Research Article

On Subtree Number Index of Generalized Book Graphs, Fan Graphs, and Wheel Graphs

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With generating function and structural analysis, this paper presents the subtree generating functions and the subtree number index of generalized book graphs, generalized fan graphs, and generalized wheel graphs, respectively. As an application, this paper also briefly studies the subtree number index and the asymptotic properties of the subtree densities in regular book graphs, regular fan graphs, and regular wheel graphs. The results provide the basis for studying novel structural properties of the graphs generated by generalized book graphs, fan graphs, and wheel graphs from the perspective of the subtree number index.

1. Introduction

The invariants of the graph and its topological indices have important significance in studying the properties of chemical molecular graphs and network graphs [1–4]. The Wiener index, for example, is defined as the sum of the distances of all disordered vertex pairs, and it can be used to estimate the boiling point of the alkanes [5, 6]. Therefore, new topological indices are constantly introduced by worldwide scholars in recent years.

As a representative structural based index, the subtree number index $STN(G)$ of a graph G (also known as ρ -index [7, 8]), which is defined as the number of nonempty labeled subtrees of G , plays an important role in areas such as biological reconstruction [9], reliable network construction [10], and machine learning [11] and therefore has attracted much attention in recent years.

The subtree number problem was first proposed by Jamison, when studying the subtree average order of trees [12, 13]. Székely and Wang [8] characterized the extremal graph among all n vertices tree. Meanwhile, they presented the subtree number formula of good binary tree and relation between the Wiener index and subtree number index of trees. Yan and Yeh [14] proposed a linear algorithm for

enumerating subtrees of trees and further characterized the trees with the second to fifth largest numbers of subtrees and the tree with the second minimum number of subtrees through graph transformation and generating function. With generating function and cycle contraction carrying weights technique, Yang et al. [15–17] solved the subtree enumeration problem and characterized the extremal graphs among all hexagonal, phenylene chains, and spiro, polyphenyl hexagonal chains of length n , respectively.

Using the generating function method, more recently, Chin et al. [18] studied the enumeration problem of subtrees and spanning subtrees of complete graphs, complete bipartite graphs, and theta graphs. Yang et al. [17] studied the expected subtree number index in random polyphenylene and spiro chains. By introducing new “middle parts” of a tree, Li et al. [19] studied several extreme ratios of distance and subtree problems in binary trees. With the combinatorial technique, Kamiński and Prałat [20] provided the upper and lower bounds for the number of subtrees of random tree. Zhang et al. [21] presented the concepts of the eccentric subtree number and studied some extremal results with respect to this topological index in a tree.

Wheel graphs, fan graphs, and book graphs are three representative cyclic graphs. Various properties of these

graphs have been studied, for example, Lin et al. [22] studied the Laplacian spectral features of vertex extension of wheel graphs and multifan graphs. Daoud [23] studied the number of spanning trees of wheel graphs and complex graphs generated by wheel graphs. Yang et al. [24] studied the subtree problem of wheel graphs and multifan graphs with the generating functions. Barioli [25] studied the complete matrix of book graphs. Based on wheel graph and chicken swarm optimization, Yu et al. [26] proposed a novel hybrid localization scheme for deep mine. Liu et al. [27] proved that multifan graphs were determined by their Laplacian spectra.

We first introduce related terminologies and notations in Section 2. The subtree generating functions and subtree number indices of generalized book graphs $B(n)$ ($n \geq 2$), fan graphs $F(n)$ ($n \geq 3$), and wheel graphs $W(n)$ ($n \geq 3$) are derived in Section 3. Some special cases are also discussed as immediate consequences. In Section 4, we analyze the asymptotic characteristics of the subtree densities of the aforementioned generalized graphs. We summarize our work in Section 5.

2. Terminology and Notation

Let $G = (V(G), E(G); f, g)$ be a weighted graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$, edge set $E(G) = \{e_1, e_2, \dots, e_m\}$, vertex weight function f , and edge weight function g . Unless otherwise stated, we specify $f: V(G) \rightarrow \mathfrak{R}$ and $g: E(G) \rightarrow \mathfrak{R}$, where $\mathfrak{R}$ is a commutative ring with a unit element 1. In what follows, we list the symbols that will be used in later discussion.

- (i) Denote by $L(G)$ the leaf set of G .
- (ii) Denote by $G \setminus X$ the graph obtained from G by removing X from G .
- (iii) Denote by $S(G)$ the set of all subtrees of G .
- (iv) Denote by $K_{1,n}$ a star with $n+1$ vertices and P_n a path with n vertices.
- (v) Denote by $S(G; X)$ the set of subtrees of G containing X , where X can be a vertex or an edge set.
- (vi) Denote by $S(G; v, P_n)$ the set of subtrees of G containing both vertex v and path P_n .

For any subtree $T \in S(G)$, the weight of T is defined as

$$w(T) = \prod_{v \in V(T), e \in E(T)} f(v)g(e), \quad (1)$$

and the subtree generating function of G is defined as $F(G; f, g) = \sum_{T \in S(G)} w(T)$. Similarly, we define $F(G; f, g; X) = \sum_{T \in S(G; X)} w(T)$, the subtree generating function of G containing X , and $F(G; f, g; v, P_n) = \sum_{T \in S(G; v, P_n)} w(T)$, the subtree generating function of G containing both vertex v and path P_n .

Letting vertex and edge weight $f = g = 1$ in the above generating functions, we have the corresponding subtree number indices, namely, $\text{STN}(G) = F(G; 1, 1)$, $\text{STN}(G; X) = F(G; 1, 1; X)$, and $\text{STN}(G; v, P_n) = F(G; 1, 1; v, P_n)$.

Next, we introduce some lemmas and definitions that will be used in our arguments. Let $T = (V(T), E(T); f, g)$ be

a weighted tree on n vertices, v_i be a vertex of T , $u \neq v_i$ be a pendant vertex, and $e = (u, v)$ be the pendant edge of T . We define a weighted tree $T' = (V(T'), E(T'); f', g')$ from T as follows: $V(T') = V(T) \setminus \{u\}$, $E(T') = E(T) \setminus \{e\}$, and

$$f'(v_s) = \begin{cases} f(v)(f(u)g(e) + 1), & \text{if } v_s = v, \\ f(v_s), & \text{otherwise,} \end{cases} \quad (2)$$

for any $v_s \in V(T')$, $g'(e) = g(e)$ ($e \in E(T')$).

Lemma 1 (see [14]). *With the above notations, we have*

$$\begin{aligned} F(T; f, g; v_i) &= F(T'; f', g'; v_i), \\ F(T; f, g) &= F(T'; f', g') + f(u). \end{aligned} \quad (3)$$

Definition 1. If the n pendant edges of the star $K_{1,n}$ are replaced by paths of length l_i ($i = 1, 2, \dots, n$), the resulting tree is called a generalized star tree $K(l_1, l_2, \dots, l_n)$ or simply $K(n)$.

Definition 2. Define $B(l_0, l_1, \dots, l_n)$ (or simply $B(n)$ ($n \geq 2$)) the n -page generalized book graph constructed by joining the end vertices u_i and v_i of the $n+1$ internally disjoint paths $u_i w_{i,1} w_{i,2} \dots w_{i,l_i} v_i$ ($i = 0, 1, \dots, n$) of length $l_i + 1$, where $u_0 = u_1 = \dots = u_n = u$ and $v_0 = v_1 = \dots = v_n = v$. Obviously, $B(n)$ has $(2 + \sum_{i=0}^n l_i)$ vertices and $\sum_{i=0}^n (l_i + 1)$ edges.

Definition 3. Define $F(l_1, \dots, l_n; m_1, \dots, m_{n-1})$ (or simply $F(n)$ ($n \geq 3$)) the generalized fan graph obtained by firstly constructing a generalized star through identifying the end vertices $v_{i,0}$ ($i = 1, 2, \dots, n$) of the n internally disjoint paths $v_{i,0} v_{i,1} v_{i,2} \dots v_{i,l_i} c_i$ ($l_i \geq 0$) together, where $v_{i,0} = c_0$ and then connecting vertex pair c_i, c_{i+1} with path $c_i v_{i,1}^* v_{i,2}^* \dots v_{i,m_i}^* c_{i+1}$ ($i = 1, 2, \dots, n-1$) of length $m_i + 1$ ($m_i \geq 0$), where $v_{i,0} = c_0$, $l_i \geq 0$ ($i = 1, 2, \dots, n$), and $m_i \geq 0$ ($i = 1, 2, \dots, n-1$) (specifying that $m_0 = 0$). For brevity, we abbreviate some notations.

