Research Article

Construction of Generalized k -Bessel–Maitland Function with Its Certain Properties

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The main motive of this study is to present a new class of a generalized k -Bessel–Maitland function by utilizing the k -gamma function and Pochhammer k -symbol. By this approach, we deduce a few analytical properties as usual differentiations and integral transforms (likewise, Laplace transform, Whittaker transform, beta transform, and so forth) for our presented k -Bessel–Maitland function. Also, the k -fractional integration and k -fractional differentiation of abovementioned k -Bessel–Maitland functions are also pointed out systematically.

1. Introduction and Preliminaries

The computation of fragmentary integrals of special functions is significant from the mark of perspective on the value of these outcomes in the assessment of generalized integrals, and the solution of differential and integral equations. Fractional integral formulas involving the Bessel function have been created and assume a significant part in a few physical problems. The Bessel function is significant in examining the solutions of differential equations, and they are related to a wide scope of problems in numerous regions of mathematical physics, likewise radiophysics, fluid dynamics, and material sciences. These contemplations have driven different specialists in the field of special functions to investigating the possible expansions and also applications for the Bessel function. Valuable speculation of the Bessel function called the k -Bessel function has also been presented by Diaz et al. [1–3] and Suthar et al. [4]. They have presented k -beta, k -gamma, k -zeta functions, and Pochhammer

k -symbol (rising factorial). Additionally, they demonstrated some of their properties and inequalities for the above-said functions. They have likewise considered k -hypergeometric functions based on k -rising factorial.

Such functions play a discernible role in a variety of appropriate fields of science and engineering. During the past several years, several researchers have obtained various k -type function (such as k -gamma, k -beta, and k -Pochhammer). This subject has received attention of various researchers and mathematicians during the last few decades. The k symbols are well known from many references related to finite difference calculus (see, [5–11], see additionally [12–16]). Recently, k -type functions and k -type operators have been considered in the literature by various authors. For this purpose, we start with the following properties in the literature.

For our current assessment, we survey here the definition of some known functions and their generalizations. The integral representations of k -gamma and k -beta functions are as follows (see [1–3]):

$$\Gamma_k(x) = k^{(x/k)-1} \Gamma\left(\frac{x}{k}\right) = \int_0^\infty t^{x-1} e^{-(t^k/k)} dt, \quad \Re(x) > 0, k > 0, \quad (1)$$

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{(x/k)-1} (1-t)^{(y/k)-1} dt, \quad x > 0, y > 0, \quad (2)$$

where

$$\begin{aligned} B_k(x, y) &= \frac{1}{k} B\left(\frac{x}{k}, \frac{y}{k}\right), \\ B_k(x, y) &= \frac{\Gamma_k(x) \Gamma_k(y)}{\Gamma_k(x+y)}. \end{aligned} \quad (3)$$

The variety of the functions likewise k -Zeta function, k -Mittag-Leffler function for two and three parameters, k -Wright, and k -hypergeometric functions could be characterized by the following formulas (see also [4, 12, 13, 16–20]):

$$\begin{aligned} \xi_k(z, p) &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(z+nk)^p}, \quad k, z > 0, p > 1, \\ E_{k,\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\alpha n + \beta)}, \quad \alpha, \beta > 0, \\ E_{k,\alpha,\beta}^\gamma(z) &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta) n!} z^n, \quad k \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \\ W_{k,\alpha,\beta}^\gamma(z) &= \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta) (n!)^2} z^n, \quad k \in \mathbb{R}, \alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \\ F_k((\beta, k); (\gamma, k); z) &= \sum_{n=0}^{\infty} \frac{(\beta)_{n,k}}{(\gamma)_{n,k} n!} z^n, \quad k \in \mathbb{R}, \beta, \gamma \in \mathbb{C}; \Re(\beta) > 0, \Re(\gamma) > 0. \end{aligned} \quad (4)$$

Definition 1. Let f be a sufficiently well-behaved function with support in $\mathbb{R}^+$ and let α be a real number $\alpha > 0$. The k -Riemann–Liouville fractional integral of order α , $I_+^\alpha f$ is given by (see [21–23])

$$I_{k,a}^\alpha(f(z)) = \frac{1}{k\Gamma_k(\alpha)} \int_a^z (z-t)^{(\alpha/k)-1} f(t) dt. \quad (5)$$

This definition unmistakably reduces the definition defined by Mubeen and Habibullah (see [14]):

$$I_k^\alpha(f(z)) = \frac{1}{k\Gamma_k(\alpha)} \int_0^z (z-t)^{(\alpha/k)-1} f(t) dt. \quad (6)$$

It is clear that the case $k = 1$ of (6) yields the traditional Riemann–Liouville fractional integral:

$$I^\alpha(f(z)) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} f(t) dt. \quad (7)$$

Definition 2. Let β be a real number. Then, k -Riemann–Liouville fractional derivative is defined by (see [21–23])

$$D_k^\beta(f(t)) = \frac{d}{dt} I_k^{1-\beta} f(t) dt, \quad (0 < \beta \leq 1), \quad (8)$$

where

$$I_k^{1-\beta}(f(x)) = \frac{1}{k\Gamma_k(1-\beta)} \int_0^z (z-t)^{(1-\beta/k)-1} f(t) dt. \quad (9)$$

Definition 3. For $u \in \varphi(R)$, the fractional Fourier transform (FFT) of order α is defined as (see [21–23])

$$\hat{u}_\alpha(w) = F_\alpha[u](w) = \int_{\mathbb{R}} e^{i w^{1/\alpha} t} u(t) dt, \quad (0 < \alpha \leq 1). \quad (10)$$

It is effectively observed that, for $\alpha = 1$, (10) reduces to the conventionally Fourier transform which is given by

$$F[\varphi](z) = \int_{-\infty}^{+\infty} e^{izt} \varphi(t) dt. \quad (11)$$

For $w > 0$, (10) easily recovers the FFT presented by Luchko et al. [24].

