Research Article

On Polynomial Solutions of Pell's Equation

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Polynomial Pell's equation is $x^2 - Dy^2 = \pm 1$, where D is a quadratic polynomial with integer coefficients and the solutions X, Y must be quadratic polynomials with integer coefficients. Let $D = a_2x^2 + a_1x + a_0$ be a polynomial in $\mathbb{Z}[x]$. In this paper, some quadratic polynomial solutions are given for the equation $x^2 - Dy^2 = \pm 1$ which are significant from computational point of view.

1. Introduction

Let D be a quadratic polynomial with integer coefficient. We consider polynomial Pell's equation

$$x^2 - Dy^2 = \pm 1, \quad (1)$$

where X, Y are quadratic polynomials with integer coefficients. Many fruitful research studies have been conducted in this direction.

In 1976, Nathanson [1] proved that when $D = x^2 + d$, equation (1) has a nontrivial solution if and only if $d = \pm 1, \pm 2$.

In 2001, Mollin et al. [2] described the background to the study of continued fraction expansions of $\sqrt{D}$ where D is a polynomial, especially quadratic, which we study herein.

2. Notation and Preliminaries

Let D be a given square-free natural number. It is well known that Pell's equation

$$x^2 - Dy^2 = 1, \quad (2)$$

always has an infinite number of integer solutions (see e.g. Nagell [3]). All the solutions of (2) with positive x and y are obtained by the formula $A_n + B_n\sqrt{D} = (a + b\sqrt{D})^n$ where

$n = 1, 2, 3, \dots$, and $a + b\sqrt{D}$ is the fundamental solution of (2).

Pell's equation

$$x^2 - Dy^2 = -1, \quad (3)$$

may not always have integer solutions. When this equation is solvable in integers for some given D , if $L + M\sqrt{D}$ is its fundamental solution, we have the relation

$$a + b\sqrt{D} = (L + M\sqrt{D})^2, \quad (4)$$

where $a + b\sqrt{D}$ is the fundamental solution of (2).

For a given value of D , in order to determine the fundamental solution of (2) or (3) (when it exists), one may use the method of continued fraction expansion of $\sqrt{D}$. The penultimate convergent of this expansion provides the fundamental solution.

From computational point of view, it would be advantageous to group the values of D into several classes such that the values of D in a specific class are associated with a similar pattern of continued fraction expansion. It is found that polynomial expressions for D come in handy for this purpose.

As usual $[a_0; \overline{a_1, \dots, a_n; 2a_0}]$ will denote the simple infinite periodic continued fraction

$$a_0 + \frac{1}{a_1 + \left(\frac{1/a}{n-1 + \left(\frac{\cdot}{a_{n-1} + \left(\frac{1}{2a_0 + \left(\frac{1}{a_1 + \cdot \right)} \right)} \right)} \right)} \right)}. \quad (5)$$

A_i/B_i will denote the i' th convergent:

$$a_0 + \frac{1}{a_1 + \left(\frac{1}{a_2 + \cdots \left(\frac{1}{a_i} \right)} \right)}. \quad (6)$$

Use will also be made of the following:

$$\begin{aligned} A_i &= a_i A_{i-1} + A_{i-2}, \\ B_i &= a_i B_{i-1} + B_{i-2}, \\ A_{-1} &= B_{-2} = 1, A_{-2} = B_{-1} = 0, \text{ where} \\ &\frac{A_i B_{i-1} - A_{i-1} B_i}{A_i B_{i-1} - A_{i-1} B_i} = (-1)^{i-1}, \end{aligned} \quad (7)$$

each of these being valid for $i = 0, 1, 2, \dots$

Definition 1 (see [4]). A polynomial $f(t) \in \mathbb{Z}[t]$ is called a Fermat – Pell polynomial in one variable if there exists polynomial $c(t), h(t) \in \mathbb{Z}[t]$ such that