- (i) Denote $\bar{e}_{i,1} = (c_0, v_{i,1})$, $\bar{e}_{i,j} = (v_{i,j-1}, v_{i,j})$, $\bar{e}_{i,l_i+1} = (v_{i,l_i}, c_i)$ ($i = 1, \dots, n; j = 2, \dots, l_i$).
- (ii) Denote $e_{i,1} = (c_i, v_{i,1}^*)$, $e_{i,j} = (v_{i,j-1}^*, v_{i,j}^*)$, $e_{i,m_i+1} = (v_{i,m_i}^*, c_{i+1})$ ($i = 1, \dots, n-1; j = 2, \dots, m_i$), specifying that $e_{0,i} = \emptyset$.

Definition 4. Define $W(l_1, \dots, l_n; m_1, \dots, m_n)$ (or simply $W(n)$ ($n \geq 3$)) the generalized wheel graph constructed from the generalized fan graph $F(n)$ (see Definition 3) by joining vertices pair c_1 and c_n of the generalized fan graph $F(n)$ with a path $v_{n,0}^* v_{n,1}^* \dots v_{n,m_n}^* v_{n,m_n+1}^*$, where $v_{n,0}^* = c_n$ and $v_{n,m_n+1}^* = c_1$. Similarly, we abbreviate some notations for the brevity.

- (i) Denote $\tilde{e}_{i,1} = (c_0, v_{i,1})$, $\tilde{e}_{i,j} = (v_{i,j-1}, v_{i,j})$, $\tilde{e}_{i,l_i+1} = (v_{i,l_i}, c_i)$ ($i = 1, \dots, n; j = 2, \dots, l_i$).
- (ii) Denote $e_{i,1}^* = (c_i, v_{i,1}^*)$, $e_{i,j}^* = (v_{i,j-1}^*, v_{i,j}^*)$, $e_{i,m_i+1}^* = (v_{i,m_i}^*, c_{i+1})$, ($i = 1, \dots, n; j = 2, \dots, m_i$), specifying that $e_{0,i}^* = \emptyset$.

3. Subtree Generating Functions for Generalized Book Graphs, Fan Graphs, and Wheel Graphs

3.1. Subtree Generating Function for Generalized Book Graphs. Before deriving the subtree generating function for generalized book graphs, we first introduce some lemmas.

Lemma 2 (see [14]). Let $P_n = (V(P_n), E(P_n); f, g)$ be a weighted path on n vertices, with $V(P_n) = \{v_0, v_1, \dots, v_{n-1}\}$, $E(P_n) = \{(v_i, v_{i+1}) | i = 0, 1, \dots, n-2\}$, $f(v) = y(v \in V(P_n))$ and $g(e) = z(e \in E(P_n))$; then,

$$F(P_n; f, g) = \sum_{i=0}^{n-1} (n-i) y^{i+1} z^i, \quad (4)$$

$$F(P_n; f, g; v_0) = \sum_{i=0}^{n-1} y^{i+1} z^i. \quad (5)$$

Lemma 3 (see [15]). Let $G' = (V(G'), E(G'); f, g)$ be a weighted graph with the common edge (u, v) , $f(v) = y(v \in V(G'))$, $g(e) = z(e \in E(G'))$, $V(G') = V(G'_1) \cup V(G'_2)$, and $E(G') = E(G'_1) \cup E(G'_2)$, $G'_1 \cap G'_2 = (u, v)$, where G'_1, G'_2 are two weighted graphs; then,

$$F(G'; f, g) = \frac{F(G'_1; f, g; (u, v)) F(G'_2; f, g; (u, v))}{y^2 z} + F(G' \setminus (u, v); f, g). \quad (6)$$

Lemma 4 (see [16]). Let $U_n = (V(U_n), E(U_n); f, g)$ be a weighted unicyclic graph with $V(U_n) = \{v_1, v_2, \dots, v_n\}$, $E(U_n) = \{(v_i, v_{i+1}) | i = 1, 2, \dots, n\}$ (where $v_{n+1} = v_1$), $f(v) = y(v \in V(U_n))$, and $g(e) = z(e \in E(U_n))$, respectively; then, for any fixed vertex $v_i \in V(U_n)$ and any continuous k edge set $\cup_{j=1}^k (v_{i+j-1}, v_{i+j})$ ($k = 1, 2, \dots, n-1$), we have

$$F(U_n; f, g; v_i) = \sum_{i=1}^n i y^i z^{i-1}, \quad (7)$$

$$F\left(U_n; f, g; \bigcup_{j=1}^k (v_{i+j-1}, v_{i+j})\right) = \sum_{i=1}^{n-k} i y^{i+k} z^{i+k-1}. \quad (8)$$

The following Lemma follows from Lemmas 1 and 2.

Lemma 5. Let $K(n) = (V(K(n)), E(K(n)); f, g)$ be a weighted generalized star tree, with $f(v) = y(v \in V(K(n)))$ and $g(e) = z(e \in E(K(n)))$; then,

$$F(K(n); f, g) = \frac{1}{y^{n-1}} \prod_{i=1}^n \sum_{j=1}^{l_i+1} y^j z^{j-1} + \sum_{i=1}^n \sum_{j=1}^{l_i} (l_i - j + 1) y^j z^{j-1}. \quad (9)$$

Theorem 1. Let $B(n) = (V(B(n)), E(B(n)); f, g)$ be an n -page weighted generalized book graph, with $f(v) = y(v \in V(B(n)))$ and $g(e) = z(e \in E(B(n)))$; then,

$$F(B(n); f, g) = \sum_{i=0}^n \frac{(y^{l_i+2} z^{l_i+1})^{1-n}}{\sum_{k=1}^{l_i+1} (k y^{k+l_i+1} z^{k+l_i})} \prod_{j=0}^n \sum_{k=1}^{l_j+1} (k y^{k+l_j+1} z^{k+l_j}) + \frac{2}{y^n} \prod_{i=0}^n \sum_{j=1}^{l_i+1} y^j z^{j-1} + \sum_{i=0}^n \sum_{j=1}^{l_i} (l_i - j + 1) y^j z^{j-1}. \quad (10)$$

Proof. We divide the subtrees of generalized book graph $B(n)$ into four categories:

- (1) Containing u but not v .
- (2) Containing v but not u .
- (3) Containing neither u nor v .
- (4) Containing both u and v .

With structural analysis and Lemma 5, we have the subtree generating function of cases (1) and (2) as

$$\frac{1}{y^n} \prod_{i=0}^n \sum_{j=1}^{l_i+1} y^j z^{j-1}. \quad (11)$$

With Lemma 1, it is easy to see that the subtree generating function of case (3) is

$$\sum_{i=0}^n \sum_{j=1}^{l_i} (l_i - j + 1) y^j z^{j-1}. \quad (12)$$

Denote by $P_{uv}^i = uw_{i,1}w_{i,2} \dots w_{i,l_i}v$ ($i = 0, 1, 2, \dots, n$) the path connecting the vertex pair u, v , and $U_{l_i+l_j+2}^{i,j} = P_{uv}^i w_{j,l_j} \dots w_{j,2}w_{j,1}u$ ($j \neq i, j \in \{0, 1, \dots, n\}$) an unicyclic graph. With structural analysis, equation (8) in Lemma 4, and Lemma 3, we have the subtree generating function of case (4) as

$$\sum_{i=0}^n \frac{\prod_{0 \leq j \neq i \leq n} F(U_{l_i+l_j+2}^{i,j}; f, g; P_{uv}^i)}{(y^{l_i+2} z^{l_i+1})^{n-1}}. \quad (13)$$

The theorem follows from equations (11)–(13). $\square$

We can further obtain the subtree generating function of special generalized book graphs.

Definition 5. Let $B(l_0, l_1, \dots, l_n)$ be an n -page generalized book graph and k be a positive integer; we call $B(l_0, l_1, \dots, l_n)$ an n -page k regular book graph, denoted by

$B_{n,k}$ ($n \geq 2$), if the lengths of its $n + 1$ internally disjoint paths are all equal to k (namely, $l_i = k$ ($i = 0, 1, \dots, n$)). Obviously, $B_{n,k}$ ($n \geq 2$) has $(n + 1)k + 2$ vertices and $(n + 1)(k + 1)$ edges.

With Theorem 1, it is not difficult to obtain the following corollary.

Corollary 1. Let $B_{n,k} = (V(B_{n,k}), E(B_{n,k}); f, g)$ be an n -page k regular book graph, with $f(v) = y$ ($v \in V(B_{n,k})$) and $g(e) = z$ ($e \in E(B_{n,k})$); then,

$$F(B_{n,k}; f, g) = (n + 1) \sum_{i=0}^{k-1} (k - i) y^{i+1} z^i + \frac{2}{y^n} \left(\sum_{i=1}^{k+1} y^i z^{i-1} \right)^{n+1} + \frac{n + 1}{(y^{k+2} z^{k+1})^{n-1}} \left(\sum_{i=1}^{k+1} i y^{i+k+1} z^{i+k} \right)^n. \quad (14)$$

By substituting $y = 1, z = 1$ into equations (10) and (14), we have the corresponding subtree number indices.

