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Research Article

Fractional Entropy-Based Test of Uniformity with Power Comparisons

Mohamed S. Mohamed,[1] **Haroon M. Barakat**,[2] **Salem A. Alyami**,[3] **and Mohamed A. Abd Elgawad**[4,5]

[1]Department of Mathematics, Faculty of Education, Ain Shams University, Cairo 11341, Egypt
[2]Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt
[3]Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 13318, Saudi Arabia
[4]Department of Mathematics, Faculty of Science, Benha University, Benha 13518, Egypt
[5]School of Computer Science and Technology, Wuhan University of Technology, Wuhan 430070, China

Correspondence should be addressed to Mohamed S. Mohamed; mohamed.said@edu.asu.edu.eg

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In the present paper, we use the fractional and weighted cumulative residual entropy measures to test the uniformity. The limit distribution and an approximation of the distribution of the test statistic based on the fractional cumulative residual entropy are derived. Moreover, for this test statistic, percentage points and power against seven alternatives are reported. Finally, a simulation study is carried out to compare the power of the proposed tests and other tests of uniformity.

1. Introduction

Rao et al. [1] suggested a nonnegative measure of uncertainty and called it the cumulative residual entropy (CRE). For any nonnegative continuous random variable (RV) X with a cumulative distribution function (CDF) $F(x) = P(X < x)$, the CRE is defined by

$$\mathrm{CRE}(F) = -\int_0^\infty \overline{F}(x)\ln(\overline{F}(x))\mathrm{d}x, \tag{1}$$

where $\overline{F}(x) = 1 - F(x)$ is the reliability function. Rao et al. [1] revealed many salient features of the CRE. For example, the CRE possesses more general mathematical properties than the Shannon entropy, and it can be easily computed from sample data, and these computations asymptotically converge to the true values. Moreover, the CRE deals with the quantity of information in residual life. For the standard uniform distribution, denoted by $U(0,1)$, Rao et al. [1] showed that the value of the CRE is 1/4. The literature abounds with many different results for Shannon's entropy and its modifications. Interested readers may refer to [1–17].

Xiong et al. [16] suggested the fractional cumulative residual entropy (FCRE) to extend the CRE to the case of fractional order. For any $0 \le q \le 1$, the FCRE for the RV X is defined by

$$\mathrm{CRE}^q(F) = \int_0^\infty \overline{F}(x)[-\ln(\overline{F}(x))]^q\mathrm{d}x. \tag{2}$$

The measure $\mathrm{CRE}^q(F)$ is a nonadditive and nonnegative. Moreover, it is a convex function of the parameter q, $\mathrm{CRE}^0(F) = \mathbb{E}(X)$, and $\mathrm{CRE}^1(F) = \mathrm{CRE}(F)$. Xiong et al. [16] derived the FCRE for some well-known distributions; for example, FCRE of the CDF $U(0,1)$ is $\Gamma(q+1)/2^{q+1}$.

Misagh et al. [15] proposed a weighted form of CRE, which is shift-dependent. This information-theoretic uncertainty measure is called the weighted cumulative residual entropy (WCRE), and it is defined by

$$\mathrm{CRE}_w(F) = -\int_0^\infty x\overline{F}(x)\ln(\overline{F}(x))\mathrm{d}x. \quad (3)$$

Later, Mirali et al. [12] and Mirali and Baratpour [13] studied several properties of this measure including its dynamic version. It is easy to observe that the WCRE of the $U(0,1)$ is 5/36.

Stephens [18] offered a practical guide to goodness-of-fit tests using statistics based on the empirical CDF. Moreover, in [18], the power comparisons of some uniformity tests were carried out. Dudewicz and Van der Meulen [9] investigated the power properties of an entropy-based test when used for testing uniformity. Moreover, via a comparison with other tests of uniformity, Dudewicz and Van der Meulen [9] showed that the entropy-based test possesses good power properties for many alternatives. Noughabi [14] constructed a test for uniformity based on the CRE and studied some of its properties. Moreover, he reported the percentage points and power comparison against seven alternative distributions. As a natural extension of the results obtained by Noughabi [14], we study the FCRE and WCRE for testing the uniformity. A result of a simulation study shows that the test based on FCRE and WCRE is competitive with the test based on CRE in terms of power. This fact gives a satisfactory motivation of our study.

