Hindawi
Journal of Mathematics
Volume 2021, Article ID 5353372, 11 pages
https://doi.org/10.1155/2021/5353372

Research Article

On q-ANALOGUE of Differential Subordination Associated with Lemniscate of Bernoulli

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Received 21 September 2021; Accepted 15 October 2021; Published 1 November 2021

Academic Editor: Fairouz Tchier

This article comprises the study of differential subordination with analogue of q-derivative. It includes the sufficient condition on γ for $1 + (\gamma \partial z_q h(z)/h^n(z))$ to be subordinated by $(1 + Az/1 + Bz)$, $-1 \leq B < A \leq 1$, and implies that $h(z) \prec \sqrt{1+z}$, where $h(z)$ is the analytic function in the open unit disk. Moreover, certain sufficient conditions for q-starlikeness of analytic functions related with lemniscate of Bernoulli are determined.

1. Introduction

Let a set $\mathscr{A}$ be considered as the class of analytic functions defined in open unit disk $\mathbb{U} = \{\varsigma\colon \varsigma \in \mathbb{C} \text{ and } |\varsigma| < 1\}$ under normalization conditions $f(0) = 0$ and $f'(0) = 1$, having

$$f(\varsigma) = \varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n, \quad \varsigma \in \mathbb{U}, \tag{1}$$

as Taylor series. The class S comprises the normalized univalent functions, defined in $\mathbb{U}$. The major subcategories of class S are C of convex functions and S^* of starlike functions. The class P is another important class of analytic univalent functions whose co-domains are restricted to the right half plane and are used to determine the convexity and starlikeness of univalent functions. For more details, see [1, 2].

Let f and g be two analytic functions in $\mathbb{U}$. Then, f is subordinated by g, denoted as $f \prec g$ if f can be written in the form of composition of g and ϖ as $f(\varsigma) = g(\varpi(\varsigma))$ subject to the existence of analytic function ϖ which satisfies the condition that $\varpi(0) = 0$ and $|\varpi(\varsigma)| < |\varsigma|$. Furthermore, if both f and g are univalent functions in $\mathbb{U}$, then $f \prec g$ implies that $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Subordination plays an important role in univalent function theory, and this concept was first introduced by Lindelöf, but Littlewood [3, 4] contributed remarkably to this field. Differential subordination is actually the generalized version of differential inequalities with real variables. Many researchers contributed in the work related to differential subordinations. Historical developments in the field of differential subordination are briefly described by Miller and Mocanu in [5].

The advancement in the field of differential subordination starts with the usage of univalent functions. It was noticed in an article by Miller et al. [6] in 1974. Furthermore, many developments in this field have been achieved with the usage of differential subordinations in past fifty years. Differential inequality was a very well-known concept of real variables, and to study it in terms of complex variables, Miller and Mocanu [7] in 1981 were the first ones to introduce the idea of differential subordination. The contribution of Ruscheweyh and Singh [8] and Ruscheweyh and Wilken [9] is also of great importance in this field. Well-known Jack's lemma [10] has brought the advancements in differential subordinations. Dziok [11] worked on some of the applications of Jack's lemma. The research work carried out by Ma and Minda [12] in the function theory is worth to mention here, as they introduced the analytic function Φ, which satisfies the conditions of normalization $\Phi(0) = 0$ and $\Phi'(0) > 1$ having real part positive. The authors in [12] utilized the function Φ and introduced the subclass $\mathscr{S}^*(\Phi)$ of starlike functions as follows:

$$\mathcal{S}^*(\Phi) = \left\{ f \in \mathcal{A}: \frac{\varsigma f'(\varsigma)}{f(\varsigma)} \prec \Phi(\varsigma); \quad \varsigma \in \mathbb{U} \right\}. \tag{2}$$

The idea presented in [12] is very useful, and it helped many researchers for further studies in this direction. Ali et al. [13, 14] worked on differential subordination for sufficiency criteria of Janowski starlikeness and evaluated several differential subordinations such as $1 + \gamma\varsigma (p'(\varsigma)/p^n(\varsigma))$ and found $p(\varsigma) \prec \sqrt{1+\varsigma}$. Also, Ravichandran et al. [15] used this concept to find the sufficient conditions for starlikeness of Bernoulli's lemniscate and Janowski functions. Sharma et al. [16] studied the differential subordinations to prove the starlikeness associated with cardioid domain and Halim et al. [17] introduced the concept for limacon domain.

Jackson [18, 19] was the one who introduced the q-derivatives and q-integrals. After Jackson, Srivastava was amongst the pioneers to contribute in the q-calculus for its usage for analytic functions and their subclasses. Not only this, but he also applied q-hypergeometric function in the functions theory. All these contributions are comprised in his book (pp. 347 in [20]). Ismail et al. [21] contributed in the q-calculus for the study of starlike functions. Anastassiu and Gal [22, 23] also played their part in the development of complex variables with q-generalization. Purohit et al. [24] have used fractional q-calculus operators to apply subordination conditions on the class of non-Bazilevic functions. Sahoo and Agrawal [25] worked on starlike functions in q-calculus and extended the idea of q-starlikeness for particular subclasses of starlike functions. The involvement of q-derivative in the class $\mathcal{S}^*(\Phi)$ gave the formation of following subclass $\mathcal{S}_q^*(\Phi)$ of starlike functions which was introduced by Aouf and Seoudy [26].

