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Research Article

A Study of New Class of Star-Like Functions Associated by Symmetric (p, q)-Calculus

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As of late quantum calculus is broadly utilized in different parts of mathematics. Uniquely, the hypothesis of univalent functions can be newly portrayed by utilizing q-calculus. In this paper, we utilize our recently presented symmetric (p, q)-number $[\widetilde{m}]_{(p,q)}$ to characterize new symmetric (p, q)-derivative $\mathfrak{D}_{(p,q)}$ of analytic function f in the open unit disk $\mathbb{U}$. Utilizing $\mathfrak{D}_{(p,q)}$, we introduce new class of analytic star-like functions and examine some fascinating results.

1. Introduction

The mathematical study of q-calculus has been a topic of great interest for researchers due to its wide applications in different fields. Some of the earlier work on the applications of q-calculus was introduced by Jackson [1]. Later on, q-calculus attained much popularity among the researchers. Recently, q-calculus has gained the attention of researchers because of its huge applications in mathematics and physics. The in-depth analysis of q-calculus was firstly discussed by Jackson [1, 2], where he defined q-integral and q-derivative in a very systematic way. Recently, authors are using these q-integral and q-derivative to define new subclasses of the class of univalent functions and obtained variety of new results. Extending the idea of q-number, which contains only one variable q, the (p, q)-number which contains two independent parameters p and q was independently considered by Chakrabarti and Jagannathan [3]. As q-calculus or quantum calculus originated by using q-number, similarly by using (p, q)-number, the (p, q)-calculus or postquantum calculus has been studied and discussed by several researchers (see, for example, Duran et al. [4] and references therein). Let $f \in \mathbb{C}$. Furthermore, f is normalized analytic, if f is single valued and differentiable for $z \in \mathbb{U}$ along $f(0) = 0$, $f'(0) = 1$ and is represented as

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m. \tag{1}$$

The symbol $\mathfrak{A}$ is used for this type of functions. Let $f \in \mathfrak{A}$ be given by (1). Then,

$$f \text{ is univalent} \Longleftrightarrow \curlywedge_1 \neq \curlywedge_2 \Longrightarrow f(\curlywedge_1) \neq f(\curlywedge_2)\, \forall \curlywedge_1, \quad \curlywedge_2 \in \mathbb{U}. \tag{2}$$

Assume that $\widetilde{p} \in \mathbb{C}$ is analytic. Furthermore,

$$\wp \in \mathfrak{P} \Longleftrightarrow Re(\wp(z)) \subset u > 0 \text{ along } \wp(0) = 1, \tag{3}$$

and is presented as

$$\wp(z) = 1 + \sum_{m=1}^{\infty} c_m z^m. \tag{4}$$

Let $D \subset \mathbb{C}$. If $w_{_0}$ is a constant in D and for all z_0 in D, we have

$$w_0 \text{ and } \forall z_0 \in D \Longrightarrow (1-t)w_0 + tz_0 \in D, \quad 0 \le t \le 1. \tag{5}$$

At that point, we state that D is a star-formed region. Geometrically, $f \in \mathfrak{A}$ is star-like if $f(\mathbb{U})$ is a star-shaped region. We meant by $\mathfrak{S}^*$ the family of these functions. Analytically, we attain

$$f \in \mathfrak{S}^* \Longleftrightarrow \frac{zf'}{f} \in \mathfrak{P}. \tag{6}$$

Assume that $D \subset \mathbb{C}$. If $\zeta_1 \ne \zeta_2 \in D$, then

$$D \text{ is convex} \Longleftrightarrow (1-t)\zeta_1 + t\zeta_2 \in D, \quad 0 \le t \le 1. \tag{7}$$

Geometrically,

$$f \in \mathfrak{A} \text{ is convex} \Longleftrightarrow f(\mathbb{U}) \text{ is convex region}. \tag{8}$$

The family containing all the convex function is represented as $\mathfrak{C}$. Analytically $f \in \mathfrak{C} \Longleftrightarrow ((zf')'/f') \in \mathfrak{P}$,