Throughout this paper, we obtain the percentage points under the WCRE and FCRE by using the Monte Carlo method via the simulation and the normality asymptotic, as well as the beta approximation, respectively. Moreover, a power comparison is performed between the FCRE and WCRE and other tests. The rest of this work is systematic as follows. In Section 2, we introduce the FCRE test statistic for uniformity and discuss some of its properties. In Section 3, we propose the methods of finding the percentage points of FCRE and illustrate the WCRE test statistics for uniformity. In addition, we calculate the percentage points of FCRE and WCRE. Then, in Section 4, we use Monte Carlo simulation to perform the power comparison of uniformity of different tests against seven alternative distributions. Section 5 is devoted to the conclusions. Everywhere in what follows, the symbols ($\xrightarrow[n]{p}$), ($\xrightarrow[n]{d}$) and ($\xrightarrow[n]{\text{a.s.}}$) stand for convergence in probability, convergence in distribution, and almost surely, as $n \longrightarrow \infty$.

2. Theoretical Aspects and Test Statistic

To establish our test of the null hypothesis H_0, we need the following theorem, which shows that, for a CDF with support $[0,1]$, one always has $0 \le \mathrm{CRE}^q(F) \le e^{-q}$, and for the distribution $U(0,1)$, we have $\mathrm{FCRE} = \Gamma(q+1)/2^{q+1}$, and this value is uniquely attained by the uniform distribution, whenever q is fixed.

Theorem 1. *Let X be a nonnegative RV with an absolutely continuous CDF F with a support $[0,1]$. From (2), it holds $0 \le CRE^q(F) \le e^{-q}$, and $CRE^q(F) = \Gamma(q+1)/2^{q+1}$ is uniquely acquired by the distribution $U(0,1)$.*

Proof. Since $0 \le \overline{F}(x)[-\ln(\overline{F}(x))]^q \le 1$, and the function $f(x) = x(-\ln x)^q$ has a maximum at $x = e^{-q}$, $0 < x \le 1$, we get $0 \le \mathrm{CRE}^q(F) \le e^{-q}$. On the other hand, using the strict convexity of $f(x) = x(-\ln x)^q$, it is easy to see that FCRE is a concave function of distribution (with support [0, 1]). This shows that $\mathrm{CRE}^q(F) = \Gamma(q+1)/2^{q+1}$ is uniquely acquired by the distribution $U(0,1)$. This completes the proof.

Let $X_1, X_2, \ldots, X_n$ be a random sample with a continuous CDF F, with support [0, 1]. Furthermore, let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the corresponding order statistics $X_1, X_2, \ldots, X_n$. According to (2), we can obtain the empirical FCRE as an estimator of $\mathrm{FCRE}(F)$ by

$$\mathrm{CRE}^q(F_n) = \int_0^\infty \overline{F}_n(x)[-\ln(\overline{F}_n(x))]^q\mathrm{d}x, \quad (4)$$

where $\overline{F}_n(x) = 1 - F_n(x)$ and $F_n(x)$ is the empirical CDF, which is defined by

$$F_n(x) = \sum_{i=1}^{n-1}\frac{i}{n}I_{[X_{(i)},X_{(i+1)})}(x) + I_{[X_{(n)},\infty)}(x), \quad x \in \mathfrak{R}, \quad (5)$$

where $I_A(x)$ is the indicator function, i.e., $I_A(x) = 1, x \in A$; $I_A(x) = 0, x \notin A$.

To perform a consistent test of the hypothesis of uniformity, we suggest the consistent statistic test

$$\begin{aligned} R_n^q = \mathrm{CRE}^q(F_n) &= \sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)\left(-\ln\left(1-\frac{i}{n}\right)\right)^q\left(X_{(i+1)} - X_{(i)}\right) \\ &= \sum_{i=1}^{n-1} A_i W_i, \end{aligned} \quad (6)$$

where $A_i = (1-(i/n))(-\ln(1-(i/n)))^q$ and $W_i = (X_{(i+1)} - X_{(i)})$, $i = 1, 2, \ldots, n-1$, $0 \le q \le 1$.