Corollary 2. The subtree number index of the n -page generalized book graph $B(n)$ is

$$\text{STN}(B(n)) = 2 \prod_{i=0}^n (l_i + 1) + \sum_{i=0}^n \sum_{j=0}^{l_i-1} (l_i - j) + \sum_{i=0}^n \frac{\prod_{j=0}^n (l_j + 1)(l_j + 2)}{2^n (l_i + 1)(l_i + 2)}. \quad (15)$$

Corollary 3. The subtree number index of the n -page k regular book graph $B_{n,k}$ is

$$\text{STN}(B_{n,k}) = (n + 1) \frac{k(k + 1)}{2} + 2(k + 1)^{n+1} + (n + 1) \frac{(k^2 + 3k + 2)^n}{2^n}. \quad (16)$$

With Corollary 3, we have the subtree number index of the $B_{n,k}$ as illustrated in Table 1.

3.2. Subtree Generating Function of Two-Tailed Generalized Fan Graph. Before solving the subtree generating functions of generalized fan graphs and generalized wheel graphs, we firstly introduce and solve the subtree generating function computing problem of the two-tailed generalized fan graphs. Firstly, we employ the convention that $\cup_{t=i}^j \bar{e}_{t,1} = \emptyset$ ($j < i$).

Definition 6. Denote by $\bar{F}_{n,i}$ ($i = 1, 2, 3, \dots, n$) the generalized fan graph of $F(n) \setminus (e_{i-1, m_{i-1}+1} \cup \cup_{j=1}^{i-1} \bar{e}_{j,1})$ that contains c_0 and c_i ; moreover, define P_{c_i, c_n} the path tree of $\bar{F}_{n,i} \setminus (\cup_{r=i}^n \bar{e}_{r, l_r+1})$ that contains vertex c_i .

With structural analysis and equations (9) and (13), we can obtain the following lemma.

Lemma 6. Keeping above symbols and definitions, we have

$$F(\bar{F}_{n,i}; f, g; c_0, P_{c_i, c_n}) = \sum_{t=i}^n \frac{(yz)^{l_t+1} \sum_{j=1}^{n-1} (m_j+1)}{y^{n-i-1} \sum_{p=1}^{l_t+1} p y^p z^{p-1}} \prod_{j=i}^n \sum_{p=1}^{l_j+1} p y^p z^{p-1}. \quad (17)$$

Definition 7. Define $\tilde{F}_{a,b}^i(l_i, \dots, l_n; m_i, \dots, m_{n-1})$ (or simply $\tilde{F}_{a,b}^{n,i}$ ($i = 1, \dots, n$)) the two-tailed generalized fan graph obtained from $\bar{F}_{n,i}$ ($i = 1, 2, 3, \dots, n$) by attaching a path $P_{a+1}' = v_0' v_1' v_2' \dots v_a'$ to vertex c_i and path $P_{b+1}'' = v_0'' v_1'' v_2'' \dots v_b''$ to vertex c_n , respectively, where $c_i = v_0'$ and $c_n = v_0''$ (see Figure 1 for an illustration).

- (i) Denote by $F_{a,b}^{n,i}$ ($i = 1, 2, \dots, n$) the k regular two-tailed generalized fan graph with $l_i = m_j = k$ ($i = 1, \dots, n; j = 1, \dots, n - 1; k \geq 0$).
- (ii) Denote by $\tilde{T}_{a,b}^{n,i} = \tilde{F}_{a,b}^{n,i} \setminus c_0$ and $T_{a,b}^{n,i} = F_{a,b}^{n,i} \setminus c_0$ ($i = 1, 2, 3, \dots, n$) the two weighted trees.

Lemma 7. Let $\tilde{F}_{a,b}^{n,i} = (V(\tilde{F}_{a,b}^{n,i}), E(\tilde{F}_{a,b}^{n,i}); f, g)$ ($i = 1, 2, \dots, n$) be a weighted two-tailed generalized fan graph, with $f(v) = y$ ($v \in V(\tilde{F}_{a,b}^{n,i})$) and $g(e) = z$ ($e \in E(\tilde{F}_{a,b}^{n,i})$), respectively; then,

$$F(\tilde{F}_{a,b}^{n,i}; f, g; c_0) = \frac{1}{y^2} \sum_{t=i}^{n-1} F(\bar{F}_{n,t+1}; f, g; c_0, P_{c_{t+1}, c_n}) \sum_{j=t/n-1}^{m_t} F(\tilde{F}_{a, m_t-j}^{t,i}; f, g; c_0) (yz)^j \cdot \sum_{j=1}^{b+1} y^j z^{j-1} + \sum_{k=1}^{a+1} (yz)^{k-1} F(\bar{F}_{n,i}; f, g; c_0, P_{c_i, c_n}) \sum_{k=1}^{b+1} (yz)^{k-1} + F(\tilde{F}_{a, m_{n-1}}^{n-1,i}; f, g; c_0) \sum_{k=1}^{l_n+b+2} (yz)^{k-1}, \quad (18)$$

with $F(\bar{F}_{n,i}; f, g; c_0, P_{c_i, c_n})$ as in equation (17) and $F(\tilde{F}_{a,b}^{n,i}; f, g; c_0) = \sum_{t=1}^{l_i+1} y^t z^{t-1} + y^{l_i} z^{l_i+1} \sum_{t=1}^{a+1} y^t z^{t-1} \sum_{t=1}^{b+1} y^t z^{t-1}$.

Proof. We categorize the subtrees of $\tilde{F}_{a,b}^{n,i}$ containing c_0 into four cases:

TABLE 1: The subtree number index $\text{STN}(B_{n,k})$ of the n -page k regular book graph $B_{n,k}$ ($k = 1, 2, \dots, 5; n = 2, 3, \dots, 11$).

n	$\text{STN}(B_{n,1})$	$\text{STN}(B_{n,2})$	$\text{STN}(B_{n,3})$	$\text{STN}(B_{n,4})$	$\text{STN}(B_{n,5})$
2	46	171	446	955	1800
3	144	1038	4536	14790	39696
4	474	6981	52078	259425	988032
5	1592	48132	608228	4587560	24598008
6	5366	330987	7032810	79890695	600922824
7	18016	2252634	80131120	1367656330	14412067680
8	60082	15155937	900524342	23069921965	340425889776
9	198888	100895088	10002097212	384453125100	7942921398312
10	653646	665482263	110008388674	6343251953235	183479416354488
11	2133968	4354627590	1200033554504	103797558593870	4203334360071504

FIGURE 1: Two-tailed generalized fan graph $\bar{F}_{1,1}^{4,i}$ ($i = 1, 2, 3, 4$). (a) $\bar{F}_{1,1}^{4,1}$. (b) $\bar{F}_{1,1}^{4,2}$. (c) $\bar{F}_{1,1}^{4,3}$. (d) $\bar{F}_{1,1}^{4,4}$.

$$S(\bar{F}_{a,b}^{n,i}; c_0) = \bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_3 \cup \bar{S}_4, \quad (19)$$

where

- (i) $\bar{S}_1$ is the set of subtrees in $S(\bar{F}_{a,b}^{n,i}; c_0)$ that does not contain edge $e_{n-1, m_{n-1}+1}$.
- (ii) $\bar{S}_2$ is the set of subtrees in $S(\bar{F}_{a,b}^{n,i}; c_0)$ that contains edge set $\cup_{k=m_{n-1}+1-j}^{m_{n-1}+1} e_{n-1, k}$, but not edge $e_{n-1, m_{n-1}-j}$ ($j = 0, 1, \dots, m_{n-1} - 1$).
- (iii) $\bar{S}_3$ is the set of subtrees in $S(\bar{F}_{a,b}^{n,i}; c_0)$ that contains edge set $\cup_{k=n-p}^{n-1} \cup_{j=1}^{m_k+1} e_{k,j} \cup \cup_{j=m_{n-1-p}+2-q}^{m_{n-1-p}+1} e_{n-1-p,j}$, but not edge $e_{n-1-p, m_{n-1-p}-q+1}$ ($p = 1, 2, \dots, n - i - 1; q = 0, 1, \dots, m_{n-p-1}$).
- (iv) $\bar{S}_4$ is the set of subtrees in $S(\bar{F}_{a,b}^{n,i}; c_0)$ that contains edge set $\cup_{k=i}^{n-1} \cup_{j=1}^{m_k+1} e_{k,j}$.

