Research Article

On Multi-Index Mittag–Leffler Function of Several Variables and Fractional Differential Equations

B. B. Jaimini ¹, Manju Sharma ¹, D. L. Suthar ² and S. D. Purohit ³

¹Department of Mathematics, Government College, Kota 324001, Rajasthan, India

²Department of Mathematics, Wollo University, P. O. Box: 1145, Amhara, Ethiopia

³Department of HEAS (Mathematics), Rajasthan Technical University, Kota 324001, Rajasthan, India

Correspondence should be addressed to D. L. Suthar; dlsuthar@gmail.com

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In this paper, we have studied a unified multi-index Mittag–Leffler function of several variables. An integral operator involving this Mittag–Leffler function is defined, and then, certain properties of the operator are established. The fractional differential equations involving the multi-index Mittag–Leffler function of several variables are also solved. Our results are very general, and these unify many known results. Some of the results are concluded at the end of the paper as special cases of our primary results.

1. Introduction

Recently, Mittag–Leffler (M-L) functions have demonstrated their special connection to fractional calculus, with a particular emphasis on fractional calculus problems arising from implementations. Several new special functions and implementations have been discovered over the last few decades. The advancement of research in the new era of special functions and their applications in mathematical modelling continues to attract many scientists from various disciplines (see recent papers; [1–13]).

The Mittag–Leffler function is extended to multi-index function in the following form [14, 15]:

$$E_{\gamma, K}[(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m); z] = \sum_{n=0}^{\infty} \frac{(\gamma)_{Kn} z^n}{\prod_{j=1}^m \Gamma(\beta_j + n\alpha_j) n!}, \quad (1)$$

where $\alpha_j, \beta_j, \gamma \in \mathbb{C}$; $\Re(\alpha_j) > 0$; $\Re(\beta_j) > 0$ ($j = 1, \dots, m$); $\Re(\sum_{j=1}^m \alpha_j) > 0$; and K is an arbitrary complex number, i.e., $K \in \mathbb{C}$.

If we make $\gamma = K = 1$ in (1) it reduces to the multi-index M-L function studied by Kiryakova [16, 17].

A multivariable extension of Mittag–Leffler function widely studied by Gautam [18], and also by Saxena et al. ([19], p. 547, Equation (7.1)), is defined and represented as follows:

$$E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}(z_1, \dots, z_r) = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{l_1 k_1} \dots (\gamma_r)_{l_r k_r}}{\Gamma(\lambda + k_1 \rho_1 + \dots + k_r \rho_r)} \frac{z_1^{k_1} \dots z_r^{k_r}}{k_1! \dots k_r!}, \quad (2)$$

where $\lambda, \gamma_j, l_j, \rho_j \in \mathbb{C}$ $\Re(\rho_j) > 0$; $\Re(l_j) > 0$; $\lambda \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$; and $j = 1, 2, \dots, r$.

Motivated by the work on these functions, we consider here the subsequent multivariable and multi-index Mittag-Leffler function:

$$E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}) ; \beta_1, \dots, \beta_m}^{(\gamma_r), (l_r)}[z_1, \dots, z_r] = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r (\gamma_i)_{l_i k_i}}{\prod_{j=1}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i)} \frac{z_1^{k_1}}{k_1!} \dots \frac{z_r^{k_r}}{k_r!}, \quad (3)$$

where $\beta_j, \gamma_i, l_i, \rho_j^{(i)} \in \mathbb{C}$; $\Re(\rho_j^{(i)}) > 0$; $\Re(l_i) > 0$; $\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$; $(\rho_j^{(r)}) \equiv \rho_j', \rho_j'', \dots, \rho_j^{(r)}$; $i = 1, \dots, r$; and $j = 1, \dots, m$.