Xiong et al. [16] proved that $\mathrm{CRE}^q(F_n) \xrightarrow[n]{p} \mathrm{CRE}^q(F)$. Moreover, under the null hypothesis H_0, we get $R_n^q \xrightarrow[n]{p} \Gamma(q+1)/2^{q+1}$. On the other hand, under the alternative hypothesis (that F is any continuous CDF with support [0, 1], which is not the uniform), we have $\mathrm{CRE}^q(F_n) \xrightarrow[n]{p} r$, where r is a smaller or larger number than $\Gamma(q+1)/2^{q+1}$. □

Theorem 2. *The test based on the sample estimate R_n^q is consistent.*

Proof. From Glivenko–Cantelli theorem (see Tucker [19]), we have $\sup_t|F_n(t) - F(t)| \xrightarrow[n]{\text{a.s.}} 0$. On the other hand, Theorem 3 in Xiong et al. [16] asserts that $\mathrm{CRE}^q(F_n) \xrightarrow[n]{\text{a.s.}} \mathrm{CRE}^q(F)$, which proves the theorem. □

Theorem 3. *Suppose that the random sample $X_1, X_2, \ldots, X_n$ has been drawn from an unknown continuous CDF F defined on $[0,1]$. Then, from (6), we have $0 \le R_n^q \le e^{-q}$.*

Proof. Since the function $f(p) = p(-\ln p)^q$, $0 < p < 1$, has a maximum value at e^{-q}, $0 \le q \le 1$; therefore,

$$0 \le R_n^q = \mathrm{CRE}^q(F_n) = \sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)\left(-\ln\left(1-\frac{i}{n}\right)\right)^q\left(X_{(i+1)}-X_{(i)}\right)$$
$$\le \sum_{i=1}^{n-1} e^{-q}\left(X_{(i+1)}-X_{(i)}\right) = e^{-q}\left(X_{(n)}-X_{(1)}\right) \le e^{-q}. \tag{7}$$

This completes the proof of the theorem. □

Theorem 4. *Under H_0, from (6), the mean and the variance of R_n^q are, respectively,*

$$\mathbb{E}\left(R_n^q\right) = \frac{1}{n+1}\sum_{i=1}^{n-1} A_i, \tag{8}$$

$$\mathrm{Var}\left(R_n^q\right) = \frac{n}{(n+1)^2(n+2)}\sum_{i=1}^{n-1} A_i^2. \tag{9}$$

Proof. The proof directly follows by noting that, for any $i = 1, 2, \ldots, n-1$, the RV $W_i = \left(X_{(i+1)} - X_{(i)}\right)$, based on the CDF $U(0,1)$, has beta distribution, i.e., $W_i \sim \mathrm{Beta}(1, n)$ (cf. [20]). This completes the proof. □

Remark 1. Under H_0, from (6), (8), and (9), we have $\lim_{n\longrightarrow\infty}\mathbb{E}\left(R_n^q\right) = \Gamma(q+1)/2^{q+1} = \mathrm{CRE}^q(U)$ and $\lim_{n\longrightarrow\infty}\mathrm{Var}\left(R_n^q\right) = 0$, where $\mathrm{CRE}^q(U)$ is the FCRE of the CDF $U(0,1)$.

The critical region, which describes the test procedure, is given by the following two inequalities:

$$\mathrm{CRE}^q(F_n) \le \mathrm{CRE}_{\alpha/2}^{*q} \coloneqq \text{lower or } \mathrm{CRE}^q(F_n) \ge \mathrm{CRE}_{1-(\alpha/2)}^{*q} \coloneqq \text{upper}, \tag{10}$$

where α is the desired level of significance, and CRE_α^{*q} is the $\alpha-$quantile of the asymptotic, or approximated, CDF of the test statistic $\mathrm{CRE}^q(F_n)$, under H_0. In the next section, we derive the asymptotic and approximated CDF of the test statistic $\mathrm{CRE}^q(F_n)$. These quantiles are computed by using the Monte Carlo method.