In 2018, Ghayasuddin and Khan [25] presented generalized Bessel–Maitland functions by

$$J_{\nu, \gamma, \delta}^{\mu, q, p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} (-z)^n}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}}, \quad (12)$$

where $\mu, \nu, \gamma, \delta \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$; $p, q > 0$, and $q < \Re(\mu) + p$.

For $b_j, j = \overline{1, q}$ different from nonpositive integers, the series (see [26, 27])

$${}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n z^n}{(b_1)_n (b_2)_n \dots (b_q)_n n!} = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n z^n}{\prod_{j=1}^q (b_j)_n n!} \quad (13)$$

is the generalized hypergeometric series, where the Pochhammer symbol

$$(a)_\mu := \frac{\Gamma(a + \mu)}{\Gamma(a)} = \begin{cases} 1, & \text{if } \mu = 0; a \in \mathbb{C} \setminus \{0\}, \\ a(a+1), \dots, (a+n-1), & \text{if } \mu = n \in \mathbb{N}; a \in \mathbb{C}, \end{cases} \quad (14)$$

and by convention $(a)_0 = 1$. When $p \leq q$, the generalized hypergeometric function converges for all complex values of z , that is, ${}_pF_q[z]$ is an entire function. When $p > q + 1$, the series converges only for $z = 0$, unless it terminates (as when one of the parameters $a_j, j = \overline{1, p}$ is a negative integer) in which case it is just a polynomial in z . When $p = q + 1$, the series converges in the open unit disk $|z| < 1$ and also for $|z| = 1$ provided that

$$\Re \left(\sum_{j=1}^q b_j - \sum_{j=1}^p a_j \right) > 0. \quad (15)$$

The summed up k -Wright function is addressed as follows (see details [7, 27]):

$$\begin{aligned} {}_p\Psi_q^k \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix} \middle| z \right] &= {}_p\Psi_q^k \left((\alpha_j, A_j)_{1,p}; (\beta_j, B_j)_{1,q}; z \right) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1 + A_1 n), \dots, \Gamma_k(\alpha_p + A_p n) z^n}{\Gamma_k(\beta_1 + B_1 n), \dots, \Gamma_k(\beta_q + B_q n) n!}, \end{aligned} \quad (16)$$

where $k \in \mathbb{R}^+$; $z \in \mathbb{C}$ and

$$\Re \left(\sum_{j=1}^q B_j - \sum_{j=1}^p A_j \right) > 0. \quad (17)$$

Motivated essentially by the demonstrated potential for applications of these extended generalized k -Wright hypergeometric functions, we extend the generalized k -Bessel–Maitland function (18) by means of the generalized k -Pochhammer symbol (1) and investigate certain basic properties including differentiation formulas, integral representations, Euler-Beta, Laplace, Whittaker, and fractional Fourier transforms with their several special cases and relations with the k -Bessel–Maitland function. We also derive

the k -fractional integration and differentiation of k -Bessel–Maitland function.

2. Generalized k -Bessel–Maitland Function

This section deals with the new development of k -Bessel–Maitland function $J_{k, \nu, q, p}^{\mu, \gamma, \delta}(z)$ and its associated properties.

Definition 4. Let $k \in \mathbb{R}$, $\mu, \nu, \gamma, \delta \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$; $p, q > 0$, and $q < \Re(\mu) + p$. The generalized k -Bessel–Maitland function is defined as

$$J_{k, \nu, q, p}^{\mu, \gamma, \delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn, k} (-z)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn, k}}. \quad (18)$$

Remark 1. We note that the case $k = 1$ in (18) leads to the generalized Bessel–Maitland function defined by Ghaya-suddin and Khan [25], which further for $\delta = p = 1$ gives the Bessel–Maitland function given by Singh et al. [20].

Theorem 1. If $k \in \mathbb{R}$, $\mu, \gamma, \delta \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\gamma) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$; $p, q > 0$ and $q < \Re(\mu) + p$, then we have

$$(\nu + 1)J_{k, \nu+k, q, p}^{\mu, \gamma, \delta}(z) + \mu z \frac{d}{dz} J_{k, \nu+k, q, p}^{\mu, \gamma, \delta}(z) = J_{k, \nu, q, p}^{\mu, \gamma, \delta}(z) \quad (19)$$

and

$$J_{k, \nu, q, p}^{\mu, \gamma+k, \delta+k}(z) - J_{k, \nu, q, p}^{\mu, \gamma, \delta}(z) = z \frac{\delta q(\gamma)_{q, k}}{\gamma p(\delta)_{p, k}} J_{k, \mu+\nu, q, p}^{\mu, \gamma+qk, \delta+pk}(z). \quad (20)$$

Proof. With the help of (18) on the L.H.S of (19), we get

$$\begin{aligned} & (\nu + 1)J_{k, \nu+k, q, p}^{\mu, \gamma, \delta}(z) + \mu z \frac{d}{dz} J_{k, \nu+k, q, p}^{\mu, \gamma, \delta}(z), \\ &= (\nu + 1) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn, k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn, k}} (-z)^n + \mu z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn, k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn, k}} (-z)^n \\ &= (\nu + 1) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn, k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn, k}} (-z)^n + \mu z \sum_{n=0}^{\infty} \frac{(\gamma)_{qn, k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn, k}} n (-z)^{n-1} \\ & (\nu + 1)J_{k, \nu+k, q, p}^{\mu, \gamma, \delta}(z) + \mu z \frac{d}{dz} J_{k, \nu+k, q, p}^{\mu, \gamma, \delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn, k}}{\Gamma_k(\mu n + \nu + k + 1)(\delta)_{pn, k}} (-z)^n (\mu n + \nu + 1). \end{aligned} \quad (21)$$

In view of $\Gamma_k(z + k) = z\Gamma_k(z)$, we acquire at our stated result (19).