We have also studied here, the integral operator involving the function defined by (3), as follows:

$$\begin{aligned} & \left(E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}) ; \beta_1, \dots, \beta_m ; a+}^{(\gamma_r), (l_r)} \psi \right)(x) \\ &= \int_a^x (x-t)^{\beta_1-1} E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}) ; \beta_1, \dots, \beta_m}^{(\gamma_r), (l_r)} \left[\omega_1 (x-t)^{\rho_1'}, \dots, \omega_r (x-t)^{\rho_1^{(r)}} \right] \psi(t) dt, \end{aligned} \quad (4)$$

with $\omega_i, \rho_j^{(i)}, \gamma_i, l_i, \beta_j \in \mathbb{C}$; $x > a$; $|\omega_i (x-t)^{\rho_1^{(i)}}| < 1$; $\Re(\rho_j^{(i)}) > 0$; $j = 1, \dots, m$; and $i = 1, \dots, r$.

The Riemann-Liouville fractional derivative operator D_{0+}^α is defined as follows [20]:

$$(D_{a+}^\alpha \psi)(x) = \left(\frac{d}{dx} \right)^n (I_{a+}^{n-\alpha} \psi)(x), \quad (\alpha \in \mathbb{C}; \Re(\alpha) > 0; n = |\Re(\alpha)| + 1), \quad (5)$$

where $(I_{a+}^\alpha \psi)(x)$ is the fractional integral operator defined by

$$(I_{a+}^\alpha \psi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\psi(t)}{(x-t)^{1-\alpha}} dt. \quad (6)$$

The elementary definitions are also required to be mentioned as follows.

The Laplace transform of fractional derivative $(D_{0+}^\alpha f)(x)$ is given as

$$\mathfrak{L}(D_{0+}^\alpha f; s) = s^\alpha F(s) - \sum_{k=1}^n s^{k-1} D_{0+}^{\alpha-k} f(0+), \quad (\Re(s) > 0; (n-1 < \alpha < n)). \quad (7)$$

Also, the formula for Laplace transform is

$$\frac{d^n}{ds^n} [\mathfrak{L}\{y(x); (s)\}] = (-1)^n \mathfrak{L}[x^n y(x)](s). \quad (8)$$

2. Results Required

The integral for the generalized M-L function defined in (3) is given by

$$\begin{aligned} & \frac{1}{\Gamma(\sigma)} \int_0^x (x-t)^{\beta_1-1} t^{\sigma-1} E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}) ; \beta_1, \dots, \beta_m}^{(\gamma_r), (l_r)} \left[\omega_1 (x-t)^{\rho_1'}, \dots, \omega_r (x-t)^{\rho_1^{(r)}} \right] dt \\ &= x^{\beta_1+\sigma-1} E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}) ; \beta_1+\sigma, \beta_2, \dots, \beta_m}^{(\gamma_r), (l_r)} \left[\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_1^{(r)}} \right]. \end{aligned} \quad (9)$$

The result in (9) is established in view of definition in (3) and using the elementary beta integral.

The Laplace transform of $E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1, \dots, \beta_m)}^{(\gamma_r), (l_r)}(\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_r'})$ defined in (3), easily obtained here, is as follows:

$$\begin{aligned} \mathfrak{L}\left\{x^{\beta_1-1} E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1, \dots, \beta_m)}^{(\gamma_r), (l_r)}[\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_r'}]\right\}(x) &= s^{-\beta_1} \prod_{j=2}^m \left(\frac{1}{\Gamma(\beta_j)}\right) \\ &\times F_{m-1:0; \dots; 0}^{0:1; \dots; 1} \left[\begin{matrix} - & : & (\gamma_1, l_1); \dots; (\gamma_r, l_r) ; \\ (\beta_j; \rho_j', \dots, \rho_j^{(r)})_{2,m} & : & -; \dots; - & ; & \omega_1 s^{-\rho_1'}, \dots, \omega_r s^{-\rho_r'} \end{matrix} \right], \end{aligned} \quad (10)$$

where $\alpha, \beta_j, \rho_j^{(i)}, \gamma_i, l_i, \omega_i \in \mathbb{C}; \Re(s) > 0; \Re(\rho_j^{(i)}) > 0; \Re(\beta_j); \Re(l_i) > 0; j = 1, \dots, m; i = 1, \dots, r;$ and $(\beta_j; \rho_j', \dots, \rho_j^{(r)})_{2,m} \equiv (\beta_2; \rho_2', \dots, \rho_2^{(r)}), \dots, (\beta_m; \rho_m', \dots, \rho_m^{(r)})$.