3. Percentage Points of the Test Statistic

In this section, we obtain the asymptotic distribution of R_n^q under H_0. From (6), we can write $R_n^q = \sum_{i=1}^{n-1} T_i$, where $T_i = A_i W_i$, $i = 1, 2, \ldots, n-1$, and $W_i \sim \mathrm{Beta}(1, n)$. Thus, we can see that T_i's have the following probability density function (PDF):

$$f_{T_i}(t) = \frac{n}{A_i}\left(1 - \frac{t}{A_i}\right)^{n-1}, \quad i = 1, 2, \ldots, n-1. \tag{11}$$

The mean and variance of T_i are, respectively,

$$\mu_i = \mathbb{E}(T_i) = A_i\mathbb{E}(W_i) = \frac{A_i}{n+1},$$
$$\sigma_i^2 = \mathrm{Var}(T_i) = A_i^2\mathrm{Var}(W_i) = \frac{nA_i^2}{(n+1)^2(n+2)}. \tag{12}$$

According to Lyapunov central limit theorem (see Billingsley [21]), we have $\sum_{i=1}^{n-1}(T_i - \mu_i)/\sqrt{\sum_{i=1}^{n-1}\sigma_i^2} = (R_n^q - \mathbb{E}(R_n^q)/\sqrt{\mathrm{Var}(R_n^q)}) \xrightarrow[n]{d} \mathcal{N}$, where $\mathcal{N}$ is the standard normal RV (in the sequel, the standard normal distribution will be denoted by $N(0,1)$). Therefore, under H_0, the percentage point ($\alpha-$quantile) CRE_α^{*q} is approximated according to the asymptotic normality of R_n^q for large n by

$$\mathrm{CRE}_\alpha^{*q} = \mathbb{E}\left(R_n^q\right) + \sqrt{\mathrm{Var}\left(R_n^q\right)}Z_\alpha, \tag{13}$$

where Z_α corresponds to the quantile $(\alpha \times 100)$ of the CDF $N(0,1)$.

Johannesson and Giri [22] proposed an approximation of the CDF of linear combination of the finite number of beta RVs. Noughabi [14] used this approximation to obtain approximately the percentage points of the CRE for finite n. By adopting the same procedure of Noughabi [14], we can obtain an approximation of R_n^q for finite n as follows:

$$R_n^q \approx \left(\sum_{i=1}^{n-1} A_i\right)Y, \tag{14}$$

where the RV Y has $\mathrm{Beta}(a, b)$ distribution,

$$a = \frac{(n+2)\left(\sum_{i=1}^{n-1} A_i\right)^2}{(n+1)\left(\sum_{i=1}^{n-1} A_i^2\right)} - \frac{1}{n+1},$$
$$b = \frac{n}{n+1}\left(\frac{(n+2)\left(\sum_{i=1}^{n-1} A_i\right)^2}{\sum_{i=1}^{n-1} A_i^2} - 1\right), \tag{15}$$

and $A_i = (1-(i/n))(-\ln(1-(i/n)))^q$, $0 \le q \le 1$, $i = 1, 2, \ldots, n-1$. According to (14), the mean and variance of R_n^q are, respectively,

$$\mathbb{E}\left(R_n^q\right) = \left(\sum_{i=1}^{n-1} A_i\right)\frac{a}{a+b},$$
$$\mathrm{Var}\left(R_n^q\right) = \left(\sum_{i=1}^{n-1} A_i\right)^2\frac{ab}{(a+b)^2(a+b+1)}. \tag{16}$$

Now, by using this approximation of R_n^q, the quantiles of order $\alpha/2$ and $1-(\alpha/2)$ of the approximated CDF of the test statistic $\mathrm{CRE}^q(F_n)$ under H_0 are, respectively,

$$\text{lower} \coloneqq \left(\sum_{i=1}^{n-1} A_i\right)F^{-1}\left(\frac{\alpha}{2}\right),$$
$$\text{upper} \coloneqq \left(\sum_{i=1}^{n-1} A_i\right)F^{-1}\left(1-\frac{\alpha}{2}\right), \tag{17}$$

where $F^{-1}(.)$ is the quantile function of the CDF F, F is the Beta (a,b) distribution, and a and b are defined in (15).