Note

$$f \in \mathfrak{C} \Longleftrightarrow zf' \in \mathfrak{S}^*. \tag{9}$$

Let f be presented as in (1), and the convolution $(f * g)$ is characterized as

$$(g * f)(z) = \sum_{m=1}^{\infty} a_m b_m z^m = (f * g)(z), \quad z \in \mathbb{U}, \tag{10}$$

where

$$g(z) = z + b_2 z^2 + \cdots = \sum_{m=1}^{\infty} b_m z^m. \tag{11}$$

Let h_1, h_2 be two functions. Then, $h_1 \prec h_2 \Longleftrightarrow \exists$, if there exist a Schwarz function ϖ analytic along $\varpi(0) = 0, |\varpi(z)| < 1$, such that $h_1(z) = (h_2 \circ \varpi)(z)$. It can be found in [5] that if $h_2 \in \mathfrak{S}$, then

$$\begin{gathered} h_1(0) = h_2(0), \\ h_1(\mathbb{U}) \subset h_2(\mathbb{U}) \Longleftrightarrow h_1 \prec h_2, \end{gathered} \tag{12}$$

and for details, see [6].

Kanas and Wisniowska [7] presented the conic regions $\Lambda_k, k \in 0, \infty)$ by

$$\Lambda_k = \left\{ \widetilde{w} = \widetilde{\mu} + i\widetilde{\nu} \in \mathbb{C}: \widetilde{\mu} > k\sqrt{(\widetilde{\mu}-1)^2 + \widetilde{\nu}^2} \right\}, \tag{13}$$

with $1 \in \Lambda_k$. Assume

$$\partial\Lambda_k = \left\{ \widetilde{w} = \widetilde{\mu} + i\widetilde{\nu} \in \mathbb{C}: \widetilde{\mu}^2 = k^2(\widetilde{\mu}-1)^2 + \widetilde{\nu}^2 \right\}. \tag{14}$$

Therefore,

$$\begin{aligned} &k = 0 \Longrightarrow \Lambda_0 = \widetilde{\mu} > 0, \text{ right half plane}, \\ &k = 1 \Longrightarrow \Lambda_1 = \widetilde{\nu}^2 < 2\widetilde{\mu} - 1, \text{ parabolic region}, \\ &k \in (0,1) \Longrightarrow \Lambda_k = \widetilde{\mu} > k\sqrt{(\widetilde{\mu}-1)^2 + \widetilde{\nu}^2}, \text{ hyperbolic region}. \end{aligned} \tag{15}$$

Identified with the region Λ_k, the accompanying functions are extremal and $p_k(\mathbb{U}) \subset \Lambda_k$:

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z}, & k = 0, \\ 1 + \dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, & k = 1, \\ 1 + \dfrac{2}{1-k^2}\sin h^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & 0 < k < 1. \end{cases} \tag{16}$$

The families $k - \mathfrak{UCB}$ and $k - \mathfrak{ST}$ are fascinating subfamilies of $\mathfrak{S}$ and are characterized in [8] for all $k \ge 0$, $z \in \mathbb{U}$, as

$$\begin{aligned} &\text{f belongs to k} - \mathfrak{UCB} \Longleftrightarrow \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > k\left|\frac{zf''(z)}{f'(z)}\right|, \\ &\text{f belongs to k} - \mathfrak{ST} \Longleftrightarrow \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > k\left|\frac{zf'(z)}{f(z)} - 1\right|. \end{aligned} \tag{17}$$

Note from [8] that

$$f \in k - \mathfrak{UCB} \Longleftrightarrow zf' \in k - \mathfrak{ST}. \tag{18}$$

Geometrically,

$$f \in k - \mathfrak{UCB} \Longleftrightarrow \left[\frac{(zf'(z))'}{f'(z)}\right](\mathbb{U}) \subset \Lambda_k. \tag{19}$$

2. Postquantum Calculus

Extending the idea of q-number, the (p,q)-number with two variables p and q was independently considered in [3].