Using Definition 3 on the L.H.S of (20), we get

$$\begin{aligned} J_{k, \nu, q, p}^{\mu, \gamma+k, \delta+k}(z) - J_{k, \nu, q, p}^{\mu, \gamma, \delta}(z) &= \sum_{n=0}^{\infty} \frac{(\gamma + k)_{qn, k}}{\Gamma_k(\mu n + \nu + 1)(\delta + k)_{pn, k}} (-z)^n - \sum_{n=0}^{\infty} \frac{(\gamma)_{qn, k}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn, k}} (-z)^n \\ &= \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma_k(\mu n + \nu + 1)} \left[\frac{(\gamma + k)_{qn, k}}{(\delta + k)_{pn, k}} - \frac{(\gamma)_{qn, k}}{(\delta)_{pn, k}} \right]. \end{aligned} \quad (22)$$

Now, by using the result given in [6], we get

$$\begin{aligned} J_{k, \nu, q, p}^{\mu, \gamma+k, \delta+k}(z) - J_{k, \nu, q, p}^{\mu, \gamma, \delta}(z) &= \sum_{n=1}^{\infty} \frac{(-z)^n}{\Gamma_k(\mu n + \nu + 1)} \left[\frac{\delta q n k (\gamma)_{qn, k}}{\gamma p n k (\delta)_{pn, k}} \right] \\ &= \sum_{n=0}^{\infty} \frac{(-z)^{n+1}}{\Gamma_k(\mu(n+1) + \nu + 1)} \left[\frac{\delta q (\gamma)_{q(n+1), k}}{\gamma p (\delta)_{p(n+1), k}} \right]. \end{aligned} \quad (23)$$

Using the result (see [6]), we get

$$\begin{aligned}
 J_{k,\nu,q,p}^{\mu,\gamma+k,\delta+k}(z) - J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) &= z \frac{\delta q}{\gamma p} \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma_k(\mu(n+1) + \nu + 1)} \left[\frac{(\gamma)_{q,k}(\gamma + qk)_{qn,k}}{(\delta)_{p,k}(\delta + pk)_{pn,k}} \right] \\
 &= z \frac{\delta q(\gamma)_{q,k}}{\gamma p(\delta)_{p,k}} \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma_k(\mu(n+1) + \nu + 1)} \left[\frac{(\gamma + qk)_{qn,k}}{(\delta + pk)_{pn,k}} \right] \\
 J_{k,\nu,q,p}^{\mu,\gamma+k,\delta+k}(z) - J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) &= z \frac{\delta q(\gamma)_{q,k}}{\gamma p(\delta)_{p,k}} J_{k,\mu+\nu,q,p}^{\mu,\gamma+qk,\delta+pk}(z),
 \end{aligned} \tag{24}$$

which is our stated result (20). $\square$ and

Theorem 2. Let $k \in \mathbb{R}$, $\mu, \nu, \gamma, \delta \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$; $p, q > 0$ and $q < \Re(\mu) + p$, then for $m \in \mathbb{N}$, we have

$$\frac{d}{dz} \left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) \right] = \frac{(\gamma)_{q,k}}{(\delta)_{p,k}} (n+1) J_{k,\mu+\nu,q,p}^{\mu,\gamma+qk,\delta+pk}(z) \tag{25}$$

Proof. With the help of (18) on the L.H.S of (25), we get

$$\begin{aligned}
 \frac{d}{dz} \left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) \right] &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k} (-z)^n}{\Gamma_k(\mu n + \nu + 1) (\delta)_{pn,k}} \\
 &= \sum_{n=1}^{\infty} \frac{(\gamma)_{qn,k} (-n) (-z)^{n-1}}{\Gamma_k(\mu n + \nu + 1) (\delta)_{pn,k}} \\
 &= \sum_{n=0}^{\infty} \frac{(\gamma)_{q(n+1),k} (-1)^{n+1} (n+1) (z)^n}{\Gamma_k(\mu n + \mu + \nu + 1) (\delta)_{p(n+1),k}} \\
 \frac{d}{dz} \left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) \right] &= \frac{(\gamma)_{q,k}}{(\delta)_{p,k}} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (\gamma + qk)_{qn,k} (n+1) z^n}{\Gamma_k(\mu n + \mu + \nu + 1) (\delta + pk)_{pn,k}} \\
 &= \frac{(\gamma)_{q,k}}{(\delta)_{p,k}} (n+1) J_{k,\mu+\nu,q,p}^{\mu,\gamma+qk,\delta+pk}(z),
 \end{aligned} \tag{27}$$

which is our stated result (25).

Now, by using Definition 3 on the L.H.S of (26), we get

$$\left(\frac{d}{dz} \right)^m J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) = \frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}} (n+1)_m J_{k,\mu+\nu,q,p}^{\mu,\gamma+qm,\delta+pm}(z). \tag{26}$$

$$\begin{aligned}
\left(\frac{d}{dz}\right)^m J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) &= \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} (-z)^n \\
&= \sum_{n=m}^{\infty} \frac{n(n-1)\dots(n-m+1)(\gamma)_{qn,k}(-1)^n z^{n-m}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \\
&= \sum_{n=0}^{\infty} \frac{(n+m)(n+m-1)\dots(n+1)(\gamma)_{q(n+m),k}(-1)^{(n+m)} z^n}{\Gamma_k(\mu(n+m) + \nu + 1)(\delta)_{p(n+m),k}} \quad (28) \\
\left(\frac{d}{dz}\right)^m J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) &= \frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}} \sum_{n=0}^{\infty} \frac{(n+1)_m (\gamma + qm)_{qn,k} (-z)^n}{\Gamma_k(\mu(n+m) + \nu + 1)(\delta + pm)_{pn,k}} \\
&= \frac{(\gamma)_{qm,k}}{(\delta)_{pm,k}} (n+1)_m J_{k,\mu m + \nu, q, p}^{\mu, \gamma + qm, \delta + pm}(z),
\end{aligned}$$

which is our stated result (26). $\square$

and

3. Integral Transform of a Generalized k -Bessel–Maitland Function

This section manages with some integral transforms likewise Laplace transform, Whittaker transform, beta transform, Hankel transform, K -transform, and fractional Fourier transform as follows.