For the case $\bar{S}_1$, we define path tree $\bar{P}_{l_n+b+2} = c_0 v_{n,1} v_{n,2} \dots v_{n,l_n} c_n P_{b+1}''$, and it is not difficult to obtain

$$\sum_{T_1 \in \bar{S}_1} w(T_1) = \frac{1}{y} F(\bar{P}_{l_n+b+2}; f, g; c_0) \times F\left(\bar{F}_{a, m_{n-1}}^{n-1,i}; f, g; c_0\right) = F\left(\bar{F}_{a, m_{n-1}}^{n-1,i}; f, g; c_0\right) \sum_{j=1}^{l_n+b+2} (yz)^{j-1}. \quad (20)$$

For the case $\bar{S}_2$, with structural analysis and equation (17), we obtain

$$\sum_{T_2 \in \bar{S}_2} w(T_2) = \frac{1}{y^2} \sum_{j=1}^{m_{n-1}} \left(F\left(\bar{F}_{a, m_{n-1}-j}^{n-1,i}; f, g; c_0\right) F(\bar{F}_{n,n}; f, g; c_0, P_{c_n c_n}) \prod_{k=1}^j f(v_{n-1, m_{n-1}-k+1}^*) g(e_{n-1, m_{n-1}-k+2}) F(P_{b+1}''; f, g; c_n) \right) \frac{1}{y^2} \sum_{j=1}^{m_{n-1}} F\left(\bar{F}_{a, m_{n-1}-j}^{n-1,i}; f, g; c_0\right) y^{l_n+2} z^{l_n+1} (yz)^j \sum_{j=1}^{b+1} y^j z^{j-1}. \quad (21)$$

Similar to analysis of case $\bar{S}_2$, we can obtain

$$\begin{aligned}
\sum_{T_3 \in \bar{S}_3} w(T_3) &= \frac{1}{y^2} \sum_{t=i}^{n-2} F(\bar{F}_{n,t+1}; f, g; c_0, P_{c_{t+1}, c_n}) \left(\sum_{j=0}^{m_t} F(\bar{F}_{a, m_t-j}^{t,i}; f, g; c_0) \prod_{p=1}^j f(v_{t, m_t-p+1}^*) g(e_{t, m_t-p+2}) F(P_{b+1}''; f, g; c_n) \right) \\
&= \frac{1}{y^2} \sum_{t=i}^{n-2} F(\bar{F}_{n,t+1}; f, g; c_0, P_{c_{t+1}, c_n}) \sum_{j=0}^{m_t} (yz)^j F(\bar{F}_{a, m_t-j}^{t,i}; f, g; c_0) \sum_{k=1}^{b+1} y^k z^{k-1}.
\end{aligned} \tag{22}$$

For the case $\bar{S}_4$, with equations (5) and (17), we can get

$$\begin{aligned}
\sum_{T_4 \in \bar{S}_4} w(T_4) &= \frac{F(\bar{F}_{n,i}; f, g; c_0, P_{c_i, c_n}) F(P_{a+1}'; f, g; c_i) F(P_{b+1}''; f, g; c_n)}{y^2} \\
&= F(\bar{F}_{n,i}; f, g; c_0, P_{c_i, c_n}) \sum_{k=1}^{a+1} (yz)^{k-1} \times \sum_{k=1}^{b+1} (yz)^{k-1}.
\end{aligned} \tag{23}$$

Clearly, $\bar{F}_{a,b}^{i,i}$ is a trivial tree, and by Lemma 1, we can get

$$\sum_{T \in S(\bar{F}_{a,b}^{i,i}; c_0)} = \sum_{t=1}^{l_i+1} y^t z^{t-1} + y^{l_i} z^{l_i+1} \times \sum_{t=1}^{a+1} y^t z^{t-1} \times \sum_{t=1}^{b+1} y^t z^{t-1}. \tag{24}$$

Combining equations (20)–(24), we can obtain equation (18), and thus the lemma is proved.

With Lemmas 1 and 7, we can obtain the subtree generating function of generalized fan graphs. $\square$

Theorem 2. Let $F(n) = (V(F(n)), E(F(n)); f, g)$ be a weighted generalized fan graph, with $f(v) = y (v \in V(F(n)))$ and $g(e) = z (e \in E(F(n)))$; then,

$$F(F(n); f, g) = F(\bar{F}_{0,0}^{n,1}; f, g; c_0) + F(\tilde{T}_{0,0}^{n,1}; f, g), \tag{25}$$

with $F(\tilde{T}_{a,b}^{n,i}; f, g; c_0)$ ($a, b \geq 0; 1 \leq i \leq n$) as in equation (18).

$$\begin{aligned}
F(\tilde{T}_{a,b}^{n,i}; f, g) &= \sum_{k=1}^{b+1} (yz)^k \sum_{k=1}^{l_n+1} (yz)^{k-1} \left(F(\tilde{T}_{a, m_{n-1}}^{n-1,i}; f, g) - F(\tilde{T}_{a, m_{n-1}-1}^{n-1,i}; f, g) \right) \\
&\quad + F(\tilde{T}_{a, m_{n-1}}^{n-1,i}; f, g) + \sum_{k=1}^{l_n+b+1} (l_n + b - k + 2) y^k z^{k-1},
\end{aligned} \tag{26}$$

and $F(\tilde{T}_{a,b}^{n,i}; f, g) = \sum_{k=0}^{b-1} (b-k) y^{k+1} z^k + \sum_{k=0}^{a-1} (a-k) y^{k+1} z^k + \sum_{k=0}^{l_i-1} (l_i-k) y^{k+1} z^k + \sum_{k=1}^{a+1} (yz)^{k-1} \times \sum_{k=1}^{l_i+1} y^k z^{k-1} \times \sum_{k=1}^{b+1} (yz)^{k-1}$ ($a, b \geq 0; 1 \leq i \leq n$).

Next, we discuss some special cases of the generalized fan graphs.

Definition 8. Let $F(l_1, \dots, l_n; m_1, \dots, m_{n-1})$ be a generalized fan graph and k be a positive integer; if $l_i = m_j = k$ ($i = 1, \dots, n; j = 1, \dots, n-1$), then the generalized fan graph is called the k regular fan graph, denoted by $F_{n,k}$ ($n \geq 3; k \geq 0$). Obviously, $F_{n,k}$ has $(2n-1)k + n + 1$ vertices and $(2n-1)(k+1)$ edges.

From Theorem 2, the following corollaries follow as immediate consequences.

Corollary 4. Let $F_{n,k} = (V(F_{n,k}), E(F_{n,k}); f, g)$ be a weighted k regular fan graph, with $f(v) = y (v \in V(F_{n,k}))$ and $g(e) = z (e \in E(F_{n,k}))$; then,

$$F(F_{n,k}; f, g) = F(\bar{F}_{0,0}^{n,1}; f, g; c_0) + F(\tilde{T}_{0,0}^{n,1}; f, g), \tag{27}$$

with

$$\begin{aligned}
F(F_{n,k}^{n,i}; f, g; c_0) &= \sum_{t=i}^{n-1} \frac{n-t}{y^{n-t}} \sum_{j=\lceil t/n-1 \rceil}^k F(F_{a,k-j}^{t,i}; f, g; c_0) (yz)^{(k+1)(n-t)+j} \times \sum_{j=1}^{b+1} y^j z^{j-1} \left(\sum_{j=1}^{k+1} j y^j z^{j-1} \right)^{n-t} \\
&\quad + F(F_{a,k}^{n-1,i}; f, g; c_0) \sum_{j=1}^{k+b+2} (yz)^{j-1} + \frac{(n-i+1)}{y^{n-i-1}} (yz)^{(k+1)(n-i+1)} \left(\sum_{j=1}^{k+1} j y^j z^{j-1} \right)^{n-i} \sum_{j=1}^{a+1} (yz)^{j-1} \sum_{j=1}^{b+1} (yz)^{j-1},
\end{aligned} \tag{28}$$

and $F(F_{a,b}^{i,i}; f, g; c_0) = \sum_{t=1}^{k+1} y^t z^{t-1} + y^k z^{k+1} \sum_{t=1}^{a+1} y^t z^{t-1}$
 $\sum_{t=1}^{b+1} y^t z^{t-1} (a, b \geq 0; 1 \leq i \leq n).$

$$F(T_{a,b}^{n,i}; f, g) = \sum_{j=1}^{b+1} (yz)^j \sum_{j=1}^{k+1} (yz)^{j-1} (F(T_{a,k}^{n-1,i}; f, g) - F(T_{a,k-1}^{n-1,i}; f, g)) \\ + F(T_{a,k}^{n-1,i}; f, g) + \sum_{j=1}^{k+b+1} (k+b-j+2) y^j z^{j-1}, \quad (29)$$

and $F(T_{a,b}^{i,i}; f, g) = \sum_{j=0}^{b-1} 1(b-j) y^{j+1} z^j + \sum_{j=0}^{a-1} (a-j) y^{j+1} z^j + \sum_{j=0}^{k-1} (k-j) y^{j+1} z^j, \\ + \sum_{j=1}^{a+1} (yz)^{j-1} \sum_{j=1}^{k+1} y^j z^{j-1} \sum_{j=1}^{b+1} (yz)^{j-1} (a, b \geq 0; 1 \leq i \leq n).$

By substituting $y = 1, z = 1$ into equations (25) and (27), we have the corresponding subtree number indices.