For $0 < q < p \le 1$, the twin basic number or (p,q)-number is defined for $m \in \mathbb{N}$ as

$$[m]_{(p,q)} = \frac{p^m - q^m}{p-q} = p^{m-1} + p^{m-2}q + \cdots + pq^{m-2} + q^{m-1}, \tag{20}$$

which is the normal speculation of q-number with the end goal that for $p \longrightarrow 1^-$, $q \longrightarrow 1^-$ we have

$$\begin{gathered}[m]_{(p,q)} = [m]_{(1,q)} = [m]_q,\\ \text{and}\\ [m]_{(p,q)} = [m]_{(1,1)} = m.\end{gathered} \tag{21}$$

Furthermore, note that $[m]_{(p,q)} = [m]_{(q,p)}$.

In 1991, Chakrabarti and Jagannathan [3] characterized (p,q)-derivative of $f \in \mathfrak{A}$ as

$$D_{(p,q)}f(z) = \begin{cases} \dfrac{f(pz) - f(qz)}{(p-q)z}, & z \neq 0,\\ f'(0), & z = 0.\end{cases} \tag{22}$$

Furthermore, if $f(z) = z^m$, then $D_{(p,q)}f(z) = [m]_{(p,q)}z^{m-1}$, and for (1), we have

$$D_{(p,q)}f(z) = 1 + \sum_{m=2}^{\infty} [m]_{(p,q)}a_m z^{m-1}, \quad z \in \mathbb{U}. \tag{23}$$

We remark here as

$$p \longrightarrow 1^- \Longrightarrow D_{(p,q)}f(z) = D_q f(z), \tag{24}$$

$$\begin{gathered} p \longrightarrow 1^-, q \longrightarrow 1^- \Longrightarrow m]_{(p,q)} = m,\\ D_{(p,q)}f(z) = f'(z).\end{gathered} \tag{25}$$

We now extend the idea in [9] and define symmetric (p,q)-number as follows:

$$\tilde{m}]_{(p,q)} = \frac{p^m - q^{-m}}{p - q^{-1}}, \quad 0 < q \leq p < 1. \tag{26}$$

Analogous to (p,q)-derivative defined by (22), the symmetric (p,q)-derivative is characterized as

$$\mathfrak{D}_{(p,q)}f(z) = \begin{cases} \dfrac{f(pz) - f\left(q^{-1}z\right)}{\left(p - q^{-1}\right)z}, & z \neq 0,\\ f'(0), & z = 0.\end{cases} \tag{27}$$

If

$$f(z) = z^m \Longrightarrow \mathfrak{D}_{(p,q)}f(z) = [\tilde{m}]_{p,q}z^{m-1}, \quad \forall z \in \mathbb{U}. \tag{28}$$

In this manner for (1), we acquire

$$\mathfrak{D}_{(p,q)}f(z) = 1 + \sum_{m=2}^{\infty} [\tilde{m}]_{(p,q)}a_m z^{m-1}, \quad z \in \mathbb{U}. \tag{29}$$

Equivalently, by using the same technique as in [9], it can be seen that

$$\begin{aligned} \mathfrak{D}_{(p,q)}(f(z) + g(z)) &= \mathfrak{D}_{(p,q)}f(z) + \mathfrak{D}_{(p,q)}g(z),\\ \mathfrak{D}_{(p,q)}(f(z)g(z)) &= g(pz)\mathfrak{D}_{(p,q)}f(z) + f\left(q^{-1}z\right)\mathfrak{D}_{(p,q)}g(z)\\ &= g\left(q^{-1}z\right)\mathfrak{D}_{(p,q)}f(z) + f(pz)\mathfrak{D}_{(p,q)}g(z),\\ \mathfrak{D}_{(p,q)}\left(\frac{f(z)}{g(z)}\right) &= \frac{g\left(q^{-1}z\right)\mathfrak{D}_{(p,q)}f(z) - f\left(q^{-1}z\right)\mathfrak{D}_{(p,q)}g(z)}{g(pz)g\left(q^{-1}z\right)}\\ &= \frac{g(pz)\mathfrak{D}_{(p,q)}f(z) - f(pz)\mathfrak{D}_{(p,q)}g(z)}{g(pz)g\left(q^{-1}z\right)}.\end{aligned} \tag{30}$$

Assume that $\mathfrak{T}$ denotes the subfamily of $\mathfrak{A}$ with negative coefficients such as

$$f(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad z \in \mathbb{U}. \tag{31}$$

For further developments about class $\mathfrak{T}$ and its subclasses, one can refer to a wide range of extraordinary articles written by famous mathematicians (see [10–13] and references therein).