3.1. Empirical Weighted Cumulative Residual Entropy. From (3), Misagh et al. [15] proposed the empirical WCRE by

$$\begin{aligned}\mathrm{CRE}_w(F_n) &= -\sum_{i=1}^{n-1}\left(\frac{X_{(i+1)}^2 - X_{(i)}^2}{2}\right)\left(1-\frac{i}{n}\right)\ln\left(1-\frac{i}{n}\right)\\ &= \sum_{i=1}^{n-1} A'_i U_i,\end{aligned} \tag{18}$$

where $A'_i = X_{(i+1)}^2 - X_{(i)}^2/2$, $U_i = -(1-(i/n))\ln(1-(i/n))$, $i = 1,2,\ldots,n-1$.

We suggest the following statistic of a consistent test based on (18):

$$T_n^w = \mathrm{CRE}_w(F_n) = \sum_{i=1}^{n-1} A'_i U_i. \tag{19}$$

Theorem 5. *The test based on the sample estimate T_n^w is consistent.*

Proof. From Mirali et al. [12] and by using Glivenko–Cantelli theorem, (see Tucker [19]), we have $\mathrm{CRE}_w(F_n) \xrightarrow[n]{\text{a.s.}} \mathrm{CRE}_w(F)$, which proves the theorem. □

Theorem 6. *Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from an unknown continuous CDF F defined on $[0,1]$. Then, from (18), we get $0 \le T_n^w \le 1/2e$.*

Proof. Since the function $f(p) = -p\ln p$, $0 < p < 1$, has a maximum value at $1/e$; therefore,

$$\begin{aligned}0 \le T_n^w = \mathrm{CRE}_w(F_n) &= -\sum_{i=1}^{n-1}\left(\frac{X_{(i+1)}^2 - X_{(i)}^2}{2}\right)\left(1-\frac{i}{n}\right)\ln\left(1-\frac{i}{n}\right)\\ &\le \frac{1}{2}\sum_{i=1}^{n-1}\frac{1}{e}\left(X_{(i+1)}^2 - X_{(i)}^2\right) = \frac{1}{2e}\left(X_{(n)}^2 - X_{(1)}^2\right) \le \frac{1}{2e}.\end{aligned} \tag{20}$$

This completes the proof. □

3.2. Percentage Points. We generate 50,000 samples of size n, where $n = 10, 20, 30, 40, 50, 70, 100$, from $U(0,1)$. Using (6), the test statistic R_n^q is estimated by the empirical R_n^q for each sample and the same for T_n^w. Moreover, we can see that $\mathrm{CRE}^{0.1}(U) = 0.4438$, $\mathrm{CRE}^{0.5}(U) = 0.3133$, $\mathrm{CRE}^{0.9}(U) = 0.2576$ and $\mathrm{CRE}_w(U) = 0.1388$, where $\mathrm{CRE}^q(U)$ and $\mathrm{CRE}_w(U)$ are the FCRE and WCRE of the CDF $U(0,1)$, respectively. Consequently, for R_n^q, we present the percentage points of the Monte Carlo method, asymptotic normality, and beta approximation by using (10), (13), and (17), respectively. The result of this study is given in Table 1, where we note that the difference between the percentage points decreases when n increases. Besides, for R_n^q, the accuracy of the Monte Carlo method is more than the other two methods.

Figures 1–4 represent the empirical PDF's of the test statistics using Monte Carlo samples with $n = 10, 20, 30, 50, 100$. When n increases, it turned out that the test statistics are nearer to the exact values, which implies that the bias and the variance decrease with increasing n.

4. Power Analysis

In this section, we study the power test of Monte Carlo study under alternative distributions. The power of R_n^q is estimated by the proportion of the generated samples falling into the critical region. Under seven alternative distributions, the power of the test statistic R_n^q is calculated by the Monte Carlo study of generating 50,000 samples each of size n, where $n = 20, 30, 50$. The alternative CDFs proposed by Stephens [18] in power study of uniformity tests are as follows:

$$\begin{aligned}&A_l\colon F(y) = 1-(1-y)^l, \quad 0 \le y \le 1, l = 1.5, 2,\\ &B_l\colon F(y) = \begin{cases} 2^{l-1}y^l, & 0 \le y \le 0.5,\\ 1-2^{l-1}(1-y)^l, & 0.5 \le y \le 1, l = 1.5, 2, 3,\end{cases}\\ &C_l\colon F(y) = \begin{cases} 0.5-2^{l-1}(0.5-y)^l, & 0 \le y \le 0.5,\\ 0.5+2^{l-1}(y-0.5)^l, & 0.5 \le y \le 1, \text{ for } l = 1.5, 2.\end{cases}\end{aligned} \tag{21}$$