Corollary 5. The subtree number index of the generalized fan graph $F(n)$ is

$$\text{STN}(F(n)) = F(F(n); 1, 1) = \text{STN}(\tilde{F}_{0,0}^{n,1}; c_0) + \text{STN}(\tilde{T}_{0,0}^{n,1}), \quad (30)$$

with

$$\text{STN}(\tilde{F}_{a,b}^{n,i}; c_0) = (b+1) \left(\sum_{t=i}^{n-1} \text{STN}(\bar{F}_{n,t+1}; c_0, P_{c_t, c_n}) \sum_{j=\lfloor t/n-1 \rfloor}^{m_t} \text{STN}(\tilde{F}_{a,m_t-j}^{t,i}; c_0) + (a+1) \text{STN}(\bar{F}_{n,i}; c_0, P_{c_i, c_n}) \right) \\ + \text{STN}(\tilde{F}_{a,m_{n-1}}^{n-1,i}; c_0) (l_n + b + 2), \quad (31)$$

and $\text{STN}(\bar{F}_{n,i}; c_0, P_{c_i, c_n}) = \sum_{t=i}^n (\prod_{j=i}^n (l_j + 1) (l_j + 2) / 2^{n-i} (l_t + 1) (l_t + 2)), \\ \text{STN}(\tilde{F}_{a,b}^{i,i}; c_0) = (a+1)(b+1) + l_i + 1 (a, b \geq 0; 1 \leq i \leq n).$

$$\text{STN}(\tilde{T}_{a,b}^{n,i}) = (b+1)(l_n+1) \left(\text{STN}(\tilde{T}_{a,m_{n-1}}^{n-1,i}) - \text{STN}(\tilde{T}_{a,m_{n-1}-1}^{n-1,i}) \right) \\ + \text{STN}(\tilde{T}_{a,m_{n-1}}^{n-1,i}) + \frac{1}{2} (l_n + b + 1) (l_n + b + 2), \quad (32)$$

and $\text{STN}(\tilde{T}_{a,b}^{i,i}) = (1/2) (a(a+1) + b(b+1) + l_i(l_i+1) + (a+1)(b+1)(l_i+1) (a, b \geq 0; 1 \leq i \leq n).$

Corollary 6. The subtree number index of the k regular fan graph $F_{n,k}$ is

$$\text{STN}(F_{n,k}) = F(F_{n,k}; 1, 1) = \text{STN}(F_{0,0}^{n,1}; c_0) + \text{STN}(T_{0,0}^{n,1}), \quad (33)$$

with

$$\text{STN}(F_{a,b}^{n,i}; c_0) = (b+1) \left(\sum_{t=i}^{n-1} (n-t) \sum_{j=\lfloor t/n-1 \rfloor}^k \text{STN}(F_{a,k-j}^{t,i}; c_0) \frac{(k^2 + 3k + 2)^{n-t}}{2^{n-t}} + (a+1)(n-i+1) \frac{(k^2 + 3k + 2)^{n-i}}{2^{n-i}} \right) \\ + \text{STN}(F_{a,k}^{n-1,i}; c_0) (k + b + 2), \quad (34)$$

and $\text{STN}(\tilde{F}_{a,b}^{i,i}; c_0) = (a+1)(b+1) + k + 1 (a, b \geq 0; 1 \leq i \leq n).$

$$\text{STN}(T_{a,b}^{n,i}) = (b+1)(k+1) \left(\text{STN}(T_{a,k}^{n-1,i}) - \text{STN}(T_{a,k-1}^{n-1,i}) \right) \\ + \text{STN}(T_{a,k}^{n-1,i}) + \frac{1}{2} (k + b + 1) (k + b + 2), \quad (35)$$

and $\text{STN}(T_{a,b}^{i,i}) = (1/2) (a(a+1) + b(b+1) + k(k+1) + (a+1)(b+1)(k+1) (a, b \geq 0; 1 \leq i \leq n).$

By Corollary 6, we can get the number of subtrees in $F_{n,k} (n = 3, 4, \dots, 12; k = 0, 1, \dots, 4)$ as shown in Table 2.

3.3. Subtree Generating Functions of Unicyclic Graphs. In order to solve the subtree generating function of the

TABLE 2: The subtree number index $\text{STN}(F_{n,k})$ of $F_{n,k}$ ($n = 3, \dots, 12; k = 0, \dots, 4$).

n	$\text{STN}(F_{n,0})$	$\text{STN}(F_{n,1})$	$\text{STN}(F_{n,2})$	$\text{STN}(F_{n,3})$	$\text{STN}(F_{n,4})$
3	25	237	942	2590	5775
4	70	1878	14889	68434	229620
5	204	16416	258651	1963014	9793890
6	616	148251	4583355	57021154	421194660
7	1901	1350654	81524559	1659516014	18133098555
8	5932	12331656	1451078262	48312035578	780774946200
9	18605	112645497	25831232655	1406527479494	33619284887595
10	58480	1029092802	459841504026	40949040340498	1447611875610240
11	183982	9401689836	8186017062132	1192173807684958	62332696770629760
12	579027	85893370743	145726106717904	34708469794621098	2683982668276977405

generalized wheel graphs, we first study the subtree generating function of the unicyclic graphs.

Lemma 8 (see [14]). *Let T be a weighted tree with more than 2 vertices, u and v be its two distinct vertices, and P_{uv} be a path of T connecting u and v with length l , vertex set $V(P_{uv}) = \{x_i | i = 1, 2, \dots, l-1\} \cup \{u, v\}$, and $E(P_{uv}) = (u, x_1) \cup \{(x_i, x_{i+1}) | i = 1, 2, \dots, l-2\} \cup (x_{l-1}, v)$; moreover, denote by T_u, T_v, T_{x_i} ($i = 1, 2, \dots, l-1$) the weighted tree of $T \setminus E(P_{uv})$ that contains u, v, x_i , respectively. Then,*

$$F(T; f, g; u, v) = f(u)f(v) \prod_{i=1}^{l-1} f(x_i) \prod_{e \in E(P_{uv})} g(e), \quad (36)$$

where $f(\tilde{v}) = F(T_{\tilde{v}}; f, g; \tilde{v})$ ($\tilde{v} \in \{u, v, x_i \mid i = 1, \dots, l-1\}$).

Theorem 3. *Let $U_n = (V(U_n), E(U_n); f, g)$ be a weighted unicyclic graph with girth n , vertex set $V(U_n) = \{v_1, v_2, \dots, v_n\}$, vertex weight function $f(v) = y$ ($v \in V(U_n)$), and edge weight function $g(e) = z$ ($e \in E(U_n)$), having trees T_i ($i = 1, 2, \dots, n$) attached to vertex v_i on the cycle; then,*

$$F(U_n; f, g) = \sum_{i=1}^{n-1} \sum_{j=1}^n \prod_{k=j}^{j+i} F(T_k; f, g; v_k) \prod_{e \in E(P_{v_j v_{j+i}})} g(e) + \sum_{i=1}^n F(T_i; f, g), \quad (37)$$

where $P_{v_j v_{j+i}} = v_j v_{j+1} v_{j+2} \dots v_{j+i}$ is the path connecting v_j and v_{j+i} ; here we specify that $T_{i+n} = T_i, v_{i+n} = v_i$.

Proof. The theorem holds through dividing the subtrees of U_n into the following three categories:

- (1) Containing no vertex and no edge on the cycle.
- (2) Containing only one vertex and no edge on the cycle.
- (3) Containing only $i+1$ vertices and i edges ($i = 1, \dots, n-1$) on the cycle. $\square$

3.4. Subtree Generating Function of the Generalized Wheel Graphs. Next, we consider the subtree generating function of the generalized wheel graph $W(n)$.

Definition 9. Let $\bar{F}_{p,q}^n(p, q \in \{1, 2, \dots, n\})$ be the generalized fan graph of $W(n)$ ($e_{p-1, m_{p-1}+1}^* \cup e_{q,1}^* \cup \bigcup_{i=q+1}^{p-1} \tilde{e}_{i,1}$) that contains c_0, c_p, c_q , and denote by P_{c_p, c_q}' the path tree of $\bar{F}_{p,q}^n \setminus (\bigcup_{i=p}^q \tilde{e}_{i,1})$ that contains c_p, c_q .

With Lemma 6, the following lemma follows immediately.