$$\mathcal{S}_q^*(\Phi) = \left\{ f \in \mathcal{A}: \frac{\varsigma D_q f(\varsigma)}{f(\varsigma)} \prec \Phi(\varsigma); \quad \varsigma \in \mathbb{U} \right\}. \tag{3}$$

The class described above has drawn the attention of many researchers. Replacing $\Phi(\varsigma)$ with different functions such as Janowski, lemniscate of Bernoulli, cardioid, and limacon, the researchers got the new directions to the study. Srivastava et al. [27] studied q-derivatives to find the relation between different classes of q-starlike functions related to Janowski function. Srivastava et al. [28] introduced the class of q-starlike functions by using general conic domains. They also obtained the bounds on Hankel and Toeplitz determinants for q-starlike functions and continued working par excellence. They produced unmatchable results that worked as great motivation for many researchers worldwide. To have an idea of their remarkable work, one can see [29–31], [20, 27, 28, 32–43]. Contributions of Haq et al. [44] and Zainab et al. [45] are also worth to mention. They studied q-analogue of differential subordinations for star-like functions related to limacon and cardioid domains, and Janowski functions. The q-derivative is the foundation of all this work in q-analogue, and it is defined as follows.

The q-derivative of a complex valued function f, defined in the domain $\mathbb{U}$, is given as follows:

$$\left(D_q f\right)(\varsigma) = \begin{cases} \dfrac{f(\varsigma) - f(q\varsigma)}{(1-q)\varsigma}, & \varsigma \neq 0, \\ \\ f'(0), & \varsigma = 0, \end{cases} \tag{4}$$

where $0 < q < 1$. This implies the following:

$$\lim_{q \longrightarrow 1^-} \left(D_q f\right)(\varsigma) = \lim_{q \longrightarrow 1^-} \frac{f(\varsigma) - f(q\varsigma)}{(1-q)\varsigma} = f'(\varsigma), \tag{5}$$

on the assumption that the function f is differentiable in $\mathbb{U}$. The q-derivative $D_q f$ of an analytic function f has Taylor series of the form

$$\left(D_q f\right)(\varsigma) = \sum_{n=0}^{\infty} [n]_q a_n \varsigma^{n-1}, \tag{6}$$

where

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \\ \displaystyle\sum_{k=0}^{n-1} q^k, & n \in \mathbb{N}. \end{cases} \tag{7}$$

For more details about q-derivative and recent work on it, we refer the readers to [29–31], [20, 27, 28, 32–43]. In addition, the q-analogue of Jack's lemma has played a vital role in this paper which states as follows.

Lemma 1 (see [46]). *Let ϖ be an analytic function in $\mathbb{U}$ with $\varpi(0) = 0$. For maximum of ϖ on $|\varsigma| = 1$ at $\varsigma_0 = ae^{i\theta}$, where $\theta \in [-\pi, \pi]$ and $0 < q < 1$, then we have*

$$\varsigma_0 D_q \varpi(\varsigma_0) = m\varpi(\varsigma_0), \tag{8}$$

where $m \in \mathbb{R}$ with $m \geq 1$.

2. Main Results

Theorem 1. *Assume that*

$$|\gamma| \geq \frac{(A-B)(\sqrt{2} + \sqrt{3-q})}{(1-|B|)}, \quad -1 < B < A \leq 1. \tag{9}$$

Consider an analytic function h on $\mathbb{U}$ with $h(0) = 1$ which satisfies

$$1 + \gamma\varsigma D_q h(\varsigma) \prec \frac{1 + A\varsigma}{1 + B\varsigma}, \quad \varsigma \in \mathbb{U}. \tag{10}$$

Also, suppose

$$1 + \gamma\varsigma D_q h(\varsigma) = \frac{1 + A\varpi(\varsigma)}{1 + B\varpi(\varsigma)}, \quad \varsigma \in \mathbb{U}. \tag{11}$$

Here, ϖ is an analytic function in $\mathbb{U}$ such that $\varpi(0) = 0$. Then, we have

$$h(\varsigma) \prec \sqrt{1+\varsigma}. \tag{12}$$

Proof. Suppose that

$$p(\varsigma) = 1 + \gamma\varsigma D_q h(\varsigma), \tag{13}$$

where p is analytic, and we have $p(0) = 1$. Also, consider that

$$h(\varsigma) = \sqrt{1 + \varpi(\varsigma)}. \tag{14}$$

Now, we prove that $|\varpi(\varsigma)| < 1$, where

$$\varpi(\varsigma) = \frac{p(\varsigma) - 1}{A - Bp(\varsigma)}. \tag{15}$$

Using (13) and (14), we obtain

$$p(\varsigma) = 1 + \gamma\varsigma \frac{D_q\varpi(\varsigma)}{\sqrt{1 + \varpi(\varsigma)} + \sqrt{1 + \varpi(\varsigma) - \varsigma D_q\varpi(\varsigma)(1 - q)}}. \tag{16}$$