Presently, utilizing the symmetric (p,q)-derivative characterized by (27) of f, we present another subclass of $\mathfrak{A}$ as follows.

Definition 1. Let $0 \leq k \leq \infty, 0 \leq \eta < 1$ and $0 < q \leq p < 1$. Assume that $f \in \mathfrak{A}$ characterized by (1). Furthermore,

$$f \in k - \widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta) \Longleftrightarrow k\left|\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right| + \eta < \mathrm{Re}\left[\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)}\right], \quad z \in \mathbb{U}. \tag{32}$$

Geometrically,

$$\left[\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)}\right](\mathbb{U}) \subset \Lambda_{k,\eta} \Longleftrightarrow f \in k - \widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta), \tag{33}$$

where $\Lambda_{k,\eta}$ is given by

$$\Lambda_{k,\eta} = (1-\eta) + \eta\Lambda_k, \tag{34}$$

and Λ_k is defined as in (13).

Utilizing the functions $p_{k,\eta}$ defined in [14], we have

$$\left(\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)}\right) \prec p_{k,\eta} \Longleftrightarrow f \in k - \widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta), \quad z \in \mathbb{U}. \tag{35}$$

Remark 1. Likewise, we set $k - \widetilde{\mathfrak{S}\mathfrak{J}}^{-}_{(p,q)}(\eta) = k - \widetilde{\mathfrak{S}\mathfrak{J}}_{(p,q)}(\eta) \cap \mathfrak{T}$.

Note that on the off chance that $p \longrightarrow 1^{-}, q \longrightarrow 1^{-}$, at that point

$$k - \widetilde{\mathfrak{S}\mathfrak{J}}_{(p,q)}(\eta) = k - \mathfrak{S}\mathfrak{J}(\eta). \tag{36}$$

3. Main Results

Now in this segment, we will demonstrate our principle results. It is worthy to mention here that our main results are extension of results studied by Kanas et al. [9]. We utilize symmetric (p,q)-derivative operator to obtain these results.

The accompanying lemmas are useful to prove main result.

Lemma 1. *Assume that $r \in (0,1)$ and $\beta > 0$. Furthermore, f and g are analytic in $\mathbb{U}$ along $f \prec g$. Then,*

$$\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\beta}\mathrm{d}\theta \le \int_0^{2\pi}\left|g\left(re^{i\theta}\right)\right|^{\beta}\mathrm{d}\theta, \quad z = \mathrm{r}e^{i\theta}, \tag{37}$$

and for details, see [15].

Lemma 2. *For the sequence $\{b_m\}_{j=1}^{\infty}$ and $\forall\, z \in \mathbb{U}$,*

$$Re\left(1 + 2\sum_{m=1}^{\infty} b_m z^m\right) > 0 \Longleftrightarrow \{b_m\}_{m=1}^{\infty}, \tag{38}$$

is a subordinating factor sequence (see [15]).

Now we will extend the existing results of [13] with the help of symmetric (p,q)-derivative.

Theorem 1. *Assume that $0 < q \le p < 1$ and let $f \in \mathfrak{S}$ be defined by (1). If the inequality*

$$\sum_{m=2}^{\infty}\left([\tilde{m}]_{(p,q)}(k+1) - (k+\eta)\right)\left|a_m\right| \le (1-\eta), \tag{39}$$

holds true, then $f \in k - \widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta)$.