In Table 2, based on the Monte Carlo study, we recorded the power values of the proposed test statistics R_n^q, T_n^w, Kolmogorov–Smirnov (K-S), Kuiper (V), Cramer-von Mises (W^2), Watson (U^2), and Anderson-Darling (A^2), for $n = 10, 20, 30$ and $\alpha = 0.05$. From Table 2, we can conclude the following:

(1) If q increases and tends to 1 ($q \longrightarrow 1$), the power of *CRE^q* test, for alternative $A_l(B_l)(C_l)$, decreases (increases) (increases), and vice versa, if q decreases and tends to 0 ($q \longrightarrow 0$).

(2) If $q \longrightarrow 1$, the CRE^q test, for alternative $A_l(B_l)$, gives the worst (best) performance compared with the other tests.

(3) To compare the performance between CRE^q and CRE$_w$ tests, we observe that:

   (a) For the alternative $A_l, q \longrightarrow 1$, CRE$_w$ performs better than CRE^q and vice versa if $q \longrightarrow 0, n$ increases.

   (b) For the alternative $B_l, q \longrightarrow 1$, CRE^q performs better than CRE$_w$, and vice versa, if $q \longrightarrow 0, n$ increases.

   (c) For the alternative $C_l, q \longrightarrow 0$, CRE$_w$ performs better than CRE^q, and vice versa, if $q \longrightarrow 1$.

Stephens [18] noted that V and U^2 tests will reveal a change at variance. Therefore, we observe the following:

(1) For alternative $A_l, q \longrightarrow 0$, CRE^q performs better than V and U^2, and vice versa, if $q \longrightarrow 1$.

Table 1: Percentage points of the proposed test statistics R_n^q and T_n^w at level $\alpha = 0.05$.

n	q	R_n^q Monte Carlo method Upper	Lower	Normal approximation Upper	Lower	Beta approximation Upper	Lower	T_n^w Upper	Lower
	0.1	0.5131	0.2282	0.6165	0.1293	0.6495	0.1672		
10	0.5	0.3522	0.1818	0.4481	0.1065	0.47003	0.1315	0.1574	0.0669
	0.9	0.2964	0.14901	0.3732	0.0873	0.3916	0.1084		
	0.1	0.50608	0.3041	0.6012	0.2144	0.6217	0.2368		
20	0.5	0.3458	0.2341	0.4282	0.1632	0.4415	0.1777	0.1544	0.0957
	0.9	0.2892	0.1901	0.3549	0.1335	0.3661	0.1458		
	0.1	0.4995	0.3357	0.58407	0.2556	0.5987	0.2715		
30	0.5	0.3422	0.2544	0.4135	0.19007	0.423	0.2002	0.15275	0.1068
	0.9	0.2853	0.2066	0.34207	0.1555	0.35009	0.1641		
	0.1	0.4945	0.3541	0.5709	0.2809	0.5824	0.2931		
40	0.5	0.3393	0.2646	0.40309	0.2064	0.4104	0.2142	0.1515	0.1126
	0.9	0.2823	0.2153	0.3331	0.16902	0.3393	0.1756		
	0.1	0.4905	0.3653	0.5607	0.2983	0.5701	0.3082		
50	0.5	0.3374	0.2712	0.3953	0.2177	0.4013	0.2241	0.1506	0.1163
	0.9	0.2808	0.2207	0.3265	0.1783	0.3316	0.1837		
	0.1	0.4854	0.3801	0.5461	0.3213	0.553	0.3285		
70	0.5	0.3345	0.2796	0.3844	0.2326	0.3888	0.2372	0.1492	0.1206
	0.9	0.2776	0.2276	0.3172	0.1906	0.32103	0.1945		
	0.1	0.4799	0.3919	0.5317	0.3418	0.5367	0.3469		
100	0.5	0.3317	0.2859	0.37404	0.2459	0.3772	0.2492	0.14805	0.1242
	0.9	0.27508	0.2335	0.3085	0.2016	0.3112	0.2044		