Also, we have

$$\left|\frac{p(\varsigma) - 1}{A - Bp(\varsigma)}\right| = \left|\frac{\gamma\varsigma D_q\varpi(\varsigma)}{(A - B)\left[\sqrt{1 + \varpi(\varsigma)} + \sqrt{1 + \varpi(\varsigma) - \varsigma D_q\varpi(\varsigma)(1 - q)}\right] - B\gamma\varsigma D_q\varpi(\varsigma)}\right|. \tag{17}$$

Consider a point $\varsigma_0 \in \mathbb{U}$ such that

$$\max_{|\varsigma| \leq |\varsigma_0|} |\varpi(\varsigma)| = |\varpi(\varsigma_0)| = 1. \tag{18}$$

Now, by using Lemma 1, we have $\varsigma_0 D_q\varpi(\varsigma_0) = m\varpi(\varsigma_0)$, $m \geq 1$. Now, consider that $\varpi(\varsigma_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$; then, for $\varsigma_0 \in \mathbb{U}$, we obtain

$$\begin{aligned} \left|\frac{p(\varsigma_0) - 1}{A - Bp(\varsigma_0)}\right| &= \left|\frac{\gamma\varsigma_0 D_q\varpi(\varsigma_0)}{(A - B)\left[\sqrt{1 + \varpi(\varsigma_0)} + \sqrt{1 + \varpi(\varsigma_0) - \varsigma_0 D_q\varpi(\varsigma_0)(1 - q)}\right] - B\gamma\varsigma_0 D_q\varpi(\varsigma_0)}\right|, \\ &\geq \frac{|\gamma| m}{(A - B)\left[\sqrt{|1| + |e^{i\theta}|} + \sqrt{|1| + |e^{i\theta}| + |me^{i\theta}(1 - q)|}\right] + |B||\gamma| m}, \\ &= \frac{|\gamma| m}{(A - B)\left[\sqrt{2} + \sqrt{2 + m(1 - q)}\right] + |B||\gamma| m}. \end{aligned} \tag{19}$$

Consider a new function

$$\Xi(m) = \frac{|\gamma| m}{(A - B)\left[\sqrt{2} + \sqrt{2 + m(1 - q)}\right] + |B||\gamma| m}. \tag{20}$$

Then,

$$\Xi'(m) = \frac{|\gamma|\left[(A - B)\left\{\sqrt{2} + \sqrt{2 + m(1 - q)}\right\}\right] - |\gamma| m\left[\left((A - B)(1 - q)/2\sqrt{2 + m(1 - q)}\right)\right]}{\left[(A - B)\left\{\sqrt{2} + \sqrt{2 + m(1 - q)}\right\} + |B||\gamma| m\right]^2} > 0. \tag{21}$$

Above expression represents that the function Ξ has increasing behavior, so we have its minimum value at $m = 1$ and

$$\Xi(1) = \frac{|\gamma|}{(A - B)\left[\sqrt{2} + \sqrt{3 - q}\right] + |B||\gamma|}. \tag{22}$$

So, we conclude that

$$\left|\frac{p(\varsigma_0) - 1}{A - Bp(\varsigma_0)}\right| \geq \frac{|\gamma|}{(A - B)\left[\sqrt{2} + \sqrt{3 - q}\right] + |B||\gamma|}. \tag{23}$$

Now, from (9), we have

$$\left|\frac{p(\varsigma_0) - 1}{A - Bp(\varsigma_0)}\right| \geq 1, \tag{24}$$

Since this result contradicts (10), therefore, $|\varpi(\varsigma)| < 1$, which completes the proof.

By taking $h(\varsigma) = (\varsigma D_q f(\varsigma)/f(\varsigma))$, we deduce the following result. □

Corollary 1. *Let* $|\gamma| \geq ((A - B)(\sqrt{2} + \sqrt{3 - q})/(1 - |B|))$, $-1 < B < A \leq 1$ *and* $f \in \mathcal{A}$, *satisfy the subordination*

$$1+\gamma\varsigma D_q\left(\frac{\varsigma D_q f(\varsigma)}{f(\varsigma)}\right)\prec\frac{1+A\varsigma}{1+B\varsigma}. \tag{25}$$

Then, $f\in\mathcal{S}_q^*(\sqrt{1+\varsigma})$.

Theorem 2. *Assume that*

$$|\gamma|\geq\frac{\sqrt{2}(A-B)(\sqrt{2}+\sqrt{3-q})}{(1-|B|)},\quad -1<B<A\leq 1. \tag{26}$$

Consider an analytic function h on $\mathbb{U}$ with $h(0)=1$ which satisfies

$$1+\frac{\gamma\varsigma D_q h(\varsigma)}{h(\varsigma)}\prec\frac{1+A\varsigma}{1+B\varsigma},\quad \varsigma\in\mathbb{U}. \tag{27}$$