Proof. To prove (39), it is adequate to prove that

$$k\left|\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right| - Re\left(\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right) < (1-\eta). \tag{40}$$

Consider

$$\begin{aligned} &k\left|\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right| - Re\left(\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right) \\ &\le k\left|\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right| + \left|\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right| \\ &= (k+1)\left|\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right|. \end{aligned} \tag{41}$$

Now using (1) and (29), we obtain

$$\begin{aligned} k\left|\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right| - \mathrm{Re}\left(\frac{z\mathfrak{D}_{(p,q)}f(z)}{f(z)} - 1\right) &\le \left|\frac{\sum_{m=2}^{\infty}[\tilde{m}]_{p,q}a_m z^m - \sum_{m=2}^{\infty} a z^m}{z + \sum_{m=2}^{\infty} a_m z^m}\right| \\ &\le (k+1)\left|\frac{\sum_{m=2}^{\infty}\left([\tilde{m}]_{p,q} - 1\right)a_m z^m}{1 + \sum_{m=2}^{\infty} a_m z^m}\right| \\ &\le (k+1)\frac{\sum_{m=2}^{\infty}\left([\tilde{m}]_{p,q} - 1\right)\left|a_m\right|}{1 - \sum_{m=2}^{\infty}\left|a_m\right|} \\ &\le (1-\eta). \end{aligned} \tag{42}$$

This can be composed as

$$(k+1)\sum_{m=2}^{\infty}\left([\tilde{m}]_{(p,q)}-1\right)|a_m| \le (1-\eta)\left(1-\sum_{m=2}^{\infty}|a_m|\right). \tag{43}$$

From (43), we obtain (39) and the proof is completed.

Remark 2. In Theorem 1, by putting $p=q$, we have the result presented in [9].

Corollary 1. *Assume that $0<q<1$ and let $f\in\mathfrak{S}$ be presented as in (1). If*

$$\sum_{m=2}^{\infty}\left[[\tilde{m}]_q(k+1)-(k+\eta)\right]|a_m| \le (1-\eta), \tag{44}$$

holds true for $0\le k<\infty$, $0\le\eta<1$, then $f\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_q(\eta)$.

Remark 3. It can be seen that the quantity

$$[\tilde{m}]_{(p,q)}(k+1)-(k+\eta)=\frac{p^m-q^{-m}}{p-q^{-1}}(k+1)-(k+\eta)>0, \tag{45}$$

for all $0\le k<\infty, 0\le\eta<1$ and $0<q\le p<1$, unless otherwise mentioned.

Now using (39), we can find some special members of $f\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta)$. One of them is the following.

Corollary 2. *Let for $0\le k<\infty, 0\le\eta<1$ and $0<q\le p<1$. If, for $f(z)=z+a_m z^m$, the following inequality:*

$$|a_m|\le\frac{(1-\eta)}{[\tilde{m}]_{(p,q)}(k+1)-(k+\eta)}, \quad m\ge 2, \tag{46}$$

holds, then $f\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta)$. Especially

$$f(z)=z+\frac{(1-\eta)q}{(pq+1)(k+1)-(k+\eta)q}z^2\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta), \quad z\in\mathbb{U}. \tag{47}$$

For $p=q$, we obtain the following known results (see [9]).

Corollary 3. *Let $0\le k<\infty$, $0\le\eta<1$ and $0<q<1$. If, for $f(z)=z+a_m z^m$, the following inequality:*

$$|a_m|\le\frac{(1-\eta)}{[\tilde{m}]_q(k+1)-(k+\eta)}, \quad m\ge 2, \tag{48}$$

holds, then $f\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_q(\eta)$. Especially

$$f(z)=z+\frac{(1-\eta)q}{\left(q^2+1\right)(k+1)-(k+\eta)q}z^2\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_q(\eta), \quad z\in\mathfrak{U}. \tag{49}$$

Also, for $k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}^{-}(\eta)$, we obtain the following.