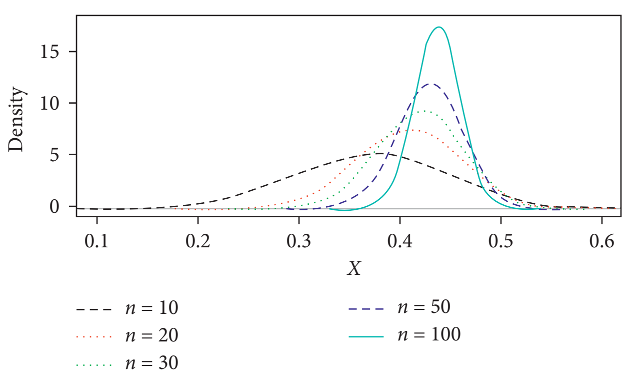

Figure 1: The estimated PDF's of $R_n^{0.1}$ based on U (0, 1).

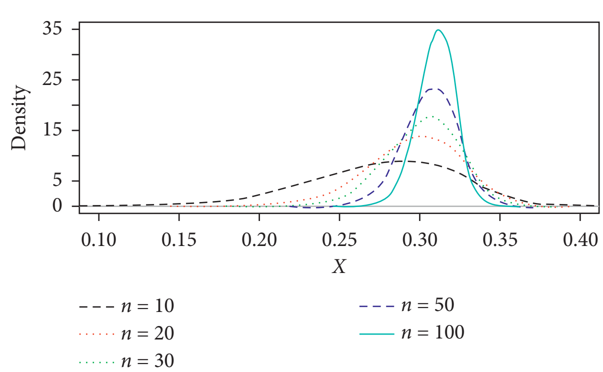

Figure 2: The estimated PDF's of $R_n^{0.5}$ based on U (0, 1).

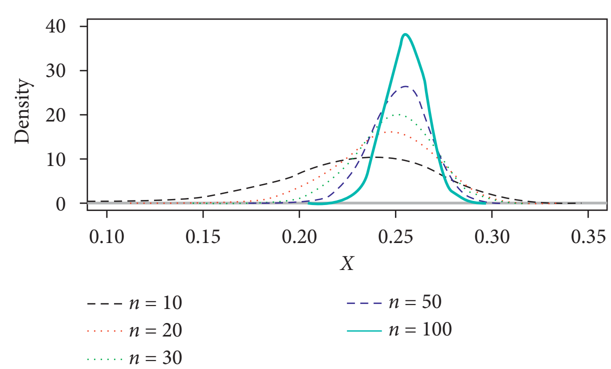

Figure 3: The estimated PDF's of $R_n^{0.9}$ based on U (0, 1).

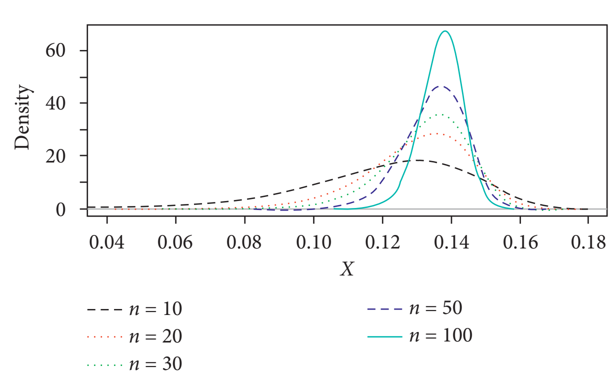

Figure 4: The estimated PDF's of T_n^w based on U (0, 1).