$$c(t)^2 - f(t)h(t)^2 = 1, \quad \text{for all } t \in \mathbb{Z}. \quad (8)$$

A triple of polynomials $(c(t), h(t), f(t))$ satisfying equation (8) constitutes a polynomial solution to Pell's equation. A Fermat – Pell polynomial is said to have a polynomial continued fraction expansion if there exists an integer T and a positive integer n such that

$$\sqrt{f(t)} = \left[a_0(t); \overline{a_1(t), \dots, a_n(t)} \right], \quad (9)$$

for all integral $t \geq T$, where $a_i(t) \in \mathbb{Z}[t]$. For a fixed Fermat – Pell polynomial $f(t)$, a pair of polynomials $(c(t), h(t))$ is said to be the fundamental polynomial solution to equation (9) if $(c(t), h(t))$ constitutes the smallest solution in positive integers to equation (8) for all non-negative values of $t \geq T$.

Definition 2 (see [5]). For real x , let the symbol x denote the greatest integer $\geq x$.

Theorem 1 (see [5]). Suppose $D = 4t^2 + 12t + 5$, where t is any natural number. Then, $a + b\sqrt{D}$ is the fundamental solutions of (2) where $a = 4t^3 + 18t^2 + 24t + 9$, $b = 2t^2 + 6t + 4$.

Theorem 2 (see [5]). If $D = 4t^2 + 4t + 5$ where t is any natural number, then $L + M\sqrt{D}$ is the fundamental solution of (3) where $L = 4t^3 + 6t^2 + 6t + 2$, $M = 2t^2 + 2t + 1$.

Theorem 3 (see [6]). Let $D \geq 1$ be a non-square integer; then:

$$\sqrt{D} = [a_0; \overline{2a_0}] \Leftrightarrow D = a_0^2 + 1, \quad (10)$$

where D is a positive integer.

3. Main Results

Let $D = a_2x^2 + a_1x + a_0$ be a polynomial in $\mathbb{Z}[x]$. In this section, we will prove our main results. As a start, we record the following propositions.

Proposition 1. Let $D = D(t)$, where t is any natural number. If $\sqrt{D} = [a_0(t); \overline{2a_0(t)}]$, then:

- (1) $L + M\sqrt{D}$ is the fundamental solution for the equation $x^2 - Dy^2 = -1$, where $L = a_0(t)$, $M = 1$.
- (2) $a + b\sqrt{D}$ is the fundamental solution for the equation $x^2 - Dy^2 = 1$, where $a = 2a_0(t)^2 + 1$, $b = 2a_0(t)$.

Proof. Let us assume $\sqrt{D} = [a_0(t); \overline{2a_0(t)}]$. Then:

- (1) The penultimate convergent is

$$\frac{a_0(t)}{1}. \quad (11)$$

Thus, we get the relation stated above, and by (10),

$$L^2 - DM^2 = a_0(t)^2 - (a_0(t)^2 + 1)(1)^2 = -1. \quad (12)$$

- (2) The penultimate convergent is

$$a_0(t) + \frac{1}{2a_0(t)} = \frac{2a_0(t)^2 + 1}{2a_0(t)}. \quad (13)$$

Thus, we get the relation stated above, and by (10),

$$a^2 - Db^2 = (2a_0(t)^2 + 1)^2 - (a_0^2 + 1)(2a_0(t))^2 = 1. \quad (14) \quad \square$$

Theorem 4. Let t, m, p be any natural numbers, p be a prime, and $a + b\sqrt{D}$ be the fundamental solution of (2); then:

- (1) If $D = m^2t^2 + m$, then $a = 2mt^2 + m$, $b = 2t$.
- (2) If $D = p^2t^2 + 2p$, then $a = pt^2 + 1$, $b = t$.
- (3) If $D = m^2t^2 + 2mt$, then $a = mt + 1$, $b = 1$.
- (4) If $D = 4t^2 + (4m + 4)t + m^2 + m$, then $a = 4t + 2m + 1$, $b = 2$.
- (5) If $D = 16(m + 1)^2t^2 + 8(m + 1)^2t + m^2 + m$, then $a = 32(m + 1)t^2 + 16(m + 1)t + 2m + 1$, $b = 8m + 2$.