Corollary 4. *Let $0\le k<\infty, 0\le\eta<1$ and $0<q\le p<1$. An essential and adequate condition of $f(z)=z-a_2z^2-\cdots$, $(a_m\ge 0)$ belongs to $k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}^{-}(\eta)$, such as*

$$\sum_{m=2}^{\infty}\left([\tilde{m}]_{(p,q)}(k+1)-(k+\eta)\right)|a_m|\le(1-\eta). \tag{50}$$

The accompanying function yields quality.

$$f(z)=z-\frac{1-\eta}{[\tilde{m}]_{(p,q,)}(k+1)-(k+\eta)}z^m. \tag{51}$$

Proof. Proof immediately follows by using Theorem 1.

On the off chance that we take $p=q$ in Corollary 4, we have the accompanying corollary (see [9]).

Corollary 5. *Let $0\le k<\infty, 0\le\eta<1$ and $0<q<1$. A necessary and sufficient condition for f of the form $f(z)=z-a_2z^2-\cdots$, $(a_m\ge 0)$, to be in the class $k-\widetilde{\mathfrak{S}\mathfrak{T}}_q^{-}(\eta)$ is that*

$$\sum_{m=2}^{\infty}\left([\tilde{m}]_{q)}(k+1)-(k+\eta)\right)|a_m|\le(1-\eta). \tag{52}$$

The result is sharp and accompanying function yields quality.

$$f(z)=z-\frac{1-\eta}{[\tilde{m}]_q(k+1)-(k+\eta)}z^m. \tag{53}$$

Theorem 2. *Let $0\le k<\infty$, $0\le\eta<1$ and $0<q\le p<1$. Assume $f(z)=z-a_2z^2-\cdots$, $(a_m\ge 0)$ belongs to $k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}^{-}(\eta)$. Furthermore, $|z|=r<1$ yields*

$$\begin{aligned} r-\frac{q(1-\eta)}{(pq+1)(k+1)-q(k+\eta)}r^2 &\le |f(z)| \\ &\le r+\frac{q(1-\eta)}{(pq+1)(k+1)-q(k+\eta)}r^2. \end{aligned} \tag{54}$$

Accompanying function yields quality for (54).

$$f(z)=z+\frac{q(1-\eta)}{(pq+1)(k+1)-q(k+\eta)}z^2, \quad z\in\mathbb{U}. \tag{55}$$

Proof. Let $f\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}^{-}(\eta)$. By using Theorem 1, we have

$$\begin{aligned} &\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)\sum_{m=2}^{\infty}a_m \\ &\le\sum_{m=2}^{\infty}\left([\tilde{m}]_{(p,q)}(k+1)-(k+\eta)\right) \\ &\le(1-\eta). \end{aligned} \tag{56}$$

This means that

$$\sum_{m=2}^{\infty} a_m \leq \frac{(1-\eta)}{\left([\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}. \tag{57}$$

Furthermore if $|z| = r < 1$,

$$\begin{aligned}|f(z)| &\leq |z| + \sum_{m=2}^{\infty} a_m |z|^m \\ &\leq r + \frac{q(1-\eta)}{(pq+1)(1-\eta)(k+1)-q(k+\eta)} r^2,\end{aligned} \tag{58}$$

and

$$\begin{aligned}|f(z)| &\geq |z| - \sum_{m=2}^{\infty} a_m |z|^m \\ &\geq r - \frac{q(1-\eta)}{(pq+1)(1-\eta)(k+1)-q(k+\eta)} r^2.\end{aligned} \tag{59}$$

Combining (58) and (59), we obtain (54).