Table 2: Power estimates of the tests at level $\alpha = 0.05$.

n	Alternative	R_n^q 0.1	0.5	0.9	T_n^w	K-S	V	W^2	U^2	A^2
10	$A_{1.5}$	0.10708	0.0908	0.07208	0.14002	0.12616	0.0756	0.1456	0.07776	0.1877
	A_2	0.2771	0.2327	0.15104	0.3414	0.30298	0.1631	0.3551	0.16308	0.4761
	$B_{1.5}$	0.10406	0.1314	0.1302	0.0896	0.07352	0.0971	0.0741	0.1017	0.1349
	B_2	0.2427	0.3379	0.3357	0.21402	0.1184	0.2307	0.1104	0.2481	0.3269
	B_3	0.5763	0.7662	0.7723	0.5516	0.2424	0.5394	0.2154	0.5699	0.72308
	$C_{1.5}$	0.0843	0.119	0.1217	0.0942	0.0342	0.0974	0.0239	0.1031	0.0222
	C_2	0.1354	0.2478	0.2543	0.1723	0.0402	0.2333	0.01114	0.2475	0.00924
20	$A_{1.5}$	0.2496	0.1975	0.0909	0.2543	0.2179	0.1226	0.25208	0.1225	0.3235
	A_2	0.6679	0.5672	0.2654	0.637	0.5616	0.3486	0.6241	0.3358	0.7538
	$B_{1.5}$	0.1449	0.2828	0.2797	0.2047	0.0869	0.1634	0.0781	0.1786	0.1774
	B_2	0.38602	0.7223	0.7222	0.5511	0.1849	0.4647	0.162	0.5067	0.52802
	B_3	0.8104	0.9923	0.9931	0.954	0.4588	0.8711	0.4615	0.8978	0.93998
	$C_{1.5}$	0.0941	0.1979	0.2101	0.1516	0.0509	0.1621	0.02406	0.1791	0.0213
	C_2	0.1509	0.4551	0.4833	0.3296	0.1162	0.4633	0.0462	0.5048	0.0338
30	$A_{1.5}$	0.4045	0.3273	0.1158	0.3686	0.3144	0.18002	0.366	0.1721	0.4498
	A_2	0.8856	0.8061	0.3854	0.8285	0.7522	0.5447	0.8105	0.5071	0.8973
	$B_{1.5}$	0.1707	0.4466	0.4481	0.3331	0.1021	0.2477	0.0873	0.2667	0.2281
	B_2	0.4794	0.9148	0.9173	0.7985	0.2706	0.6695	0.25108	0.7076	0.7002
	B_3	0.9007	0.99994	0.99998	0.9983	0.6701	0.97506	0.7237	0.9819	0.99104
	$C_{1.5}$	0.1021	0.2759	0.2938	0.2084	0.07	0.2492	0.0303	0.2678	0.0271
	C_2	0.1627	0.6123	0.6513	0.4736	0.2077	0.6711	0.1258	0.7111	0.1105

(2) For the alternative $B_l, q \longrightarrow 1$, CRE^q performs better than V and U^2, and vice versa, if $q \longrightarrow 0, n$ increases.

(3) For the alternative $C_l, q \longrightarrow 0$, V and U^2 performs better than CRE^q.

(4) CRE_w performs better than V and U^2 against the alternative A_l.

(5) CRE_w performs better than V and U^2 against the alternative B_l, n increases. But, V and U^2 perform better than CRE_w against the alternative C_l.

Consequently, based on alternatives with a change toward a smaller variance, the tests CRE_w and CRE^q, $q \longrightarrow 1$, are the best. Meanwhile, under alternatives with a change toward a larger variance, the tests CRE_w and CRE^q, $q \longrightarrow 0$, are weaker.

5. Conclusion

For the CDFs with support $[0, 1]$, we exhibited that the values of CRE^q and CRE_w are within $[0, e^{-q}]$ and $[0, 1/2e]$, respectively. Moreover, the test of uniformity was proposed by calculating the percentage points and power analysis of CRE^q and CRE_w. Besides, for CRE^q, we obtained the percentage points by using the Monte Carlo method via the simulation and the normality asymptotic, as well as the beta approximation. Moreover, for CRE_w the percentage points were derived by using the Monte Carlo method via the simulation. A power comparison was performed between the FCRE and WCRE and other tests, where, by changing the value of q, we indicated when the test has higher and lower power compared with the other tests.

Data Availability

The simulated data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest concerning the publication of this article.

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