Also, suppose

$$1+\frac{\gamma\varsigma D_q h(\varsigma)}{h(\varsigma)}=\frac{1+A\varpi(\varsigma)}{1+B\varpi(\varsigma)},\quad \varsigma\in\mathbb{U}, \tag{28}$$

where ϖ is analytic function on $\mathbb{U}$ with $\varpi(0)=0$. Then,

$$h(\varsigma)\prec\sqrt{1+\varsigma}. \tag{29}$$

Proof. We define a function

$$p(\varsigma)=1+\frac{\gamma\varsigma D_q h(\varsigma)}{h(\varsigma)}, \tag{30}$$

where p is analytic and $p(0)=1$. Now, consider that

$$h(\varsigma)=\sqrt{1+\varpi(\varsigma)}. \tag{31}$$

To obtain the result, we have to show that $|\varpi(\varsigma)|<1$. Using (30) and (31), we obtain the result

$$p(\varsigma)=1+\gamma\varsigma\frac{D_q\varpi(\varsigma)}{\sqrt{1+\varpi(\varsigma)}\left[\sqrt{1+\varpi(\varsigma)}+\sqrt{1+\varpi(\varsigma)-\varsigma D_q\varpi(\varsigma)(1-q)}\right]}. \tag{32}$$

Also, we have

$$\left|\frac{p(\varsigma)-1}{A-Bp(\varsigma)}\right|=\left|\frac{\gamma\varsigma D_q\varpi(\varsigma)}{(A-B)\sqrt{1+\varpi(\varsigma)}\left[\sqrt{1+\varpi(\varsigma)}+\sqrt{1+\varpi(\varsigma)-\varsigma D_q\varpi(\varsigma)(1-q)}\right]-B\gamma\varsigma D_q\varpi(\varsigma)}\right|. \tag{33}$$

Consider a point $\varsigma_0\in\mathbb{U}$ such that

$$\max_{|\varsigma|\leq|\varsigma_0|}|\varpi(\varsigma)|=|\varpi(\varsigma_0)|=1. \tag{34}$$

Now, by using Lemma 1, we have $\varsigma_0 D_q\varpi(\varsigma_0)=m\varpi(\varsigma_0)$, $m\geq 1$. Now, consider that $\varpi(\varsigma_0)=e^{i\theta}$, $\theta\in[-\pi,\pi]$; then, for $\varsigma_0\in\mathbb{U}$, we obtain

$$\begin{aligned}\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right|&=\left|\frac{\gamma m e^{i\theta}}{(A-B)\sqrt{1+e^{i\theta}}\left[\sqrt{1+e^{i\theta}}+\sqrt{1+e^{i\theta}-me^{i\theta}(1-q)}\right]-B\gamma m e^{i\theta}}\right|,\\ &\geq\frac{|\gamma|m}{(A-B)\sqrt{|1|+|e^{i\theta}|}\left[\sqrt{|1|+|e^{i\theta}|}+\sqrt{|1|+|e^{i\theta}|+|me^{i\theta}(1-q)|}\right]+|B||\gamma|m},\\ &=\frac{|\gamma|m}{(A-B)\sqrt{2}[\sqrt{2}+\sqrt{2+m(1-q)}]+|B||\gamma|m}.\end{aligned} \tag{35}$$

Consider

$$\Xi_1(m) = \frac{|\gamma| m}{(A-B)\sqrt{2}[\sqrt{2} + \sqrt{2+m(1-q)}] + |B||\gamma| m}. \tag{36}$$

Then,

$$\Xi_1'(m) = \frac{|\gamma|[(A-B)\sqrt{2}\{\sqrt{2} + \sqrt{2+m(1-q)}\}] - |\gamma| m [(\sqrt{2}(A-B)(1-q)/2\sqrt{2+m(1-q)})]}{[(A-B)\sqrt{2}\left\{\sqrt{2} + \sqrt{2+m(1-q)}\right\} + |B||\gamma| m]^2} > 0. \tag{37}$$

Above expression represents that function Ξ_1 has increasing behavior, so we have its minimum value at $m = 1$ and

$$\Xi_1(1) = \frac{|\gamma|}{(A-B)\sqrt{2}[\sqrt{2} + \sqrt{3-q}] + |B||\gamma|}. \tag{38}$$

So, we conclude that

$$\left|\frac{p(\varsigma_0) - 1}{A - Bp(\varsigma_0)}\right| \geq \frac{|\gamma|}{(A-B)\sqrt{2}[\sqrt{2} + \sqrt{3-q}] + |B||\gamma|}. \tag{39}$$

Now, from (26), we have

$$\left|\frac{p(\varsigma_0) - 1}{A - Bp(\varsigma_0)}\right| \geq 1, \tag{40}$$

which contradicts (27), and hence, $|\varpi(\varsigma)| < 1$, which completes the proof.

By taking $h(\varsigma) = (\varsigma D_q f(\varsigma)/f(\varsigma))$,we deduce the following result. □

Corollary 2. *Let* $|\gamma| \geq \sqrt{2}((A-B)(\sqrt{2} + \sqrt{3-q})/(1-|B|))$, $-1 < B < A \leq 1$ *and* $f \in \mathcal{A}$, *which satisfies the subordination*

$$1 + \gamma\varsigma\left(\frac{f(\varsigma)}{\varsigma D_q f(\varsigma)}\right) D_q\left(\frac{\varsigma D_q f(\varsigma)}{f(\varsigma)}\right) \prec \frac{1 + A\varsigma}{1 + B\varsigma}. \tag{41}$$

Then, $f \in \mathcal{S}_q^*(\sqrt{1+\varsigma})$.