Theorem 3. *Suppose that $f \in k-\widetilde{\mathfrak{S}\mathfrak{T}}^{-}_{(p,q)}(\eta)$. Moreover,*

$$f_2(z) = z - \frac{(1-\eta)}{[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)} z^2. \tag{60}$$

If

$$\begin{aligned}&\forall z = re^{i\theta}, \\ &0 < r < 1 \Longrightarrow \int_0^{2\pi} |f(z)|^{\eta} d\theta \leq \int_0^{2\pi} \left|f_2(z)\right|^{\eta} d\theta.\end{aligned} \tag{61}$$

Proof. Note that to prove (61) for $f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m$, it is equivalent to prove

$$\begin{aligned}&\int_0^{2\pi} \left|1 - \sum_{m=2}^{\infty} |a_m| z^{m-1}\right|^{\eta} d\theta \\ &\leq \int_0^{2\pi} \left|1 - \frac{(1-\eta)}{[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)}\right|^{\eta} d\theta.\end{aligned} \tag{62}$$

By Lemma 1, it is sufficient to show that

$$1 - \sum_{m=2}^{\infty} |a_m| z^{m-1} \prec 1 - \frac{(1-\eta)}{[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)}. \tag{63}$$

Set

$$1 - \sum_{m=2}^{\infty} |a_m| z^{m-1} = 1 - \frac{(1-\eta)}{[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)} w(z). \tag{64}$$

This means that

$$|w(z)| = \left|\sum_{m=2}^{\infty} \frac{[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)}{(1-\eta)} |a_m| z^{m-1}\right|. \tag{65}$$

Now using (39), we get $|w(z)| \leq |z| < 1$ and this finishes the proof.

Theorem 4. *Let $f \in k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta)$ along $\in \mathfrak{C}$. Then,*

$$\frac{\left([\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{2\left\{(1-\eta)+[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}} (f * g)(z) \prec g(z), \tag{66}$$

that is,

$$\operatorname{Re}(f(z)) > -\frac{\left\{(1-\eta)+[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}{\left([\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)} > 0, \quad z \in \mathbb{U}. \tag{67}$$

The constant $\left(\left([\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)/2\{(1-\eta)+[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\}\right)$ cannot be increased.

Proof. Let $f \in k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta)$ and $g \in \mathfrak{C}$.
Then, consider

$$\begin{aligned}&\frac{\left([\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{2\left\{(1-\eta)+[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}} (f * g)(z) \\ &= \frac{\left([\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{2\left\{(1-\eta)+[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}} \left(z + \sum_{m=2}^{\infty} a_m b_m z^m\right).\end{aligned} \tag{68}$$

□

Therefore, by using the definition of subordinating factor sequence, (66) holds true, if

$$\left(\frac{\left([\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{2\left\{(1-\eta)+[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}} a_m\right)_{n=1}^{\infty}, \tag{69}$$

is a subordinating factor sequence with $a_1 = 1$. Equivalently, by Lemma 2, we have

$$\operatorname{Re}\left(1 + \sum_{m=1}^{\infty} \frac{\left([\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{\left\{(1-\eta)+[\widetilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}} a_m z^m\right) > 0, \quad z \in \mathbb{U}. \tag{70}$$

Now for $|z| = r < 1$, we can write

$$\begin{aligned}
&\operatorname{Re}\left(1+\frac{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}\sum_{m=1}^{\infty}a_m z^m\right)\\
&=\operatorname{Re}\left(1+\frac{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}z+\frac{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)\sum_{m=2}^{\infty}a_m z^m}{\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}\right)\\
&>1-\frac{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}r-\frac{\sum_{m=2}^{\infty}\left([\tilde{m}]_{(p,q)}(k+1)-(k+\eta)\right)a_m r^j}{\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}.
\end{aligned} \tag{71}$$

It can be seen that the function defined by

$$\psi_{(p,q)}(n,k,\eta)=[\tilde{m}]_{(p,q)}(k+1)-(k+\eta), \tag{72}$$

is increasing for $n \geq 2$; furthermore, by utilizing affirmation (39) of Theorem 1, we have

$$\begin{aligned}
&\operatorname{Re}\left(1+\frac{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}\sum_{m=1}^{\infty}a_m z^m\right)\\
&>1-\frac{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}r-\frac{1-\eta}{\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}r\\
&>0.
\end{aligned} \tag{73}$$

This equivalently proves the subordination condition given by (66). The inequality given by (67) can be obtained from (66) by taking $g(z) = (z/1-z) \in \mathfrak{C}$. Now from (47), we obtained the following function:

$$F(z)=z+\frac{(1-\eta)}{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta). \tag{74}$$

Function (66) becomes

$$\frac{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{2\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}F(z)\prec\frac{z}{1-z}. \tag{75}$$

It is easily verified that

$$\min\left\{\operatorname{Re}\left(\frac{\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)}{2\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}}F(z)\right)\right\}=-\frac{1}{2},\quad z\in\mathbb{U}. \tag{76}$$

This implies that the constant $\left(\left([\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right)/2\left\{(1-\eta)+[\tilde{2}]_{(p,q)}(k+1)-(k+\eta)\right\}\right)$ is possible.