Here, $F[\omega_1 s^{-\rho_1'}, \dots, \omega_r s^{-\rho_r'}]$ is the generalized Lauricella function ([21], p. 37, Equations (21–23)).

3. Main Results

Theorem 1. Let $a \in \mathbb{R}_+; \alpha, \beta_j, \rho_j^{(i)}, \gamma_i, l_i, \omega_i \in \mathbb{C}; \Re(\alpha) > 0; \Re(\rho_j^{(i)}) > 0; \Re(\beta_j) > 0; \text{ and } \Re(l_i) > 0 (j = 1, \dots, m; i = 1, \dots, r).$

Then, for $x > a$, we have

$$\begin{aligned} D_{a+}^{\alpha} \left[(t-a)^{\beta_1-1} E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1, \dots, \beta_m)}^{(\gamma_r), (l_r)} \left\{ \omega_1 (t-a)^{\rho_1'}, \dots, \omega_r (t-a)^{\rho_r'} \right\} \right](x) \\ = (x-a)^{\beta_1-\alpha-1} E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1-\alpha, \beta_2, \dots, \beta_m)}^{(\gamma_r), (l_r)} \left[\omega_1 (x-a)^{\rho_1'}, \dots, \omega_r (x-a)^{\rho_r'} \right] \end{aligned} \quad (11)$$

and

$$\begin{aligned} I_{a+}^{\alpha} \left[(t-a)^{\beta_1-1} E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1, \dots, \beta_m)}^{(\gamma_r), (l_r)} \left\{ \omega_1 (t-a)^{\rho_1'}, \dots, \omega_r (t-a)^{\rho_r'} \right\} \right](x) \\ = (x-a)^{\beta_1+\alpha-1} E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1+\alpha, \beta_2, \dots, \beta_m)}^{(\gamma_r), (l_r)} \left[\omega_1 (x-a)^{\rho_1'}, \dots, \omega_r (x-a)^{\rho_r'} \right]. \end{aligned} \quad (12)$$

If $\alpha, \beta_j, \gamma_i, l_i, \omega_i \in \mathbb{C}; \Re(\alpha) > 0; \Re(\rho_j^{(i)}) > 0; \Re(\beta_j) > 0; \Re(l_i) > 0; j = 1, \dots, m; \text{ and } i = 1, \dots, r$ with the initial condition $(I_{0+}^{1-\alpha} y)(0+) = c$ (c is an arbitrary constant) and solution of differential equations existing in the space $L(0, \infty)$, then Theorems 2–4 are stated in the following form.

Theorem 2. If

$$(D_{0+}^{\alpha} y)(x) = \lambda \left(E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1, \dots, \beta_m; \omega_r; 0_+)}^{(\gamma_r), (l_r)} \right)(x) + f(x), \quad (13)$$

then its solution is given by

$$\begin{aligned} y(x) &= \frac{cx^{\alpha-1}}{\Gamma(\alpha)} + \lambda x^{\beta_1+\alpha} E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1+\alpha+1, \beta_2, \dots, \beta_m)}^{(\gamma_r), (l_r)} [\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_r'}] \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt. \end{aligned} \quad (14)$$

Theorem 3. *If*

then its solution is given by

$$\begin{aligned} (D_{0+}^{\alpha} y)(x) &= \lambda \left(E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1, \dots, \beta_m; \omega_r; 0+}^{(\gamma_r), (l_r)} \right) (x) \\ &\quad + p x^{\beta_1} E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1+1, \beta_2, \dots, \beta_m}^{(\gamma_r), (l_r)} \left[\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_1^{(r)}} \right], \end{aligned} \quad (15)$$

$$y(x) = \frac{c x^{\alpha-1}}{\Gamma(\alpha)} + (\lambda + p) x^{\beta_1+\alpha} E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1+\alpha+1, \beta_2, \dots, \beta_m}^{(\gamma_r), (l_r)} \left[\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_1^{(r)}} \right]. \quad (16)$$