Theorem 3 (k -beta transform). Let $k \in \mathbb{R}$, $\mu, \nu, \beta, \gamma, \delta \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\beta) > 0$, $p, q > 0$, and $q < \Re(\mu) + p$, then we have

$$\frac{1}{\Gamma_k(\beta)} \int_0^1 t^{\nu/k} (1-t)^{(\beta/k)-1} J_{k,\nu,q,p}^{\mu,\gamma,\delta}(zt^{\mu/k}) dt = J_{k,\nu+\beta,q,p}^{\mu,\gamma,\delta}(z) \quad (29)$$

$$\begin{aligned}
&\frac{1}{\Gamma_k(\beta)} \int_t^x (x-s)^{(\beta/k)-1} (s-t)^{\nu/k} J_{k,\nu,q,p}^{\mu,\gamma,\delta}[z(s-t)^{\mu/k}] ds \\
&= (x-t)^{\beta+\nu/k} J_{k,\nu+\beta,q,p}^{\mu,\gamma,\delta}[z(x-t)^{\mu/k}]. \quad (30)
\end{aligned}$$

Proof. By using (18) on the L.H.S of (29) and rearranging in reference to integration and summation (which is ensured under the condition), we acquire

$$\begin{aligned}
\frac{1}{\Gamma_k(\beta)} \int_0^1 t^{\nu/k} (1-t)^{(\beta/k)-1} J_{k,\nu,q,p}^{\mu,\gamma,\delta}(zt^{\mu/k}) dt &= \frac{1}{\Gamma_k(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} (-z)^n \int_0^1 t^{\nu+\mu n/k} (1-t)^{\beta/k-1} dt \\
&= \frac{1}{\Gamma_k(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} (-z)^n B_k(\mu n + \nu + 1, \beta) \\
&= J_{k,\nu+\beta,q,p}^{\mu,\gamma,\delta}(z), \quad (31)
\end{aligned}$$

which is our stated result (29).

In the event that we set the transformation $w = s - t/x - t$ on the L.H.S of equation (30) and using Definition 3, we acquire

$$\begin{aligned}
 & (x-t)^{\beta+\nu/k} \int_0^1 w^{\nu/k} (1-w)^{(\beta/k)-1} J_{k,\nu,q,p}^{\mu,\gamma,\delta} [z(w(x-t))^{\mu/k}] dw \\
 &= (x-t)^{\beta+\nu/k} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k} [-z(x-t)^{\mu/k}]^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \int_0^1 w^{\mu n + \nu/k} (1-w)^{(\beta/k)-1} dw \\
 &= (x-t)^{\beta+\nu/k} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k} [-z(x-t)^{\mu/k}]^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} B_k(\mu n + \nu + 1, \beta) \\
 &= (x-t)^{\beta+\nu/k} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k} [-z(x-t)^{\mu/k}]^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \frac{\Gamma_k(\mu n + \nu + 1) \Gamma_k(\beta)}{\Gamma_k(\mu n + \beta + \nu + 1)},
 \end{aligned} \tag{32}$$

which is our stated result (30).

□ **Theorem 4** (Laplace Transform). Let $k \in \mathbb{R}$, $\mu, \nu, \alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $p, q > 0$, and $q < \Re(\mu) + p$, then we have

$$\int_0^{\infty} z^{\alpha-1} e^{-sz} J_{k,\nu,q,p}^{\mu,\gamma,\delta} (xz^{\beta}) dz = \frac{\Gamma(\delta/k)}{s^{\alpha} \Gamma(\gamma/k)_3} \Psi_2 \left[\begin{matrix} ((\gamma/k), q), (\alpha, \beta), (1, 1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k} \right), \left(\frac{\delta}{k}, p \right) \end{matrix} ; \frac{-x k^{q-p-\mu/k}}{s^{\beta}} \right]. \tag{33}$$

Proof. By using (18) and the definition of Laplace transform we get

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt, \tag{34}$$

$$\begin{aligned}
 \int_0^{\infty} z^{\alpha-1} e^{-sz} J_{k,\nu,q,p}^{\mu,\gamma,\delta} (xz^{\beta}) dz &= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k} (-x)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \int_0^{\infty} e^{-sz} z^{\alpha+\beta n-1} dz \\
 &= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k} (-x)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \frac{\Gamma(\alpha + \beta n)}{s^{\alpha+\beta n}} \\
 \int_0^{\infty} z^{\alpha-1} e^{-sz} J_{k,\nu,q,p}^{\mu,\gamma,\delta} (xz^{\beta}) dz &= \frac{\Gamma(\delta/k)}{s^{\alpha} \Gamma(\gamma/k)} \sum_{n=0}^{\infty} \frac{k^{qn} \Gamma((\gamma/k) + qn) (-x)^n}{k^{\mu n/k} \Gamma(\mu n + \nu + 1/k) k^{pn} \Gamma((\delta/k) + pn)} \frac{\Gamma(\alpha + \beta n)}{s^{\beta n}}.
 \end{aligned} \tag{35}$$