Proof. (1) We have

$$mt < \sqrt{D} < mt + 1, \quad (15)$$

and so $\sqrt{D} = mt$. Now, expanding $\sqrt{D}$ into a continued fraction expansion, we obtain

$$\begin{aligned}
\sqrt{D} &= mt + \sqrt{m^2 t^2 + m} - mt \\
&= mt + \frac{1}{\left(\left(\sqrt{m^2 t^2 + m} + mt \right) / (m^2 t^2 + m - m^2 t^2) \right)} \\
&= mt + \frac{1}{2t + \left(\left(\sqrt{m^2 t^2 + m} - mt \right) / m \right)} \\
&= mt + \frac{1}{2t + \left(1 / \left(m \left(\sqrt{m^2 t^2 + m} + mt \right) / (m^2 t^2 + m - m^2 t^2) \right) \right)} \\
&= mt + \frac{1}{2t + \left(1 / \left(2mt + \sqrt{m^2 t^2 + m} - mt \right) \right)} \\
&= [mt; \overline{2t, 2mt}].
\end{aligned} \tag{16}$$

The penultimate convergent is

$$mt + \frac{1}{2t} = \frac{2mt^2 + 1}{2t}. \tag{17}$$

Thus, we get the relation

$$(2mt^2 + 1)^2 - (m^2 t^2 - m)(2t)^2 = 1. \tag{18}$$

Similarly, (2)–(4) and (7) can all be proven. $\square$

Theorem 5. Suppose $D = 16k^2 t^2 + 8k^2 t + k^2 + k$, where t, k are any natural numbers. Then:

- (1) $a + b\sqrt{D}$ is the fundamental solution of (2) where $a = 2k(4t + 1)^2 + 1, b = 2(4t + 1)$.
- (2) If $k = 1$, then $L + M\sqrt{D}$ is the fundamental solution of (3) where $L = 4t + 1, M = 1$.

Proof. (1) Expanding $\sqrt{D}$ into a continued fraction expansion, we obtain

$$\sqrt{D} = [4kt + k; \overline{2(4t + k), 2(4kt + k)}]. \tag{19}$$

The penultimate convergent is

$$4kt + k + \frac{1}{2(4t + k)} = \frac{2k(4t + 1)^2 + 1}{2(4t + 1)}. \tag{20}$$

Thus, we get the relation

$$(2k(4t + 1)^2 + 1)^2 - (16k^2 t^2 + 8k^2 t + k^2 + k)(2(4t + 1))^2 = 1. \tag{21}$$

(2) For $k = 1$, $D = 16t^2 + 8t + 2$. Now, expanding $\sqrt{D}$ into a continued fraction expansion, we obtain

$$\sqrt{D} = [4t + 1; \overline{2(4t + 1)}]. \tag{22}$$

By Proposition 1, $L = 4t + 1, M = 1$; then, $(4t + 1) + \sqrt{16t^2 + 8t + 2}$ is the fundamental solution of (3). $\square$

Theorem 6. Suppose $D = t^2 + 2ct + c^2 + 1$, where t, c are any natural numbers. Then:

- (1) $L + M\sqrt{D}$ is the fundamental solution of (3) where $L = t + c, M = 1$.
- (2) $a + b\sqrt{D}$ is the fundamental solution of (2) where $a = 2(t + c)^2 + 1, b = 2(t + c)$.

Example 1. Let $D = 4t^2 + 2$; then, $(4t^2 + 1) + 2t\sqrt{4t^2 + 2}$ is the fundamental solution of (2), by (1) in Theorem 3.

Example 2. Let $D = t^2 + 4t + 5$; then, $(t + 2) + \sqrt{t^2 + 4t + 5}$ is the fundamental solution of (3) and $(2(t + 2)^2 + 1) + 2(t + 2)\sqrt{t^2 + 4t + 5}$ is the fundamental solution of (2), by Theorem 5.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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