

Assuming that $A\ne 0$, we have $u^{\lambda-\lambda_2-\lambda_1}=(B/A)$ a. e. in R_+, and then, $\lambda-\lambda_2-\lambda_1=0$, that is, $\lambda_1+\lambda_2=\lambda$. The proof of Theorem 2 is complete. □

4. Some Applications

Theorem 3. *Under the assumptions described in (C1), we have the following inequality which is equivalent to inequality (39):*

$$J:=\left\{\sum_{m=1}^\infty m^{q\alpha\widehat{\lambda}_1-1}\left[\sum_{n=1}^\infty\frac{b_n}{(m^\alpha+n^\beta)^\lambda}\right]^q\right\}^{(1/q)}$$
$$<\lambda\left(\frac{1}{\beta}k_{\lambda+1}(\lambda_2+1)\right)^{(1/p)}\left(\frac{1}{\alpha}k_{\lambda+1}(\lambda_1)\right)^{(1/q)} \tag{58}$$
$$\cdot\left\{\sum_{n=1}^\infty n^{q\left[1-\beta\left(1+\widehat{\lambda}_2\right)\right]-1}B_n^q\right\}^{(1/q)}.$$

In particular, for $\lambda_1+\lambda_2=\lambda$ with

$$\lambda_1\in\left(0,\frac{2}{\alpha}\right]\cap(0,\lambda),$$
$$\lambda_2\in\left(0,\frac{2}{\beta}-1\right]\cap(0,\lambda),\quad(\lambda\in(0,5]), \tag{59}$$

we have the following inequality which is equivalent to (41):

$$\left\{\sum_{m=1}^\infty m^{q\alpha\widehat{\lambda}_1-1}\left[\sum_{n=1}^\infty\frac{b_n}{(m^\alpha+n^\beta)^\lambda}\right]^q\right\}^{(1/q)}$$
$$<\frac{\lambda_2}{\beta^{(1/p)}\alpha^{(1/q)}}B(\lambda_1,\lambda_2)\left\{\sum_{n=1}^\infty n^{q[1-\beta(1+\lambda_2)]-1}B_n^q\right\}^{(1/q)}. \tag{60}$$

Proof. Assuming that (58) is valid, by using Hölder's inequality ([12]), we obtain

$$I = \sum_{m=1}^{\infty} m^{(1/q)-\alpha\widehat{\lambda}_1} a_m \left[m^{-(1/q)+\alpha\widehat{\lambda}_1} \sum_{n=1}^{\infty} \frac{b_n}{\left(m^{\alpha}+n^{\beta}\right)^{\lambda}} \right] \leq \left[\sum_{m=1}^{\infty} m^{p\left(1-\alpha\widehat{\lambda}_1\right)-1} a_m^p \right]^{(1/p)} J. \quad (61)$$

Then, by (58), we have (39). On the other hand, assuming that (39) is valid, we set

$$a_m := m^{q\alpha\widehat{\lambda}_1 - 1} \left[\sum_{n=1}^{\infty} \frac{b_n}{\left(m^{\alpha}+n^{\beta}\right)^{\lambda}} \right]^{q-1}, \quad (m \in N). \quad (62)$$

If $J = 0$, then (58) is naturally valid; if $J = \infty$, then it is impossible that makes (58) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (39), we have

$$\sum_{m=1}^{\infty} m^{p\left(1-\alpha\widehat{\lambda}_1\right)-1} a_m^p = J^q = I < \lambda \left(\frac{1}{\beta} k_{\lambda+1}(\lambda_2+1) \right)^{(1/p)} \left(\frac{1}{\alpha} k_{\lambda+1}(\lambda_1) \right)^{(1/q)} J^{q-1} \left\{ \sum_{n=1}^{\infty} n^{q\left[1-\beta\left(1+\widehat{\lambda}_2\right)\right]-1} B_n^q \right\}^{(1/q)},$$
$$J = \left\{ \sum_{m=1}^{\infty} m^{p\left(1-\alpha\widehat{\lambda}_1\right)-1} a_m^p \right\}^{(1/q)} < \lambda \left(\frac{1}{\beta} k_{\lambda+1}(\lambda_2+1) \right)^{(1/p)} \left(\frac{1}{\alpha} k_{\lambda+1}(\lambda_1) \right)^{(1/q)} \left\{ \sum_{n=1}^{\infty} n^{q\left[1-\beta\left(1+\widehat{\lambda}_2\right)\right]-1} B_n^q \right\}^{(1/q)}. \quad (63)$$

Hence, (58) is valid. Therefore, inequality (58) is equivalent to inequality (39). Theorem 3 is proved. □

Remark 3. Putting $\beta = 1$ in (58), we get the following inequality which is equivalent to (44).

$$\left\{ \sum_{m=1}^{\infty} m^{q\alpha\widehat{\lambda}_1 - 1} \left[\sum_{n=1}^{\infty} \frac{b_n}{(m^{\alpha}+n)^{\lambda}} \right]^q \right\}^{(1/q)} < \frac{\lambda}{\alpha^{(1/q)}} \left(k_{\lambda+1}(\lambda_2+1)\right)^{(1/p)} \left(k_{\lambda+1}(\lambda_1)\right)^{(1/q)} \cdot \left(\sum_{n=1}^{\infty} n^{-q\widehat{\lambda}_2 - 1} B_n^q \right)^{(1/q)}. \quad (64)$$