Lemma 9. *Keeping above symbols and definitions, we have*

$$F(\bar{F}_{p,q}^n; f, g; c_0, P_{c_p, c_q}') = \sum_{i=p}^x \frac{(yz)^{l_i+1+\sum_{k=p}^{x-1} (m_k+1)}}{y^{x-p-1} \sum_{j=1}^{l_i+1} j y^j z^{j-1}} \prod_{j=p}^x \sum_{k=1}^{l_j+1} k y^k z^{k-1}, \quad (38)$$

with

$$x = \begin{cases} q, & \text{if } q \geq p, \\ n+q, & \text{otherwise.} \end{cases} \quad (39)$$

Theorem 4. *Let $W(n) = (V(W(n)), E(W(n)); f, g)$ be a weighted generalized wheel graph with vertex weight function $f(v) = y$ ($v \in V(W(n))$) and edge weight function $g(e) = z$ ($e \in E(W(n))$) (see Definition 4); then,*

$$\begin{aligned}
F(W(n); f, g) = & \sum_{p=1}^n \sum_{k \equiv [p/n]}^{m_p} \left(\frac{1}{y} \sum_{i=1}^{p-1} \sum_{j=0}^{m_i} (yz)^{k+j} F(\bar{F}_{p+1,i}^n; f, g; c_0, P_{c_{p+1}, c_i}') \times F(\bar{F}_{m_i-j, m_p-k}^{p,i+1}; f, g; c_0) \right) \\
& + F(\bar{F}_{p+1,p}^n; f, g; c_0, P_{c_{p+1}, c_p}') \sum_{i=0}^{m_p-k} (yz)^{k+i} \\
& + \sum_{i=1}^n \sum_{j \equiv [i/n]}^{m_i} (yz)^j \left(\sum_{k=0}^{m_i-j} (yz)^k \right)^{1-[i-1/n]} \left(\left(F(\bar{T}_{0, m_i-j}^{i,1}; f, g) - F(\bar{T}_{m_i, m_i-j}^{i,2}; f, g) - \sum_{k=0}^{l_i-1} (l_i-k) y^{k+1} z^k \right) \right)^{[i-1/n]} \\
& \prod_{j=i-[i-1/n]}^n \sum_{t=1}^{l_j+1} y^t z^{t-1} \prod_{j=i+1}^n y^{m_j} z^{m_j+1} + F(\bar{T}_{0, m_n}^{n,1}; f, g) + F(\bar{F}_{0, m_n}^{n,1}; f, g; c_0),
\end{aligned} \tag{40}$$

with $F(\bar{T}_{a,b}^{n,i}; f, g; c_0)$, $F(\bar{T}_{a,b}^{n,i}; f, g)$, $F(\bar{F}_{p,q}^n; f, g; c_0, P_{c_p, c_q}')$ as in equations (18), (26), and (38), respectively, and $\bar{F}_{i+n,q}^n = \bar{F}_{i,q}^n$ ($1, 2, \dots, n$).

Proof. We first divide the subtrees of generalized wheel graph into two categories:

- (1) Not containing the center vertex c_0 .
- (2) Containing the center vertex c_0 .

Obviously, $W(n) \setminus c_0$ is a unicyclic graph; with Theorem 3, we have the subtree generating function of case (1) as

$$\begin{aligned}
F(W(n) \setminus c_0; f, g) = & \sum_{i=1}^n \sum_{j \equiv [i/n]}^{m_i} (yz)^j \left(\left(F(\bar{T}_{0, m_i-j}^{i,1}; f, g) - F(\bar{T}_{m_i, m_i-j}^{i,2}; f, g) - \sum_{k=0}^{l_i-1} (l_i-k) y^{k+1} z^k \right) \right)^{[i-1/n]} \\
& \left(\sum_{k=0}^{m_i-j} (yz)^k \right)^{1-[i-1/n]} \prod_{j=i+1}^n y^{m_j} z^{m_j+1} \prod_{j=i-[i-1/n]}^n \sum_{t=1}^{l_j+1} y^t z^{t-1} + F(\bar{T}_{0, m_n}^{n,1}; f, g).
\end{aligned} \tag{41}$$

We further partition the subtree of case (2) into four categories.

$$S(W(n); c_0) = S_1 \cup S_2 \cup S_3 \cup S_4, \tag{42}$$

where

- (i) S_1 is the set of subtrees in $S(W(n); c_0)$ that does not contain edge e_{n, m_n+1}^* .
- (ii) S_2 is the set of subtrees in $S(W(n); c_0)$ that contains edge set $\cup_{j=m_n+1-i}^{m_n+1} e_{i,j}^*$, but not e_{n, m_n-i}^* ($i = 0, 1, \dots, m_n - 1$).
- (iii) S_3 is the set of subtrees in $S(W(n); c_0)$ that contains edge set $\cup_{i=n-p+1}^n \cup_{j=1}^{m_i+1} e_{i,j}^* \cup \cup_{j=m_{n-p}+2-q}^{m_{n-p}+1} e_{n-p,j}^*$, but not $e_{n-p, m_{n-p}-q+1}^*$ ($p = 1, 2, \dots, n-2; q = 0, 1, \dots, m_{n-p}$).
- (iv) S_4 is the set of subtrees in $S(W(n); c_0)$ that contains edge set $\cup_{i=2}^n \cup_{j=1}^{m_i+1} e_{i,j}^* \cup \cup_{j=m_i+2-q}^{m_i+1} e_{1,j}^*$, but not e_{1, m_1-q+1}^* ($q = 0, 1, \dots, m_1$).

With equation (18), we have the subtree generating function of case S_1 as

$$\sum_{T_1 \in S_1} w(T_1) = F(\bar{F}_{0, m_n}^{n,1}; f, g; c_0). \tag{43}$$

We further categorize the subtrees in set S_2 into two cases:

$$S_2 = S_{2,1} \cup S_{2,2}. \tag{44}$$

- (v) $S_{2,1}$ is the set of subtrees in S_2 that contains edge set $(\cup_{i=1}^{p-1} \cup_{j=1}^{m_i+1} e_{i,j}^*) \cup \cup_{j=1}^{q-1} e_{p,j}^*$ ($p = 1, 2, \dots, n-1; q = 1, 2, \dots, m_p + 1$), but not $e_{p,q}^*$.
- (vi) $S_{2,2}$ is the set of subtrees in S_2 that contains edge set $\cup_{i=1}^{n-1} \cup_{j=1}^{m_i+1} e_{i,j}^*$.

Then, we have

TABLE 3: The subtree number index $\text{STN}(W_{n,k})$ of the k regular wheel graphs $W_{n,k}$ ($n = 3, \dots, 10; k = 0, 1, \dots, 4$).

n	$\text{STN}(W_{n,0})$	$\text{STN}(W_{n,1})$	$\text{STN}(W_{n,2})$	$\text{STN}(W_{n,3})$	$\text{STN}(W_{n,4})$
3	38	854	6012	25508	80810
4	112	7221	101139	713728	3388080
5	332	64245	1784919	20793884	146828850
6	1007	582668	31779252	607320008	6352788995
7	3110	5314153	566171601	17710247828	274109511640
8	9704	48533965	10083174375	515947702112	11812025781160
9	30431	443386514	179530660440	15024717676460	508755146972555
10	95643	4050782145	3196186084833	437460940097784	21908623666992075

$$\begin{aligned}
\sum_{T_2 \in \mathcal{S}_2} w(T_2) &= \sum_{p=1}^{m_n} \prod_{j=1}^p f(v_{n,m_n-j+1}^*) g(e_{n,m_n-j+2}^*) \left(\sum_{T_{2,1} \in \mathcal{S}_{2,1}} w(T_{2,1}) + \sum_{T_{2,2} \in \mathcal{S}_{2,2}} w(T_{2,2}) \right) \\
&= \sum_{p=1}^{m_n} \frac{(yz)^p}{y} \sum_{j=1}^{n-1} F(\bar{F}_{1,j}^n; f, g; c_0, P_{c_1, c_j}') \sum_{i=0}^{m_j} F(\bar{F}_{m_j-i, m_n-p}^{n, j+1}; f, g; c_0) \prod_{s=1}^i f(v_{j,s}^*) g(e_{j,s}^*) \\
&\quad + F(\bar{F}_{1, n}^n; f, g; c_0, P_{c_1, c_n}') \times F(P_{m_n+1-p}''; f, g; c_n) \\
&= \sum_{p=1}^{m_n} y^{p-1} z^p \left(\sum_{j=1}^{n-1} F(\bar{F}_{1,j}^n; f, g; c_0, P_{c_1, c_j}') \sum_{i=0}^{m_j} (yz)^i F(\bar{F}_{m_j-i, m_n-p}^{n, j+1}; f, g; c_0) + F(\bar{F}_{1, n}^n; f, g; c_0, P_{c_1, c_n}') \sum_{q=1}^{m_n+1-p} y^q z^{q-1} \right).
\end{aligned} \tag{45}$$