Summing up the above the series with the help of (1), we easily arrive at our stated result (33). $\square$

Theorem 5 (Hankel transform). If $k \in \mathbb{R}$, $\mu, \nu, \beta, \gamma, \delta, \eta$, $\lambda \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\beta) > 0$, $\Re(\eta) > 0$, $\Re(\lambda) > 0$; $a, b > 0$, $p, q > 0$, and $q < \Re(\mu) + p$, then we have

$$\int_0^\infty z^{\eta-1} J_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}(bz^\beta) dz = \frac{2^{\eta-1} \Gamma(\delta/k)}{a^\eta \Gamma(\gamma/k) k^{(\nu+1/k)-1}} \times_3 \Psi_3 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), \left(\frac{\lambda+\eta}{2}, \frac{\beta}{2}\right), (1, 1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right), \left(1 + \frac{\lambda-\eta}{2}, -\frac{\beta}{2}\right) \end{matrix} \middle| k^{q-p-\mu/k} - b \left(\frac{2}{a}\right)^\beta \right]. \quad (36)$$

Proof. Applying Definition 3, we have

$$\begin{aligned} \int_0^\infty z^{\eta-1} J_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}(bz^\beta) dz \\ = \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} (-b)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \int_0^\infty z^{\eta+\beta n-1} J_\lambda(az) dz. \end{aligned} \quad (37)$$

By following the given formula [13],

$$\int_0^\infty t^{s-1} J_\nu(at) dt = \frac{2^{s-1} \alpha^{-s} \Gamma(\nu + s/2)}{\Gamma(1 + \nu - s/2)}, \Re(\nu) < \Re(s) < \frac{3}{2}, \alpha > 0, \quad (38)$$

we get

$$\begin{aligned} \int_0^\infty z^{\eta-1} J_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}(bz^\beta) dz &= \frac{2^{\eta-1} \Gamma(\delta/k)}{a^\eta \Gamma(\gamma/k) k^{(\nu+1/k)-1}} \\ &\cdot \sum_{n=0}^\infty \frac{k^{qn} \Gamma(\gamma/k + qn) (-b)^n}{k^{\mu n/k} \Gamma(\mu n + \nu + 1/k) k^{pn} \Gamma(\delta/k + pn)} \\ &\times \frac{2^{\beta n} \Gamma(\lambda + \eta + \beta n/2)}{a^{\beta n} \Gamma(1 + (\lambda - \eta - \beta n/2))}. \end{aligned} \quad (39)$$

In view of (16), we get our stated result (36). $\square$

Theorem 6 (K -transform). Let $k \in \mathbb{R}$, $\mu, \nu, \gamma, \delta, \eta$, $\lambda \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\eta \pm \lambda) > 0$; $a > 0$, $b > 0$, $w > 0$, $p, q > 0$, and $q < \Re(\mu) + p$, then we have

$$\int_0^\infty z^{\eta-1} K_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}(bz^w) dz = \frac{2^{\eta-2} \Gamma(\delta/k)}{a^\eta \Gamma(\gamma/k) k^{\nu+1/k-1}} \times_3 \Psi_2 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), \left(\frac{\eta \pm \lambda}{2}, \frac{w}{2}\right), (1, 1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right) \end{matrix} \middle| -bk^{q-p-\mu/k} \left(\frac{2}{a}\right)^w \right]. \quad (40)$$

Proof. Applying Definition 3, we have

$$\begin{aligned} \int_0^\infty z^{\eta-1} K_\lambda(az) J_{k,\nu,q,p}^{\mu,\gamma,\delta}(bz^w) dz \\ = \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} (-b)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \int_0^\infty z^{\eta+wn-1} K_\lambda(az) dz. \end{aligned} \quad (41)$$

By using the following integral (given in [13])

$$\int_0^\infty x^{\rho-1} K_\nu(x) dx = 2^{\rho-2} \Gamma\left(\frac{\rho \pm \nu}{2}\right), \quad (42)$$

in the above equation, we arrive at

$$\int_0^\infty z^{\eta-1} K_\lambda(az) J_{k,\gamma,q,p}^{\mu,\gamma,\delta}(bz^w) dz = \frac{2^{\eta-2} \Gamma(\delta/k)}{a^\eta \Gamma(\gamma/k) k^{(\gamma+1/k)-1}} \times \sum_{n=0}^\infty \frac{k^{qn} \Gamma(\gamma/k + qn) (-b)^n}{k^{\mu n/k} \Gamma(\mu n + \gamma + 1/k) k^{pn} \Gamma(\delta/k + pn)} \frac{2^{wn}}{a^{wn}} \Gamma\left(\frac{\eta + wn \pm \lambda}{2}\right). \quad (43)$$

In view of (16), we get our stated result (40). $\square$

$(\gamma) > 0, \Re(\delta) > 0, \Re(\alpha \pm m) > -(1/2); \quad p, q > 0; \quad \text{and}$
 $q < \Re(\alpha) + p$, then we have

Theorem 7 (Whittaker transform). Let $k \in \mathbb{R}$,
 $\mu, \nu, \alpha, \gamma, \delta, \lambda, \rho \in \mathbb{C}, \Re(\mu) > 0, \Re(\nu) \geq -1, \Re$

$$\int_0^\infty t^{\alpha-1} e^{-st/2} W_{\lambda,m}(st) J_{k,\gamma,q,p}^{\mu,\gamma,\delta}(wt^\rho) dt = \frac{\Gamma(\delta/k)}{s^\alpha \Gamma(\gamma/k) k^{\gamma+1/k-1}} \times {}_4\Psi_3 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), \left(\frac{1}{2} \pm m + \alpha, \rho\right), (1, 1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right), (1 - \lambda + \alpha, \rho) \end{matrix} ; k^{q-p-\mu/k} \left(-\frac{w}{s^\rho}\right) \right]. \quad (44)$$