Theorem 3. *Assume that*

$$|\gamma| \geq \frac{2(A-B)(\sqrt{2} + \sqrt{3-q})}{(1-|B|)}, \quad -1 < B < A \leq 1. \tag{42}$$

Consider an analytic function h on $\mathbb{U}$ with $h(0) = 1$ which satisfies

$$1 + \frac{\gamma\varsigma D_q h(\varsigma)}{h^2(\varsigma)} \prec \frac{1 + A\varsigma}{1 + B\varsigma}, \quad \varsigma \in \mathbb{U}. \tag{43}$$

Also, suppose

$$1 + \frac{\gamma\varsigma D_q h(\varsigma)}{h^2(\varsigma)} = \frac{1 + A\varpi(\varsigma)}{1 + B\varpi(\varsigma)}, \quad \varsigma \in \mathbb{U}, \tag{44}$$

where ϖ is analytic function on $\mathbb{U}$ with $\varpi(0) = 0$. Then,

$$h(\varsigma) \prec \sqrt{1+\varsigma}. \tag{45}$$

Proof. We define a function

$$p(\varsigma) = 1 + \frac{\gamma\varsigma D_q h(\varsigma)}{h^2(\varsigma)}, \tag{46}$$

where p is analytic, and we have $p(0) = 1$. Now, consider that

$$h(\varsigma) = \sqrt{1 + \varpi(\varsigma)}. \tag{47}$$

To obtain the result, we have to show that $|\varpi(\varsigma)| < 1$. Using (46) and (47), we obtain the result

$$p(\varsigma) = 1 + \gamma\varsigma \frac{D_q\varpi(\varsigma)}{\sqrt{1+\varpi(\varsigma)}\left[\sqrt{1+\varpi(\varsigma)} + \sqrt{1+\varpi(\varsigma) - \varsigma D_q\varpi(\varsigma)(1-q)}\right]}. \tag{48}$$

Also, we have

$$\left|\frac{p(\varsigma) - 1}{A - Bp(\varsigma)}\right| = \left|\frac{\gamma\varsigma D_q\varpi(\varsigma)}{(A-B)\sqrt{1+\varpi(\varsigma)}\left[\sqrt{1+\varpi(\varsigma)} + \sqrt{1+\varpi(\varsigma) - \varsigma D_q\varpi(\varsigma)(1-q)}\right] - B\gamma\varsigma D_q\varpi(\varsigma)}\right|. \tag{49}$$

Consider a point $\varsigma_0 \in \mathbb{U}$ such that

$$\max_{|\varsigma|\leq|\varsigma_0|} |\varpi(\varsigma)| = |\varpi(\varsigma_0)| = 1. \tag{50}$$

Now, by using Lemma 1, we have $\varsigma_0 D_q \varpi(\varsigma_0) = m\varpi(\varsigma_0)$, $m \geq 1$. Now, consider that $\varpi(\varsigma_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$; then, for $\varsigma_0 \in \mathbb{U}$, we obtain

$$\begin{aligned}\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right| &= \left|\frac{\gamma m e^{i\theta}}{(A-B)\sqrt{1+e^{i\theta}}\left[\sqrt{1+e^{i\theta}}+\sqrt{1+e^{i\theta}-me^{i\theta}(1-q)}\right]-B\gamma m e^{i\theta}}\right|,\\ &\geq \frac{|\gamma|m}{(A-B)\sqrt{|1|+|e^{i\theta}|}\left[\sqrt{|1|+|e^{i\theta}|}+\sqrt{|1|+|e^{i\theta}|+|me^{i\theta}(1-q)|}\right]+|B||\gamma|m},\\ &= \frac{|\gamma|m}{(A-B)2[\sqrt{2}+\sqrt{2+m(1-q)}]+|B||\gamma|m}.\end{aligned} \tag{51}$$

Consider a function

$$\Xi_2(m) = \frac{|\gamma|m}{(A-B)2[\sqrt{2}+\sqrt{2+m(1-q)}]+|B||\gamma|m}. \tag{52}$$

Then,

$$\Xi_2'(m) = \frac{|\gamma|[(A-B)2\{\sqrt{2}+\sqrt{2+m(1-q)}\}]-|\gamma|m[(2(A-B)(1-q)/2\sqrt{2+m(1-q)})]}{[(A-B)2\left\{\sqrt{2}+\sqrt{2+m(1-q)}\right\}+|B||\gamma|m]^2} > 0. \tag{53}$$

Here, Ξ_2 is clearly an increasing function, so we have its minimum value at $m = 1$ and

$$\Xi_2(1) = \frac{|\gamma|}{2(A-B)[\sqrt{2}+\sqrt{3-q}]+|B||\gamma|}. \tag{54}$$

So, we conclude that

$$\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right| \geq \frac{|\gamma|}{2(A-B)[\sqrt{2}+\sqrt{3-q}]+|B||\gamma|}. \tag{55}$$

Now, from (42), we have

$$\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right| \geq 1, \tag{56}$$

which contradict (43), and hence, $|\varpi(\varsigma)| < 1$, which completes the proof.