Theorem 4. *If*

Proof. In Theorem 1, let the left-hand side of result (11) be Δ_1 , i.e.,

$$x(D_{0+}^{\alpha} y)(x) = \lambda \left(E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1, \dots, \beta_m; \omega_r; 0+}^{(\gamma_r), (l_r)} \right) (x), \quad (17)$$

then its solution is given by

$$\begin{aligned} y(x) &= \frac{c x^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} (x-t)^{\beta_1-1} \\ &\quad \times E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1+1, \beta_2, \dots, \beta_m}^{(\gamma_r), (l_r)} \left[\omega_1 (x-t)^{\rho_1'}, \dots, \omega_r (x-t)^{\rho_1^{(r)}} \right] dt. \end{aligned} \quad (18)$$

$$\Delta_1 = D_{a+}^{\alpha} \left[(t-a)^{\beta_1-1} E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1, \dots, \beta_m}^{(\gamma_r), (l_r)} \left\{ \omega_1 (t-a)^{\rho_1'}, \dots, \omega_r (t-a)^{\rho_1^{(r)}} \right\} \right] (x). \quad (19)$$

Having used the definition of $E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1, \dots, \beta_m}^{(\gamma_r), (l_r)} [\cdot]$ given in (3), we obtain the following form:

$$\Delta_1 = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}]}{\prod_{j=1}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i) \prod_{i=1}^r (k_i!)} D_{a+}^{\alpha} \left[(t-a)^{\beta_1 + \sum_{i=1}^r \rho_1^{(i)} k_i - 1} \right] (x). \quad (20)$$

On using the fractional derivative of power function $(t-a)^{\beta_1 + \sum_{i=1}^r \rho_1^{(i)} k_i - 1}$ ([20], p. 36, Equation (2.26)), we have

$$\Delta_1 = (x-a)^{\beta_1 - \alpha - 1} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}] (x-a)^{\sum_{i=1}^r \rho_1^{(i)} k_i}}{\prod_{j=2}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_1^{(i)} k_i) \Gamma(\beta_1 - \alpha + \sum_{i=1}^r \rho_1^{(i)} k_i) \prod_{i=1}^r (k_i!)} \quad (21)$$

On interpreting multiple series by the definition of $E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1, \dots, \beta_m}^{(\gamma_r), (l_r)}[\cdot]$, we at once arrive at (11).

The proof of (12) follows the proof of (11) using (6) and ([20], p. 40, Equation (2.44)) therein.

Theorem 2 is proved as follows.

Use the definition of operator $(E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1, \dots, \beta_m; a+}^\psi)$ (x) (at $a = 0$ and $\psi(x) = 1$) and result (9) (at $\sigma = 1$) in (13), we have

$$y(s) = cs^{-\alpha} + \lambda s^{-\beta_1-1-\alpha} \prod_{j=2}^m \left(\frac{1}{\Gamma(\beta_j)} \right) \times F_{m-1}^0 \begin{matrix} 1; \dots; 1 \\ 0; \dots; 0 \end{matrix} \begin{matrix} - & : & (\gamma_1, l_1); \dots; (\gamma_r, l_r) \\ (\beta_j; \rho_j', \dots, \rho_j^{(r)})_{2,m} & : & -; \dots; - \end{matrix} \begin{matrix} \omega_1 s^{-\rho_1'}, \dots, \omega_r s^{-\rho_1^{(r)}} \\ \end{matrix} \Bigg] + f(s)s^{-\alpha}. \quad (23)$$

In view of the definition of generalized Lauricella function ([21], p. 37, Equations (21–23)), we have the form

$$y(s) = cs^{-\alpha} + \lambda \frac{\sum_{k_1, \dots, k_r=0}^{\infty} \prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}] s^{(-\beta_1-1-\alpha - \sum_{i=1}^r \rho_j^{(i)} k_i)}}{\prod_{j=2}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i) \prod_{i=1}^r (k_i!)} + f(s)s^{-\alpha}. \quad (24)$$

Applying inverse Laplace transform on both sides of (24) and using convolution theorem, we find

$$y(x) = c \frac{x^{\alpha-1}}{\Gamma(\alpha)} + \lambda \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}]}{\prod_{j=1}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i) \prod_{i=1}^r (k_i!)} \frac{x^{\beta_1+\alpha + \sum_{i=1}^r \rho_j^{(i)} k_i}}{\Gamma(\beta_1 + 1 + \alpha + \sum_{i=1}^r \rho_j^{(i)} k_i)} + \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt. \quad (25)$$