Similar to the analysis of case $\mathcal{S}_2$, we have

$$\begin{aligned}
\sum_{T_3 \in \mathcal{S}_3} w(T_3) &= \frac{1}{y} \sum_{p=2}^{n-1} \sum_{k=0}^{m_p} \prod_{j=1}^k f(v_{p, m_p-j+1}^*) g(e_{p, m_p-j+2}^*) \left(\sum_{i=1}^{p-1} F(\bar{F}_{p+1,i}^n; f, g; c_0, P_{c_{p+1}, c_i}') \sum_{j=0}^{m_i} F(\bar{F}_{m_i-j, m_p-k}^{p, j+1}; f, g; c_0) \prod_{s=1}^j f(v_{i,s}^*) g(e_{i,s}^*) \right) \\
&\quad + F(\bar{F}_{p+1, p}^n; f, g; c_0, P_{c_{p+1}, c_p}') F(P_{m_p+1-k}''; f, g; c_p) \\
&= \sum_{p=2}^{n-1} \sum_{k=0}^{m_p} y^{k-1} z^k \left(\sum_{i=1}^{p-1} F(\bar{F}_{p+1,i}^n; f, g; c_0, P_{c_{p+1}, c_i}') \sum_{j=0}^{m_i} F(\bar{F}_{m_i-j, m_p-k}^{p, j+1}; f, g; c_0) (yz)^j + F(\bar{F}_{p+1, p}^n; f, g; c_0, P_{c_{p+1}, c_p}') \sum_{q=1}^{m_p+1-k} y^q z^{q-1} \right),
\end{aligned} \tag{46}$$

$$\begin{aligned}
\sum_{T_4 \in \mathcal{S}_4} w(T_4) &= \frac{1}{y} F(\bar{F}_{2,1}^n; f, g; c_0, P_{c_2, c_1}') \sum_{p=0}^{m_1} \prod_{i=1}^p f(v_{1, m_1-i+1}^*) g(v_{1, m_1-i+2}^*) F(P_{m_1+1-p}''; f, g; c_1) \\
&= F(\bar{F}_{2,1}^n; f, g; c_0, P_{c_2, c_1}') \sum_{p=0}^{m_1} y^{p-1} z^p \times \sum_{j=1}^{m_1+1-p} y^j z^{j-1}.
\end{aligned} \tag{47}$$

Equation (40) follows from equations (41) to (47). $\square$

With Theorem 4, we can also obtain the subtree generating functions of some special generalized wheel graphs.

Definition 10. Let $W(l_1, \dots, l_n; m_1, \dots, m_n)$ be a generalized wheel graph and k be a positive integer; the generalized

wheel graph is called k regular wheel graph, denoted by $W_{n,k}$ ($n \geq 3; k \geq 0$), if $l_i = m_i = k$ ($i = 1, \dots, n$). Clearly, $W_{n,k}$ has $n(2k+1)+1$ vertices and $2n(k+1)$ edges.

Corollary 7. Let $W_{n,k} = (V(W_{n,k}), E(W_{n,k}); f, g)$ be a weighted k regular wheel graph, with $f(v) = y$ ($v \in V(W_{n,k})$) and $g(e) = z$ ($e \in E(W_{n,k})$); then,

FIGURE 2: Subtree densities of $B_{n,k}$ ($n = 2, 3, \dots, 101; k = 1, 2, \dots, 100$).TABLE 4: Related data for $B_{n,k}$ ($n = 2, 3, \dots, 11; k = 1, 2, 3$).

n	$k = 1$		$k = 2$		$k = 3$	
	$P_y(B_{n,1})$	$D(B_{n,1})$	$P_y(B_{n,2})$	$D(B_{n,2})$	$P_y(B_{n,3})$	$D(B_{n,3})$
$n = 2$	160	0.695652	948	0.692982	3434	0.699959
$n = 3$	640	0.740741	7738	0.745472	47624	0.749937
$n = 4$	2524	0.760699	63416	0.757007	667458	0.753912
$n = 5$	9752	0.765704	507894	0.753722	9081980	0.746593
$n = 6$	36880	0.763656	3954124	0.746654	119376902	0.738012
$n = 7$	136704	0.758792	29977970	0.739331	1521704016	0.730391
$n = 8$	497716	0.753086	222103008	0.732726	18907602266	0.724007
$n = 9$	1783768	0.747392	1613730478	0.727007	230033554532	0.718704
$n = 10$	6305512	0.742052	11533135796	0.722104	2750146800750	0.714269
$n = 11$	22023584	0.737177	81280358058	0.717896	32400637534328	0.71052

$P_y(B_{n,k})$ stands for $(\partial F(B_{n,k}; y, 1)/\partial y)|_{y=1}$ ($n = 2, 3, \dots, 11; k = 1, 2, 3$).

$$\begin{aligned}
 F(W_{n,k}; f, g) &= F(F_{0,k}^{n,1}; f, g; c_0) + (y^k z^{k+1})^{n-1} \sum_{i=0}^k \sum_{j=0}^{k-i} (yz)^{i+j} \left(\sum_{j=1}^{k+1} y^j z^{j-1} \right)^n \\
 &+ \sum_{p=1}^n \sum_{t=\lfloor p/n \rfloor}^k \left(\sum_{i=1}^{p-1} \frac{(n-p+i)}{y^{n-p+i-1}} \sum_{j=0}^k F(F_{k-j,k-t}^{p,i+1}; f, g; c_0) (yz)^{(k+1)(n-p+i)+t+j} \times \left(\sum_{s=1}^{k+1} s y^s z^{s-1} \right)^{n-p+i-1} \right. \\
 &\quad \left. + \frac{n(yz)^{n(k+1)}}{y^{n-2}} \left(\sum_{s=1}^{k+1} s y^s z^{s-1} \right)^{n-1} \sum_{i=0}^{k-t} (yz)^{t+i} \right) \\
 &+ \sum_{i=2}^n \sum_{j=\lfloor i/n \rfloor}^k (yz)^j \left(F(T_{0,k-j}^{i,1}; f, g) - F(T_{k,k-j}^{i,2}; f, g) - \sum_{t=0}^{k-1} (k-t) y^{t+1} z^t \right) \times \left(y^k z^{k+1} \sum_{t=1}^{k+1} y^t z^{t-1} \right)^{n-i} \\
 &+ F(T_{0,k}^{n,1}; f, g),
 \end{aligned} \tag{48}$$

with $F(F_{a,b}^{n,i}; f, g; c_0)$ and $F(T_{a,b}^{n,i}; f, g)$ as in equations (28) and (29), respectively.

Substituting $y = 1, z = 1$ into equations (40) and (48), we have the following corollaries.

Corollary 8. *The subtree number index of generalized wheel graph $W(n)$ is*

$$\begin{aligned} \text{STN}(W(n)) &= \sum_{p=1}^n \sum_{k \neq [p/n]}^{m_p} \left(\sum_{i=1}^{p-1} \sum_{j=0}^{m_i} \text{STN}(\bar{F}_{p+1,i}^n; c_0, P_{c_{p+1}, c_i}) \text{STN}(\bar{F}_{m_i-j, m_p-k}^{p,i+1}; c_0) + \text{STN}(\bar{F}_{p+1,p}^n; c_0, P_{c_{p+1}, c_p}) (m_p - k + 1) \right) \\ &\quad + \text{STN}(\tilde{T}_{0, m_n}^{n,1}) + \text{STN}(\bar{F}_{0, m_n}^{n,1}; c_0) + \sum_{i=1}^n \sum_{j \neq [i/n]}^{m_i} \left(\text{STN}(\tilde{T}_{0, m_i-j}^{i,1}) - \text{STN}(\tilde{T}_{m_i, m_i-j}^{i,2}) - \frac{l_1}{2} (l_1 + 1) \right)^{[i-1/n]} \\ &\quad \times (m_1 - j + 1)^{1-[i-1/n]} \prod_{j=i+[i-1/n]}^n (l_j + 1), \end{aligned} \quad (49)$$

with $\text{STN}(\bar{F}_{a,b}^{n,i}; c_0)$, $\text{STN}(\tilde{T}_{a,b}^{n,i})$ as in equations (31) and (32), respectively, and $\text{STN}(\bar{F}_{p+1,i}^n; c_0, P_{c_{p+1}, c_i}) = F(\bar{F}_{p+1,i}^n; 1, 1; c_0, P_{c_{p+1}, c_i})$ by substituting $y = z = 1$ into equation (38).