Proof. Applying Definition 3 on the L.H.S of (44) and by setting $st = z$, we get

$$\begin{aligned} & \int_0^\infty e^{-(z/2)} \left(\frac{z}{s}\right)^{\alpha-1} W_{\lambda,m}(z) J_{k,\gamma,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right) \frac{dz}{s} \\ &= s^{-\alpha} \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} (-w/s^\rho)^n}{\Gamma_k(\mu n + \gamma + 1) (\delta)_{pn,k}} \\ & \times \int_0^\infty e^{-(z/2)} z^{\alpha+pn-1} W_{\lambda,m}(z) dz. \end{aligned} \quad (45)$$

By using the following formula (given in [11])

$$\int_0^\infty e^{-(x/2)} x^{\nu-1} W_{\lambda,m}(x) dx = \frac{\Gamma(1/2 \pm m + \nu)}{\Gamma(1 - \lambda + \nu)}, \quad (46)$$

we get

$$\begin{aligned} & \int_0^\infty e^{-(z/2)} \left(\frac{z}{s}\right)^{\alpha-1} W_{\lambda,m}(z) J_{k,\gamma,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right) \frac{dz}{s} = \frac{\Gamma(\delta/k)}{s^\alpha \Gamma(\gamma/k) k^{(\gamma+1/k)-1}} \\ & \times \sum_{n=0}^\infty \frac{k^{qn} \Gamma(\gamma/k + qn) (-w/s^\rho)^n}{k^{\mu n/k} \Gamma(\mu n + \gamma + 1/k) k^{pn} \Gamma(\delta/k + pn)} \frac{\Gamma(1/2 \pm m + \alpha + \rho n)}{\Gamma(1 - \lambda + \alpha + \rho n)}. \end{aligned} \quad (47)$$

In view of (16), we get our stated result. $\square$

Theorem 8. Let $k \in \mathbb{R}$, $\mu, \nu, \alpha, \gamma, \delta, \lambda, \rho \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\alpha + m) > -(1/2)$, $p, q > 0$, and $q < \Re(\mu) + p$, then we have

$$\begin{aligned} & \int_0^\infty t^{\alpha-1} e^{-st/2} M_{\lambda,m}(st) J_{k,\nu,q,p}^{\mu,\gamma,\delta}(wt^\rho) dt \\ &= \frac{\Gamma(2m+1)\Gamma(\delta/k)}{s^\alpha \Gamma(\gamma/k) \Gamma(m+\lambda+1/2) k^{(\nu+1/k)-1}} \\ & \quad \times {}_4\Psi_3 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), \left(m+\alpha+\frac{1}{2}, \rho\right), (\lambda-\alpha, -\rho), (1, 1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right), \left(m-\alpha+\frac{1}{2}, -\rho\right) \end{matrix} \middle| k^{q-p-\mu/k} \left(\frac{w}{s^\rho}\right) \right]. \end{aligned} \quad (48)$$

Proof. Applying Definition 3 on the L.H.S of (48) and by setting $st = z$, we get

$$\begin{aligned} & \int_0^\infty e^{-(z/2)} \left(\frac{z}{s}\right)^{\alpha-1} M_{\lambda,m}(z) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right) \frac{dz}{s} \\ &= s^{-\alpha} \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} (-w/s^\rho)^n}{\Gamma_k(\mu n + \nu + 1) (\delta)_{pn,k}} \int_0^\infty e^{-(z/2)} z^{\alpha+pn-1} M_{\lambda,m}(z) dz. \end{aligned} \quad (49)$$

By using the following integral (given in [13])

$$\begin{aligned} & \int_0^\infty e^{-(x/2)} x^{\nu-1} M_{\lambda,m}(x) dx \\ &= \frac{\Gamma(2m+1)\Gamma(m+\nu+1/2)\Gamma(\lambda-\nu)}{\Gamma(m-\nu+1/2)\Gamma(m+\lambda+1/2)}, \end{aligned} \quad (50)$$

we get

$$\begin{aligned} & \int_0^\infty e^{-(z/2)} \left(\frac{z}{s}\right)^{\alpha-1} M_{\lambda,m}(z) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right) \frac{dz}{s} \\ &= s^{-\alpha} \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} (-w/s^\rho)^n}{\Gamma_k(\mu n + \nu + 1) (\delta)_{pn,k}} \frac{\Gamma(2m+1)\Gamma(m+\alpha+pn+1/2)\Gamma(\lambda-\alpha-pn)}{\Gamma(m-\alpha-pn+1/2)\Gamma(m+\lambda+1/2)} \\ & \quad \int_0^\infty e^{-(z/2)} \left(\frac{z}{s}\right)^{\alpha-1} M_{\lambda,m}(z) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right) \frac{dz}{s} \\ &= \frac{\Gamma(2m+1)\Gamma(\delta/k)}{s^\alpha \Gamma(\gamma/k) \Gamma(m+\lambda+1/2) k^{(\nu+1/k)-1}} \sum_{n=0}^\infty \frac{k^{qn} \Gamma((\gamma/k) + qn) (-w/s^\rho)^n}{k^{\mu n/k} \Gamma(\mu n + \nu + 1/k) k^{pn} \Gamma(\delta/k + pn)} \\ & \quad \times \frac{\Gamma(m+\alpha+pn+1/2)\Gamma(\lambda-\alpha-pn)}{\Gamma(m-\alpha-pn+1/2)}. \end{aligned} \quad (51)$$