Theorem 4. *If $\lambda_1 + \lambda_2 = \lambda\,(\in (0,5])$, then the constant factor*

$$\frac{\lambda}{\alpha^{(1/q)}} \left(k_{\lambda+1}(\lambda_2+1)\right)^{(1/p)} \left(k_{\lambda+1}(\lambda_1)\right)^{(1/q)}, \quad (65)$$

in (64) is the best possible. Moreover, if

$$\lambda_1, \lambda - \lambda_2 \in \left(0, \frac{2}{\alpha}\right] \cap (0,\lambda), \qquad \lambda_2, \lambda - \lambda_1 \in (0,1] \cap (0,\lambda), \quad (66)$$

and the constant factor $(\lambda/\alpha^{(1/q)})$ $(k_{\lambda+1}(\lambda_2+1))^{(1/p)}$ $(k_{\lambda+1}(\lambda_1))^{(1/q)}$ in (64) is the best possible, then we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. If the constant factor $(\lambda/\alpha^{(1/q)})(k_{\lambda+1}(\lambda_2+1))^{(1/p)}$ $(k_{\lambda+1}(\lambda_1))^{(1/q)}$ in (64) is not the best possible, then by (60) (for $\beta = 1$), we would reach a contradiction that the same constant factor in (44) is not the best possible. On the other hand, if

$$\lambda_1, \lambda - \lambda_2 \in \left(0, \frac{2}{\alpha}\right] \cap (0,\lambda), \qquad \lambda_2, \lambda - \lambda_1 \in (0,1] \cap (0,\lambda), \quad (67)$$

and the constant factor $(\lambda/\alpha^{(1/q)})(k_{\lambda+1}(\lambda_2+1))^{(1/p)}$ $(k_{\lambda+1}(\lambda_1))^{(1/q)}$ in (64) is the best possible, then, in view of the equivalency of (44) and (64) and $I = J^q$, the same constant factor in (20) is the best possible. Furthermore, by Theorem 2, we have $\lambda_1 + \lambda_2 = \lambda$. This completes the proof of Theorem 4. □

Remark 4. If we take $\beta = 1$ in (41) and (59), respectively, then for

$$\lambda_1 \in \left(0, \frac{2}{\alpha}\right] \cap (0,\lambda), \qquad \lambda_2 \in (0,1] \cap (0,\lambda), \quad (\lambda \in (0,5]), \quad (68)$$

we have the following inequalities with the best possible constant factor $(\lambda_2/\alpha^{(1/q)})B(\lambda_1,\lambda_2)$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n)^{\lambda}}<\frac{\lambda_2}{\alpha^{(1/q)}}B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\alpha\lambda_1)-1}a_m^p\right]^{(1/p)}\left(\sum_{n=1}^{\infty}n^{-q\lambda_2-1}B_n^q\right)^{(1/q)}, \tag{69}$$

$$\left\{\sum_{m=1}^{\infty}m^{q\alpha\lambda_1-1}\left[\sum_{n=1}^{\infty}\frac{b_n}{(m^{\alpha}+n)^{\lambda}}\right]^q\right\}^{(1/q)}<\frac{\lambda_2}{\alpha^{(1/q)}}B(\lambda_1,\lambda_2)\left(\sum_{n=1}^{\infty}n^{-q\lambda_2-1}B_n^q\right)^{(1/q)}. \tag{70}$$

(i) Taking $\lambda=\alpha=1,\lambda_1=(1/q),\lambda_2=(1/p)$ in (69) and (70), respectively, we obtain the following inequalities with the best possible constant factor $(\pi/p\sin(\pi/p))$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{p\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{(1/p)}\left[\sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^q\right]^{(1/q)}, \tag{71}$$

$$\left[\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{b_n}{m+n}\right)^q\right]^{(1/q)}<\frac{\pi}{p\sin(\pi/p)}\left[\sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^q\right]^{(1/q)}. \tag{72}$$

(ii) Taking $\lambda=\alpha=1,\lambda_1=(1/p),\lambda_2=(1/q)$ in (69) and (70), respectively, we get the following inequalities with the best possible constant factor $(\pi/q\sin(\pi/p))$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{q\sin(\pi/p)}\left(\sum_{m=1}^{\infty}m^{p-2}a_m^p\right)^{(1/p)}\left(\sum_{n=1}^{\infty}\frac{B_n^q}{n^2}\right)^{(1/q)}, \tag{73}$$

$$\left[\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{b_n}{m+n}\right)^q\right]^{(1/q)}<\frac{\pi}{q\sin(\pi/p)}\left(\sum_{n=1}^{\infty}\frac{B_n^q}{n^2}\right)^{(1/q)}. \tag{74}$$

In particular, if we take $p=q=2$, then both (71) and (73) reduce to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{2}\left[\sum_{m=1}^{\infty}a_m^2\sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^2\right]^{(1/2)}, \tag{75}$$

and both (72) and (74) reduce to

$$\left[\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{b_n}{m+n}\right)^2\right]^{(1/2)}<\frac{\pi}{2}\left[\sum_{n=1}^{\infty}\left(\frac{B_n}{n}\right)^2\right]^{(1/2)}, \tag{76}$$

which is the equivalent form of inequality (75).

5. Conclusions

In this study, by using the weight coefficients, idea of introducing parameters, Euler–Maclaurin summation formula, and Abel's partial summation formula, a multiparameter Hardy–Hilbert-type inequality is established, which contains partial sums as terms in one of the series. As applications, several equivalent statements of the best possible constant factor related to several parameters are provided. Moreover, we illustrate that our main results can generate more Hardy–Hilbert-type inequalities such as the inequalities (69)–(76) asserted by Remark 4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

BY carried out the mathematical studies and drafted the manuscript. SW and JL participated in the design of the study and performed the numerical analysis. All authors contributed equally in the preparation of this article. All authors read and approved the final manuscript.

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