By taking $h(\varsigma) = (\varsigma D_q f(\varsigma)/f(\varsigma))$, we deduce the following result. □

Corollary 3. *Let $|\gamma| \geq 2((A-B)(\sqrt{2}+\sqrt{3-q})/(1-|B|))$, $-1 < B < A \leq 1$ and $f \in \mathcal{A}$, satisfies the subordination*

$$1+\gamma\varsigma\left(\frac{f(\varsigma)}{\varsigma D_q f(\varsigma)}\right)^2 D_q\left(\frac{\varsigma D_q f(\varsigma)}{f(\varsigma)}\right) \prec \frac{1+A\varsigma}{1+B\varsigma}. \tag{57}$$

Then, $f \in \mathcal{S}_q^*(\sqrt{1+\varsigma})$.

Theorem 4. *Assume that*

$$|\gamma| \geq \frac{2\sqrt{2}(A-B)(\sqrt{2}+\sqrt{3-q})}{(1-|B|)}, \quad -1 < B < A \leq 1. \tag{58}$$

Consider an analytic function h on $\mathbb{U}$ with $h(0) = 1$ which satisfies

$$1+\frac{\gamma\varsigma D_q h(\varsigma)}{h^3(\varsigma)} \prec \frac{1+A\varsigma}{1+B\varsigma}, \quad \varsigma \in \mathbb{U}. \tag{59}$$

Also, suppose

$$1+\frac{\gamma\varsigma D_q h(\varsigma)}{h^3(\varsigma)} = \frac{1+A\varpi(\varsigma)}{1+B\varpi(\varsigma)}, \quad \varsigma \in \mathbb{U}, \tag{60}$$

where ϖ is analytic function on $\mathbb{U}$ with $\varpi(0) = 0$. Then,

$$h(\varsigma) \prec \sqrt{1+\varsigma}. \tag{61}$$

Proof. We define a function

$$p(\varsigma) = 1+\frac{\gamma\varsigma D_q h(\varsigma)}{h^3(\varsigma)}, \tag{62}$$

where p is analytic, and we have $p(0) = 1$. Now, consider that

$$h(\varsigma) = \sqrt{1 + \varpi(\varsigma)}. \tag{63}$$

To obtain the result, we have to show that $|\varpi(\varsigma)| < 1$. Using (62) and (63), we obtain

$$p(\varsigma) = 1 + \gamma\varsigma \frac{D_q\varpi(\varsigma)}{(1 + \varpi(\varsigma))^{(3/2)}\left[\sqrt{1 + \varpi(\varsigma)} + \sqrt{1 + \varpi(\varsigma) - \varsigma D_q\varpi(\varsigma)(1 - q)}\right]}. \tag{64}$$

Also, we have

$$\left|\frac{p(\varsigma) - 1}{A - Bp(\varsigma)}\right| = \left|\frac{\gamma\varsigma D_q\varpi(\varsigma)}{(A - B)(1 + \varpi(\varsigma))^{(3/2)}\left[\sqrt{1 + \varpi(\varsigma)} + \sqrt{1 + \varpi(\varsigma) - \varsigma D_q\varpi(\varsigma)(1 - q)}\right] - B\gamma\varsigma D_q\varpi(\varsigma)}\right|. \tag{65}$$

Consider a point $\varsigma_0 \in \mathbb{U}$ such that

$$\max_{|\varsigma| \le |\varsigma_0|} |\varpi(\varsigma)| = |\varpi(\varsigma_0)| = 1. \tag{66}$$

Now, by using Lemma 1, we have $\varsigma_0 D_q\varpi(\varsigma_0) = m\varpi(\varsigma_0)$, $m \ge 1$. Now, consider that $\varpi(\varsigma_0) = e^{i\theta}$, $\theta \in [-\pi, \pi]$; then, for $\varsigma_0 \in \mathbb{U}$, we obtain

$$\begin{aligned}
\left|\frac{p(\varsigma_0) - 1}{A - Bp(\varsigma_0)}\right| &= \left|\frac{\gamma m e^{i\theta}}{(A - B)\left(1 + e^{i\theta}\right)^{(3/2)}\left[\sqrt{1 + e^{i\theta}} + \sqrt{1 + e^{i\theta} - me^{i\theta}(1 - q)}\right] - B\gamma m e^{i\theta}}\right|, \\
&\ge \frac{|\gamma| m}{(A - B)\left(|1| + \left|e^{i\theta}\right|\right)^{(3/2)}\left[\sqrt{|1| + \left|e^{i\theta}\right|} + \sqrt{|1| + \left|e^{i\theta}\right| + \left|me^{i\theta}(1 - q)\right|}\right] + |B||\gamma| m}, \\
&= \frac{|\gamma| m}{(A - B)2^{(3/2)}\left[\sqrt{2} + \sqrt{2 + m(1 - q)}\right] + |B||\gamma| m}.
\end{aligned} \tag{67}$$

Consider a function

$$\Xi_3(m) = \frac{|\gamma| m}{2^{(3/2)}(A - B)\left[\sqrt{2} + \sqrt{2 + m(1 - q)}\right] + |B||\gamma| m}. \tag{68}$$

Then,

$$\Xi_3'(m) = \frac{|\gamma|\left[(A - B)2^{(3/2)}\left\{\sqrt{2} + \sqrt{2 + m(1 - q)}\right\}\right] - |\gamma| m\left[\left(2^{(3/2)}(A - B)(1 - q)/2\sqrt{2 + m(1 - q)}\right)\right]}{\left[(A - B)2^{(3/2)}\left\{\sqrt{2} + \sqrt{2 + m(1 - q)}\right\} + |B||\gamma| m\right]^2} > 0. \tag{69}$$