Finally, by applying Definition 1.17, we get our stated result. $\square$

Theorem 9. Let $k \in \mathbb{R}$, $\mu, \nu, \alpha, \gamma, \delta, \rho \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\alpha) > -1$, $\Re(\alpha/2 \pm m) > -1$, $\Re(\alpha/2 \pm \lambda) > -1$, $p, q > 0$; and $q < \Re(\mu) + p$, then we have

$$\int_0^\infty t^{\alpha-1} W_{\lambda,m}(st) W_{-\lambda,m}(st) J_{k,\nu,q,p}^{\mu,\gamma,\delta}(wt^\rho) dt$$

$$= \frac{\Gamma(\delta/k)}{s^\alpha \Gamma(\gamma/k) k^{(\nu+1/k)-1}} {}_5\Psi_4 \left[\begin{matrix} \left(\frac{\gamma}{k}, q\right), \left(\frac{\alpha}{2} \pm m + 1, \frac{\rho}{2}\right), (\alpha+1, \rho), (1, 1) \\ \left(\frac{\nu+1}{k}, \frac{\mu}{k}\right), \left(\frac{\delta}{k}, p\right), \left(1 + \frac{\alpha}{2} \pm \lambda, \frac{\rho}{2}\right) \end{matrix} \middle| k^{q-p-\mu/k} \left(\frac{w}{s^\rho}\right) \right]. \quad (52)$$

Proof. Applying Definition 3 on the L.H.S of (44) and by setting $st = z$, we get

$$\int_0^\infty \left(\frac{z}{s}\right)^{\alpha-1} W_{\lambda,m}(z) W_{-\lambda,m}(z) J_{k,\nu,q,p}^{\mu,\gamma,\delta}\left(w\left(\frac{z}{s}\right)^\rho\right) \frac{dz}{s}$$

$$= s^{-\alpha} \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} (-w/s^\rho)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \quad (53)$$

$$\cdot \int_0^\infty z^{\alpha+\rho n-1} W_{\lambda,m}(z) W_{-\lambda,m}(z) dz.$$

By using the integral given in [11]

$$\int_0^\infty x^{\nu-1} W_{\lambda,m}(x) W_{-\lambda,m}(x) dx = \frac{\Gamma((\nu+1/2) \pm m) \Gamma(\nu+1)}{\Gamma(1 + (\nu/2) \pm \lambda)}, \quad (54)$$

we get

$$= \frac{\Gamma(\delta/k)}{s^\alpha \Gamma(\gamma/k) k^{\nu+1/k-1}} \sum_{n=0}^\infty \frac{k^{qn} \Gamma(\gamma/k + qn) (-w/s^\rho)^n}{k^{\mu n/k} \Gamma(\mu n + \nu + 1/k) k^{pn} \Gamma(\delta/k + pn)}$$

$$\cdot \frac{\Gamma(\alpha + \rho n + 1/2 \pm m) \Gamma(\alpha + \rho n + 1)}{\Gamma(1 + (\alpha + \rho n/2) \pm \lambda)}. \quad (55)$$

Now, by summing up the above series with the help of (16), we get our stated result. $\square$

Theorem 10 (fractional Fourier transform). *The FFT of the generalized k -Bessel–Maitland function for $t < 0$ is given by*

$$F_\alpha \left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) \right] = \sum_{n=0}^\infty \frac{n! (\gamma)_{qn,k} i^{-n-1} w^{-(n+1)/\alpha}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}}. \quad (56)$$

Proof. From (11) and (18), we have

$$F_\alpha \left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) \right] = \int_{\mathbb{R}} e^{iw^{1/\alpha} z} \sum_{n=0}^\infty \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} (-z)^n dz$$

$$= \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} (-1)^n}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \int_{-\infty}^0 e^{iw^{1/\alpha} z} z^n dz. \quad (57)$$

On changing variables $i w^{1/\alpha} z = -t$ and $i w^{1/\alpha} dz = -dt$, we arrive at

$$F_\alpha \left[J_{k,\nu,q,p}^{\mu,\gamma,\delta}(z) \right] = \sum_{n=0}^\infty \frac{(\gamma)_{qn,k} i^{-n-1} w^{-(n+1)/\alpha}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}} \int_0^\infty e^{-t} t^n dt$$

$$= \sum_{n=0}^\infty \frac{n! (\gamma)_{qn,k} i^{-n-1} w^{-(n+1)/\alpha}}{\Gamma_k(\mu n + \nu + 1)(\delta)_{pn,k}}, \quad (58)$$

which is our stated result (56). $\square$

4. K -Fractional Integration and K -Fractional Differentiation

Recently, k -fractional calculus gained more attention due to its wide variety of applications in various fields [14, 17]. The k -fractional calculus of various types of special functions is used in many research papers [4, 28]. For more details about the recent works in the field of dynamic system theory, stochastic systems, non-equilibrium statistical mechanics, and quantum mechanics, we refer the interesting readers to [9, 17, 24]. In this section, we deduce the outcomes for k -fractional integration and k -fractional differentiation of the above-said function in an orderly way.

Theorem 11 (K -fractional integration). *If $k, \eta \in \mathbb{R}$; $\gamma, \delta, \mu, \nu \in \mathbb{C}$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $p, q > 0$, and $q < \Re(\mu) + p$, then*

$$I_k^\eta \left[z^{(v/k)-1} J_{k,v-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}) \right] = z^{(v+\eta/k)-1} J_{k,v+\eta-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}). \quad (59)$$

Proof. From (9) and (18), we have

$$\begin{aligned} I_k^\eta \left[z^{v/k-1} J_{k,v-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}) \right] &= \frac{1}{k\Gamma_k(\eta)} \int_0^z (z-t)^{\eta/k-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} t^{(\mu n + \nu/k)-1} dt \\ &= \frac{1}{k\Gamma_k(\eta)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} \int_0^z (z-t)^{(\eta/k)-1} t^{(\mu n + \nu/k)-1} dt. \end{aligned} \quad (60)$$