Corollary 9. *The subtree number index of the k regular wheel graph $W_{n,k}$ is*

$$\begin{aligned} \text{STN}(W_{n,k}; f, g) &= \text{STN}(T_{0,k}^{n,1}) + \sum_{p=1}^n \sum_{t \neq [p/n]}^k \left(\sum_{i=1}^{p-1} (n-p+i) \sum_{j=0}^k \text{STN}(F_{k-j, k-t}^{p,i+1}; c_0) \frac{(k^2 + 3k + 2)^{n-p+i-1}}{2^{n-p+i-1}} + \frac{n}{2^{n-1}} (k+1-t)(k^2 + 3k + 2)^{n-1} \right) \\ &\quad + \sum_{i=2}^n \sum_{j \neq [i/n]}^k \left(\text{STN}(T_{0, k-j}^{i,1}) - \text{STN}(T_{k, k-j}^{i,2}) - \frac{k}{2} (k+1) \right) (k+1)^{n-i} + \text{STN}(F_{0,k}^{n,1}; c_0) + \frac{(k+2)}{2} (k+1)^{n+1}, \end{aligned} \quad (50)$$

with $\text{STN}(\bar{F}_{a,b}^{n,i}; c_0)$ and $\text{STN}(\tilde{T}_{a,b}^{n,i})$ as in equations (34) and (35), respectively.

With Corollary 9, we can get the subtree number index of the k regular wheel graph $W_{n,k}$ ($n = 3, \dots, 10; k = 0, \dots, 4$) as shown in Table 3.

4. Subtree Density for Graphs

The subtree generating function can be used to reveal more structure information of the graph than just subtree numbers. Next, we study the asymptotic characteristics of the subtree density of three types of graphs, namely, k regular book graph $B_{n,k}$, k regular fan graph $F_{n,k}$, and k regular wheel graph $W_{n,k}$. The formal definition of the subtree density is given below.

Definition 11 (see [12]). Assume G is a graph with n vertices and k subtrees of order $n_1, n_2, \dots, n_k$. Then, $u(G) = (1/k) \sum_{i=1}^k n_i$ denotes the average order of subtrees of G , and the subtree density of G is defined as $D(G) = (u(G)/n)$.

By Definition 11, the subtree density of G^* is simply

$$D(G^*) = \frac{(\partial F(G^*; y, 1)/\partial y)|_{y=1}}{F(G^*; 1, 1) \times n(G^*)}, \quad (51)$$

where $F(G^*; y, 1)$ represents the vertex generating function of G^* , $F(G^*; 1, 1)$ represents the number of subtrees of G^* , and $n(G^*)$ represents the number of vertices of G^* .

4.1. Subtree Density of the k Regular Book Graph. According to definition of k regular book graph, it is easy to know that the number of vertices in $B_{n,k}$ is

$$n(B_{n,2}) = nk + 2. \quad (52)$$

With Corollary 1, we obtain the vertex generating function of $B_{n,k}$:

$$\begin{aligned} F(B_{n,k}; f, 1) &= (n+1) \sum_{i=0}^{k-1} (k-i) y^{i+1} \\ &\quad + \frac{2}{y^n} \left(\sum_{i=1}^{k+1} y^i \right)^{n+1} + \frac{n+1}{(y^{k+2})^{n-1}} \left(\sum_{i=1}^{k+1} i y^{i+k+1} \right)^n. \end{aligned} \quad (53)$$

TABLE 5: Related data for $F_{n,k}$ ($n = 3, 4, \dots, 15; k = 0, 1, 2$).

n	$k = 0$		$k = 1$		$k = 2$	
	$P_y(F_{n,0})$	$D(F_{n,0})$	$P_y(F_{n,1})$	$D(F_{n,1})$	$P_y(F_{n,2})$	$D(F_{n,2})$
$n = 3$	70	0.7	1469	0.688701	9230	0.699879
$n = 4$	258	0.737143	16804	0.745651	213112	0.753336
$n = 5$	937	0.765523	188646	0.766106	4770342	0.768465
$n = 6$	3375	0.782699	2064131	0.773512	102963491	0.774643
$n = 7$	12041	0.791754	22039684	0.777037	2157456112	0.778349
$n = 8$	42510	0.796246	230629962	0.779261	44198270871	0.780998
$n = 9$	148559	0.798490	2375047265	0.780899	889968667136	0.783027
$n = 10$	514425	0.799691	24148356952	0.782189	17679691108012	0.784641
$n = 11$	1767141	0.800414	243005062626	0.783241	347426791890924	0.785954
$n = 12$	6028731	0.80091	2424615958415	0.784117	6766882927525253	0.787045

$P_y(F_{n,k})$ stands for $(\partial F(F_{n,k}; y,)/\partial y)|_{y=1}$ ($n = 3, 4, \dots, 12; k = 0, 1, 2$).

(a)

(b)

FIGURE 3: Subtree densities of $F_{n,k}$ ($n = 3, 4, \dots, 15; k = 0, 1, \dots, 15$). (a) Subtree densities of $F_{n,k}$ ($n = 3, 4, \dots, 15; k = 0, 1, 2$). (b) Subtree densities of $F_{n,k}$ ($n = 3, 4, \dots, 15; k = 0, 1, \dots, 15$).

Together with equations (16) and (51), we can obtain the subtree densities of $B_{n,k}$ as plotted in Figure 2, and the related data are listed in Table 4.

Observing Table 4 and Figure 2, we see that the $B_{2,100}$ has the maximal subtree density among $B_{n,k}$ ($n = 2, 3, \dots, 101; k = 1, 2, \dots, 100$). On the whole, it appears that the subtree density of $B_{n,k}$ ($n = 2, 3, \dots, 101; k = 1, 2, \dots, 100$) decreases with the increase of n and k .

4.2. Subtree Density of the k Regular Fan Graph. Similarly, with Corollary 4, we can obtain the vertex generation function of $F_{n,k}$, and combining the subtree density definition (see equation (51)), we can obtain the subtree densities of $F_{n,k}$ as shown in Table 5 and Figures 3(a) and 3(b).

From Figure 3(a), when $k \in \{0, 1, 2\}$ is fixed, we can observe that the subtree density of $F_{n,k}$ ($3 \leq n \leq 15$) gradually

TABLE 6: Related data for $W_{n,k}$ ($n = 3, 4, \dots, 10; k = 0, 1, 2$).

n	$k = 0$		$k = 1$		$k = 2$	
	$P_y(W_{n,0})$	$D(W_{n,0})$	$P_y(W_{n,1})$	$D(W_{n,1})$	$P_y(W_{n,2})$	$D(W_{n,2})$
$n = 3$	116	0.763158	6518	0.763232	74760	0.777196
$n = 4$	444	0.792857	74597	0.794659	1701059	0.800906
$n = 5$	1617	0.811747	825165	0.802752	37355674	0.804942
$n = 6$	5789	0.821251	8896844	0.803639	792991830	0.80494
$n = 7$	20519	0.824719	93866421	0.802885	16393751516	0.804318
$n = 8$	72064	0.825135	973031117	0.801938	332260056199	0.803706
$n = 9$	250841	0.824294	9945496874	0.801099	6633082879524	0.803191
$n = 10$	865923	0.823064	100509266345	0.800397	130855615202033	0.802768

$P_y(W_{n,k})$ stands for $(\partial F(W_{n,k}; y, 1)/\partial y)|_{y=1}$ ($n = 3, 4, \dots, 10; k = 0, 1, 2$).

FIGURE 4: Subtree densities of $W_{n,k}$ ($n = 3, 4, \dots, 10; k = 0, 1, \dots, 12$). (a) Subtree densities of $W_{n,k}$ ($n = 3, 4, \dots, 10; k = 0, 1, \dots, 12$). (b) Subtree densities of $W_{n,k}$ ($n = 3, 4, 5; k = 0, 1, \dots, 12$).

increases with the increase of n . As illustrated in Figure 3(b), when $n \in \{3, 4, 5\}$ is fixed, the subtree density of $F_{n,k}$ ($0 \leq k \leq 15$) also increases with the increase of k .

4.3. Subtree Density of the k Regular Wheel Graph. With Corollary 7 and equations (50) and (51), we can also obtain the subtree density of $W_{n,k}$ as shown in Table 6 and Figure 4; here, we skip the repetitive analysis details.

From Figure 4(a), when $k \in \{0, 1, 2\}$ is fixed, we notice a very interesting observation: the subtree density of

$W_{n,k}$ ($3 \leq n \leq 10$) gradually decreases with the increase of n . This trend needs to be further verified through combinatoric approaches. When $n \in \{3, 4, 5\}$ is fixed, by observing Figure 4(b), it appears that the subtree density of $W_{n,k}$ ($0 \leq k \leq 12$) increases with the increase of k .

5. Conclusion

As an important topological index of graphs and networks, the subtree number index is closely related to the analysis of the hybrid local reliable network and discovery of new

properties of compounds. We present the subtree generating functions and subtree number indices of the generalized book graph $B(n)$ ($n \geq 2$), generalized fan graph $F(n)$ ($n \geq 3$), and generalized wheel graph $W(n)$ ($n \geq 3$). Furthermore, we also briefly discuss the subtree density asymptotic characteristics of the k regular cases of these three graphs: $B_{n,k}$, $F_{n,k}$, and $W_{n,k}$. Our study provides some basis for exploring hidden properties of complex cyclic graphs and chemical compounds from a new metric.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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