Theorem 5. *Let* $0 \leq k < \infty, 0 \leq \eta < 1$ *and* $0 < q \leq p < 1$. *If we set*

$$f_1(z)=zf_m(z)=z-\frac{1-\eta}{[\tilde{m}]_{(p,q)}(k+1)-(k+\eta)}z^m, \tag{77}$$

then

$$f\in k-\widetilde{\mathfrak{S}\mathfrak{T}}_{(p,q)}(\eta)\Longleftrightarrow f(z)=\sum_{m=1}^{\infty}\delta_m f_m(z),\ \sum_{m=1}^{\infty}\delta_m=1. \tag{78}$$

Proof. Suppose that

$$f(z)=\sum_{m=1}^{\infty}\delta_m f_m(z)=\delta_1 f_1(z)+\sum_{m=2}^{\infty}\delta_m f_m(z). \tag{79}$$

Utilizing (77), we get

$$\begin{aligned}
f(z)&=\delta_1 f_1(z)+\sum_{m=2}^{\infty}\delta_m\left\{z-\frac{(1-\eta)}{[\tilde{m}]_{(p,q)}(k+1)-(k+\eta)}z^m\right\}\\
&=\left(\sum_{m=1}^{\infty}\delta_m\right)z-\sum_{m=2}^{\infty}\delta_m\frac{(1-\eta)}{[\tilde{m}]_{(p,q)}(k+1)-(k+\eta)}z^m\\
&=z-\sum_{m=2}^{\infty}\delta_m\frac{(1-\eta)}{[\tilde{m}]_{(p,q)}(k+1)-(k+\eta)}z^m.
\end{aligned} \tag{80}$$

Since $\sum_{m=1}^{\infty}\delta_m = 1$, this means that $\sum_{m=2}^{\infty}\delta_m = \sum_{m=1}^{\infty}\delta_m - \delta_1 = 1 - \delta_1 \le 1$, and using the assertion (51) of Corollary 4, we obtain $f \in k - \widetilde{\mathfrak{S}\mathfrak{T}}^{-}_{(p,q)}(\eta)$.

Conversely, suppose that $f \in k - \widetilde{\mathfrak{S}\mathfrak{T}}^{-}_{(p,q)}(\eta)$. Making use of (39), we may set

$$\delta_m = \frac{[\widetilde{m}]_{(p,q)}(k+1) - (k+\eta)}{(1-\eta)}\left|a_m\right|, \sum_{m=1}^{\infty}\delta_m = 1. \tag{81}$$

Then,

$$\begin{aligned}
f(z) &= z - \sum_{m=2}^{\infty} a_m z^m \\
&= z - \sum_{m=2}^{\infty}\delta_m \frac{(1-\eta)}{[\widetilde{m}]_{(p,q)}(k+1) - (k+\eta)} z^m \\
&= z - \sum_{m=2}^{\infty}\delta_m\left(z - f_m(z)\right) \\
&= \left(1 - \sum_{m=2}^{\infty}\delta_m\right)z + \sum_{m=2}^{\infty}\delta_m f_m(z) \\
&= \delta_1 z + \sum_{m=2}^{\infty}\delta_m f_m(z), \\
\delta_1 &= 1 - \sum_{m=2}^{\infty}\delta_m \\
&= \sum_{m=1}^{\infty}\delta_m f_m(z), \quad z \in \mathbb{U},
\end{aligned} \tag{82}$$

which is required.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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