Now, on interpreting the multiple series using (3), we at once arrive at desired result (14). $\square$

$$(D_{0+}^\alpha y)(x) = \lambda x^{\beta_1} E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1+1, \beta_2, \dots, \beta_m}^{(\gamma_r), (l_r)} \left[\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_1^{(r)}} \right] + f(x). \quad (22)$$

Taking Laplace transform of (22) and then using formula (7) (for $n = 1$) and (10), therein we have

$$(D_{0+}^\alpha y)(x) = (\lambda + p) x^{\beta_1} E_{(\rho_1^{(r)}), \dots, (\rho_m^{(r)}); \beta_1+1, \beta_2, \dots, \beta_m}^{(\gamma_r), (l_r)} \left[\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_1^{(r)}} \right]. \quad (26)$$

Proof. of Theorem 3. We use (at $a = 0$ and $\psi(x) = 1$) and (9) (at $\sigma = 1$) in (15), and it takes the following form:

On both sides of (26), we take Laplace transform and then using formula (7) (for $n = 1$) and (10) therein, we obtain

$$y(s) = cs^{-\alpha} + (\lambda + p)s^{-\beta_1-1-\alpha} = \prod_{j=2}^m \left(\frac{1}{\Gamma(\beta_j)} \right) \times F_{m-1; 0; \dots; 1}^{0: 1; \dots; 1} \left[\begin{matrix} - & : & (\gamma_1, l_1); \dots; (\gamma_r, l_r) & ; \\ (\beta_j; \rho_j', \dots, \rho_j^{(r)})_{2,m} & : & -; \dots; - & ; \end{matrix} \omega_1 s^{-\rho_1'}, \dots, \omega_r s^{-\rho_1^{(r)}} \right]. \quad (27)$$

In view of the definition of generalized Lauricella function ([21], p. 37, Equations (21–23)), we have

$$y(s) = cs^{-\alpha} + (\lambda + p) \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}] s^{(-\beta_1-1-\alpha - \sum_{i=1}^r \rho_j^{(i)} k_i)}}{\prod_{j=1}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i) \prod_{i=1}^r (k_i!)}. \quad (28)$$

Applying inverse Laplace transform on (28), we have

$$y(x) = c \frac{x^{\alpha-1}}{\Gamma(\alpha)} + (\lambda + p) \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}]}{\prod_{j=2}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i) \prod_{i=1}^r (k_i!)} \times \frac{x^{(\beta_1+\alpha + \sum_{i=1}^r \rho_j^{(i)} k_i)}}{\Gamma(\beta_1 + 1 + \alpha + \sum_{i=1}^r \rho_j^{(i)} k_i)}. \quad (29)$$

On interpreting the multiple series using (3), we at once arrive at result (16). $\square$

Proof. of Theorem 4. We use operator $(E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1, \dots, \beta_m; \omega_r; a+)}^{(\gamma_r, l_r)}) \psi(x)$ (at $a = 0$ and $\psi(x) = 1$) and (9) (at $\sigma = 1$) in (17), and we have the following form:

$$x(D_{0+}^{\alpha} y)(x) = \lambda x^{\beta_1} E_{(\rho_1^{(r)}, \dots, \rho_m^{(r)}; \beta_1+1, \beta_2, \dots, \beta_m)}^{(\gamma_r, l_r)} \left[\omega_1 x^{\rho_1'}, \dots, \omega_r x^{\rho_1^{(r)}} \right]. \quad (30)$$

On both sides of (30), we take Laplace transform and use formulae (8) and (10) (for $n = 1$); then, we obtain

$$\frac{d}{ds} y(s) + \frac{\alpha}{s} y(s) = -\lambda s^{-\beta_1-1-\alpha} \prod_{j=2}^m \left(\frac{1}{\Gamma(\beta_j)} \right) \times F_{m-1; 0; \dots; 1}^{0: 1; \dots; 1} \left[\begin{matrix} - & : & (\gamma_1, l_1); \dots; (\gamma_r, l_r) & ; \\ (\beta_j; \rho_j', \dots, \rho_j^{(r)})_{2,m} & : & -; \dots; - & ; \end{matrix} \omega_1 s^{-\rho_1'}, \dots, \omega_r s^{-\rho_1^{(r)}} \right]. \quad (31)$$