Putting $t = zx$ and $dt = zdx$ in the above equation, we get

$$\begin{aligned} &= \frac{1}{k\Gamma_k(\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} z^{(\mu n + \nu + \eta/k)-1} \int_0^1 (1-x)^{(\eta/k)-1} x^{(\mu n + \nu/k)-1} dx \\ &= \frac{1}{k\Gamma_k(\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} z^{\mu n + \nu + \eta/k-1} B\left(\frac{\eta}{k}, \frac{\mu n + \nu}{k}\right) \\ &= \frac{1}{k\Gamma_k(\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} z^{\mu n + \nu + \eta/k-1} \frac{\Gamma(\eta/k)\Gamma(\mu n + \nu/k)}{\Gamma(\mu n + \nu + \eta/k)}. \end{aligned} \quad (61)$$

By using Definition 2, we have

$$\begin{aligned} I_k^\eta \left[z^{(v/k)-1} J_{k,v-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}) \right] &= \frac{1}{k\Gamma_k(\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} z^{(\mu n + \nu + \eta/k)-1} \frac{k^{(\mu n + \nu + \eta/k)-1} \Gamma_k(\eta) \Gamma_k(\mu n + \nu)}{k^{(\mu n + \nu + \eta/k)-2} \Gamma_k(\mu n + \nu + \eta)} \\ I_k^\eta \left[z^{(v/k)-1} J_{k,v-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}) \right] &= z^{(v+\eta/k)-1} J_{k,v+\eta-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}), \end{aligned} \quad (62)$$

which is our stated result. $\square$

$$D_k^\eta \left[z^{(v/k)-1} J_{k,v-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}) \right] = \frac{z^{(v-\eta+1/k)-2}}{k} J_{k,v-\eta-k-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}). \quad (63)$$

Theorem 12 (K -fractional differentiation). If $k, \eta \in \mathbb{R}$; $\gamma, \delta, \mu, \nu \in \mathbb{C}$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $p, q > 0$, and $q < \Re(\mu) + p$, then

Proof. From (8), (9), and (18), we have

$$\begin{aligned} D_k^\eta \left[z^{v/k-1} J_{k,v-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}) \right] &= \frac{1}{k\Gamma_k(1-\eta)} \frac{d}{dz} \int_0^z (z-t)^{(1-\eta/k)-1} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} t^{(\mu n + \nu/k)-1} dt \\ &= \frac{1}{k\Gamma_k(1-\eta)} \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} \int_0^z (z-t)^{(1-\eta/k)-1} t^{(\mu n + \nu/k)-1} dt. \end{aligned} \quad (64)$$

Putting $t = zx$ and $dt = zdx$ in the above equation, we get

$$\begin{aligned}
&= \frac{1}{k\Gamma_k(1-\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} \frac{d}{dz} z^{(\mu n + \nu - \eta + 1/k) - 1} \int_0^1 (1-x)^{(1-\eta/k)-1} x^{(\mu n + \nu/k) - 1} dx \\
&= \frac{1}{k\Gamma_k(1-\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} \frac{d}{dz} z^{\mu n + \nu - \eta + 1/k - 1} B\left(\frac{1-\eta}{k}, \frac{\mu n + \nu}{k}\right) \\
&= \frac{1}{k\Gamma_k(1-\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} \left(\frac{\mu n + \nu - \eta + 1}{k} - 1\right) z^{\mu n + \nu - \eta + 1/k - 2} \frac{\Gamma(1-\eta/k)\Gamma(\mu n + \nu/k)}{\Gamma(\mu n + \nu - \eta + 1/k - 1 + 1)}.
\end{aligned} \tag{65}$$

Using Definition 2 and the result $\Gamma(n+1) = n\Gamma(n)$ in the above expression, we get

$$\begin{aligned}
&= \frac{1}{\Gamma_k(1-\eta)} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu)(\delta)_{pn,k}} \frac{(\mu n + \nu - \eta + 1/k - 1)}{(\mu n + \nu - \eta + 1/k - 1)} z^{\mu n + \nu - \eta + 1/k - 2} \\
&\quad \cdot \frac{k^{(\mu n + \nu - \eta + 1/k - 1)} \Gamma_k(1-\eta) \Gamma_k(\mu n + \nu)}{k^{(\mu n + \nu - \eta + 1/k - 1)} \Gamma_k(\mu n + \nu - \eta + 1 - k)} \\
&= \frac{z^{(\nu - \eta + 1/k) - 2}}{k} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma)_{qn,k}}{\Gamma_k(\mu n + \nu - \eta - k + 1)(\delta)_{pn,k}} z^{\mu n/k} \\
D_k^\eta \left[z^{\nu/k - 1} J_{k,\nu-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}) \right] &= \frac{z^{(\nu - \eta + 1/k) - 2}}{k} J_{k,\nu-\eta-k-1,q,p}^{\mu,\gamma,\delta} (z^{\mu/k}).
\end{aligned} \tag{66}$$

This completes the proof. $\square$

5. Concluding Remarks

In the present article, we have established generalized k -Bessel–Maitland function $J_{k,\nu,\delta,p}^{\mu,\gamma,q}(z)$ and its intriguing properties. Also, we have pointed out several integral transform such as beta transform, Laplace transform, Whittaker transform, K -transform, and fractional Fourier transform. In the last section, we deduced the outcomes for k -fractional integration and k -fractional differentiation of k -Bessel–Maitland function. Various special cases of the papers related results may be analyzed by taking appropriate values of the relevant parameters. For example, as given in Remarks 1.5, 1.6, and 1.7, we obtain the undeniable result due to Nisar et al. [15]. For several other special cases, we refer to [4, 12, 23, 24, 26, 28, 29] and leave the findings to interested readers.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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