Above expression represents that function Ξ_3 has increasing behavior, so we have its minimum value at $m=1$ and

$$\Xi_3(1)=\frac{|\gamma|}{2^{(3/2)}(A-B)[\sqrt{2}+\sqrt{3-q}]+|B||\gamma|}. \tag{70}$$

So, we conclude that

$$\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right|\geq\frac{|\gamma|}{2^{(3/2)}(A-B)[\sqrt{2}+\sqrt{3-q}]+|B||\gamma|}. \tag{71}$$

Now, from (58), we have

$$\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right|\geq 1, \tag{72}$$

which contradicts (59), and hence, $|\varpi(\varsigma)|<1$, which completes the proof.

By taking $h(\varsigma)=(\varsigma D_q f(\varsigma)/f(\varsigma))$, we deduce the following result. □

Corollary 5. *Let* $|\gamma|\geq 2\sqrt{2}((A-B)(\sqrt{2}+\sqrt{3-q})/(1-|B|))$, $-1<B<A\leq 1$ *and* $f\in\mathscr{A}$, *satisfy the subordination*

$$1+\gamma\varsigma\left(\frac{f(\varsigma)}{\varsigma D_q f(\varsigma)}\right)^3 D_q\left(\frac{\varsigma D_q f(\varsigma)}{f(\varsigma)}\right)\prec\frac{1+A\varsigma}{1+B\varsigma}. \tag{73}$$

Then, $f\in\mathcal{S}_q^*(\sqrt{1+\varsigma})$.

Theorem 6. *Assume that*

$$|\gamma|\geq\frac{2^{(n/2)}(A-B)(\sqrt{2}+\sqrt{3-q})}{(1-|B|)},\quad -1<B<A\leq 1. \tag{74}$$

Consider an analytic function h on $\mathbb{U}$ with $h(0)=1$ which satisfies

$$1+\frac{\gamma\varsigma D_q h(\varsigma)}{h^n(\varsigma)}\prec\frac{1+A\varsigma}{1+B\varsigma},\quad \varsigma\in\mathbb{U}. \tag{75}$$

Also, suppose

$$1+\frac{\gamma\varsigma D_q h(\varsigma)}{h^n(\varsigma)}=\frac{1+A\varpi(\varsigma)}{1+B\varpi(\varsigma)},\quad \varsigma\in\mathbb{U}, \tag{76}$$

where ϖ is analytic function on $\mathbb{U}$ with $\varpi(0)=0$. Then,

$$h(\varsigma)\prec\sqrt{1+\varsigma}. \tag{77}$$

Proof. We define a function

$$p(\varsigma)=1+\frac{\gamma\varsigma D_q h(\varsigma)}{h^n(\varsigma)}, \tag{78}$$

where p is analytic, and we have $p(0)=1$. Now, consider that

$$h(\varsigma)=\sqrt{1+\varpi(\varsigma)}. \tag{79}$$

To obtain the result, we have to show that $|\varpi(\varsigma)|<1$. Using (78) and (79), we obtain

$$p(\varsigma)=1+\gamma\varsigma\frac{D_q\varpi(\varsigma)}{(1+\varpi(\varsigma))^{(n/2)}\left[\sqrt{1+\varpi(\varsigma)}+\sqrt{1+\varpi(\varsigma)-\varsigma D_q\varpi(\varsigma)(1-q)}\right]}. \tag{80}$$

Also, we have

$$\begin{aligned}\left|\frac{p(\varsigma)-1}{A-Bp(\varsigma)}\right|&=\left|\frac{\gamma\varsigma\left(D_q\varpi(\varsigma)/(1+\varpi(\varsigma))^{(n/2)}\left[\sqrt{1+\varpi(\varsigma)}+\sqrt{1+\varpi(\varsigma)-\varsigma D_q\varpi(\varsigma)(1-q)}\right]\right)}{A-B\left[1+\gamma\varsigma\left(D_q\varpi(\varsigma)/(1+\varpi(\varsigma))^{(n/2)}\left[\sqrt{1+\varpi(\varsigma)}+\sqrt{1+\varpi(\varsigma)-\varsigma D_q\varpi(\varsigma)(1-q)}\right]\right)\right]}\right|\\&=\left|\frac{\gamma\varsigma D_q\varpi(\varsigma)}{(A-B)(1+\varpi(\varsigma))^{(n/2)}\left[\sqrt{1+\varpi(\varsigma)}+\sqrt{1+\varpi(\varsigma)-\varsigma D_q\varpi(\varsigma)(1-q)}\right]-B\gamma\varsigma D_q\varpi(\varsigma)}\right|.\end{aligned} \tag{81}$$

Consider a point $\varsigma_0\in\mathbb{U}$ such that

$$\max_{|\varsigma|\leq|\varsigma_0|}|\varpi(\varsigma)|=|\varpi(\varsigma_0)|=1. \tag{82}$$