In view of the definition of generalized Lauricella function ([21], p. 37, Equations (21–23)), we have

$$\frac{d}{ds} y(s) + \frac{\alpha}{s} y(s) = -\lambda \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}]}{\prod_{j=2}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i) \prod_{i=1}^r (k_i!)} s^{(-\beta_1-1-\alpha-\sum_{i=1}^r \rho_j^{(i)} k_i)}. \quad (32)$$

Since this is a linear differential equation of first order and first degree,

$$y(s) = \lambda \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}]}{\prod_{j=2}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i) \prod_{i=1}^r (k_i!)} \frac{(s)^{(-\beta_1-\alpha-\sum_{i=1}^r k_1 \rho_1^{(i)})}}{(\beta_1 + \sum_{i=1}^r k_i \rho_1^{(i)})} + c s^{-\alpha}. \quad (33)$$

Taking inverse Laplace transform of (33), we have

$$y(x) = \lambda \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}]}{\prod_{j=2}^m \Gamma(\beta_j + \sum_{i=1}^r \rho_j^{(i)} k_i) \prod_{i=1}^r (k_i!) 1/(\beta_1 + \sum_{i=1}^r k_i \rho_1^{(i)})} \\ \times \mathfrak{L}^{-1} \left[(s)^{\left(-\alpha-\beta_1-\sum_{i=1}^r k_1 \rho_1^{(i)} \right)} \right] + c \mathfrak{L}^{-1} (s^{-\alpha}). \quad (34)$$

In view of convolution theorem, we obtain

$$y(x) = \frac{\lambda}{\Gamma(\alpha)} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{\prod_{i=1}^r [(\gamma_i)_{l_i k_i} (\omega_i)^{k_i}]}{\prod_{j=2}^m \Gamma(\beta_j + \sum_{i=1}^r k_i \rho_j^{(i)}) \prod_{i=1}^r (k_i!) (1/\Gamma(\beta_1 + 1 + \sum_{i=1}^r k_i \rho_1^{(i)}))}. \\ \times \int_0^x t^{\alpha-1} (x-t)^{\beta_1-1+\sum_{i=1}^r k_i \rho_1^{(i)}} dt + \frac{c}{\Gamma(\alpha)} x^{\alpha-1}. \quad (35)$$

Now, on interpreting the multiple series in view of (3), we obtain the result in (18).

4. Conclusion

Here, we conclude further interesting known results:

- (1) Our main results for $m = 1$, respectively, give the known results provided by Gupta and Jaimini ([22], pp. 145–146, Equations (1–10)).
- (2) If in result (10) and in Theorem 2, we take $r = 1$ (i.e., $\omega_2 = \dots = \omega_r = 0$), then result (10) reduces to the known result provided by Saxena et al.

([23], p. 10, Equation (50)) and Theorem 2 gives the correct form (at $\nu = 0$) of the theorem provided by Saxena et al. ([23], p. 10, Theorem (5.1)).

- (3) For $m = 1$ and $r = 1$, Theorems 1 to 4 reduce, respectively, to the known results (at $\nu = 0$) provided by Srivastava and Tomovski [24].
- (4) If in Theorem 1, we take $m = 1$ and $l_1 = l_2 = \dots = l_r = 1$, then these results, respectively, reduce to the known results provided by Gautam ([18], pp. 201–202, Equations (4.64)–(4.65)).
- (5) If in Theorems 1 to 4, we take $m = 1$, $l_1 = l_2 = \dots = l_r = 1$, and $\rho_1' = \rho_1'' = \dots = \rho_1^{(r)} = 1$,

then these, respectively, reduce to the results for function $\phi_2^{(r)}(\cdot)$ provided by Gupta ([25], pp. 250–253, Equations (4.9.19)–(4.9.27)).

Therefore, the results presented in the article would immediately yield a large number of results that include a wide range of special functions occurring in issues of scientific research, computer science, and applied mathematics, among others.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this article.

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