Now, by using Lemma 1, we have $\varsigma_0 D_q\varpi(\varsigma_0)=m\varpi(\varsigma_0)$, $m\geq 1$. Now, consider that $\varpi(\varsigma_0)=e^{i\theta}$, $\theta\in[-\pi,\pi]$; then, for $\varsigma_0\in\mathbb{U}$, we obtain

$$\begin{aligned}\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right| &= \frac{\gamma\varsigma_0 D_q\varpi(\varsigma_0)}{(A-B)(1+\varpi(\varsigma_0))^{(n/2)}\left[\sqrt{1+\varpi(\varsigma_0)}+\sqrt{1+\varpi(\varsigma_0)-\varsigma_0 D_q\varpi(\varsigma_0)(1-q)}\right]-B\gamma\varsigma_0 D_q\varpi(\varsigma_0)},\\ &= \left|\frac{\gamma m e^{i\theta}}{(A-B)\left(1+e^{i\theta}\right)^{(n/2)}\left[\sqrt{1+e^{i\theta}}+\sqrt{1+e^{i\theta}-me^{i\theta}(1-q)}\right]-B\gamma m e^{i\theta}}\right|,\\ &\geq \frac{|\gamma| m}{(A-B)\left(1+e^{i\theta}\right)^{(n/2)}\left[\left|\sqrt{1+e^{i\theta}}\right|+\left|\sqrt{1+e^{i\theta}-me^{i\theta}(1-q)}\right|\right]-|B||\gamma| m},\\ &= \frac{|\gamma| m}{(A-B)\left(|1|+\left|e^{i\theta}\right|\right)^{(n/2)}\left[\sqrt{\left|1+e^{i\theta}\right|}+\sqrt{\left|1+e^{i\theta}-me^{i\theta}(1-q)\right|}\right]+|B||\gamma| m},\\ &\geq \frac{|\gamma| m}{(A-B)\left(|1|+\left|e^{i\theta}\right|\right)^{(n/2)}\left[\sqrt{|1|+\left|e^{i\theta}\right|}+\sqrt{|1|+\left|e^{i\theta}\right|+\left|me^{i\theta}(1-q)\right|}\right]+|B||\gamma| m},\\ &= \frac{|\gamma| m}{(A-B)2^{(n/2)}\left[\sqrt{2}+\sqrt{2+m(1-q)}\right]+|B||\gamma| m}.\end{aligned} \tag{83}$$

Consider

$$\Xi_4(m) = \frac{|\gamma| m}{2^{(n/2)}(A-B)\left[\sqrt{2}+\sqrt{2+m(1-q)}\right]+|B||\gamma| m}. \tag{84}$$

Then,

$$\Xi_4'(m) = \frac{|\gamma|\left[(A-B)2^{(n/2)}\left\{\sqrt{2}+\sqrt{2+m(1-q)}\right\}\right]-|\gamma| m\left[\left(2^{(n/2)}(A-B)(1-q)/2\sqrt{2+m(1-q)}\right)\right]}{\left[(A-B)2^{(n/2)}\left\{\sqrt{2}+\sqrt{2+m(1-q)}\right\}+|B||\gamma| m\right]^2} > 0. \tag{85}$$

Above expression represents that function Ξ_4 has increasing behavior, so we have its minimum value at $m=1$ and

$$\Xi_4(1) = \frac{|\gamma|}{2^{(n/2)}(A-B)\left[\sqrt{2}+\sqrt{3-q}\right]+|B||\gamma|}. \tag{86}$$

So, we conclude that

$$\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right| \geq \frac{|\gamma|}{2^{(n/2)}(A-B)\left[\sqrt{2}+\sqrt{3-q}\right]+|B||\gamma|}. \tag{87}$$

Now, from (74), we have

$$\left|\frac{p(\varsigma_0)-1}{A-Bp(\varsigma_0)}\right| \geq 1, \tag{88}$$

which contradict (75), and hence, $|\varpi(\varsigma)|<1$, which completes the proof.

By taking $h(\varsigma) = (\varsigma D_q f(\varsigma)/f(\varsigma))$, we deduce the following result. □

Corollary 7. *Let $|\gamma| \geq (2^{(n/2)}(A-B)(\sqrt{2}+\sqrt{3-q})/(1-|B|))$, $-1<B<A\leq 1$ and $f\in\mathscr{A}$, satisfy the subordination*

$$1+\gamma\varsigma\left(\frac{f(\varsigma)}{\varsigma D_q f(\varsigma)}\right)^n D_q\left(\frac{\varsigma D_q f(\varsigma)}{f(\varsigma)}\right) \prec \frac{1+A\varsigma}{1+B\varsigma}. \tag{89}$$

Then, $f\in\mathcal{S}_q^*(\sqrt{1+\varsigma})$.

3. Conclusion

In this article, we have investigated the q-differential subordination by using q-version of well-known Jack's Lemma. We have found the condition on γ such that $1 + (\gamma\varsigma D_q h(\varsigma)/h^n(\varsigma)) \prec (1 + A\varsigma/1 + B\varsigma)$ implies that $h(\varsigma) \prec \sqrt{1+\varsigma}$. These results have been utilized to find sufficient conditions for star-like functions related to lemniscate of Bernoulli. This method can further be applied to find sufficient conditions for star-like functions of Ma–Minda type.

Data Availability

No data were used in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and approved the final manuscript.

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