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Research Article

Gob's Circles of a Triangle in the Isotropic Plane

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In this paper, we consider Gob's circles of an allowable triangle in the isotropic plane and derive their equations in the case of a standard triangle. We prove that the potential axes of the circumscribed circle and Gob's circles are the angle bisectors of the given triangle. We investigate some other interesting properties of Gob's circles and some other circles related to the given triangle.

1. Introduction

Let ABC be a given triangle in Euclidean plane. Denote by A_b the point on the half-line BA and by A_c the point on the half-line CA such that $|BA_b| = |CA_c| = |BC|$. In a similar way, construct the points B_c on the half-line CB and B_a on the half-line AB such that $|CB_c| = |AB_a| = |CA|$, and C_a on the half-line AC and C_b on the half-line BC such that $|AC_a| = |BC_b| = |AB|$. The three circles AA_bA_c, BB_cB_a, and CC_aC_b are the so-called Gob's circles of the triangle ABC [1, 2]. The radii of these circles are equal to $|OI|$, where O is the center of the circumscribed circle and I is the center of the inscribed circle of the triangle ABC. The tangents at points A, B, and C to these three circles meet at one point on the circumscribed circle of the triangle ABC, and lines A_bA_c, B_cB_a, and C_aC_b are perpendicular to the line OI. These and some other properties of Gob's circles in Euclidean plane have their analogues in the isotropic plane, and that is what we will study here.

The isotropic plane is the projective-metric plane, where the absolute consists of a line, the absolute line ω, and a point on that line, the absolute point Ω. Lines through the point Ω are isotropic lines, and points on the line ω are isotropic points.

The *distance* between two points $P_i = (x_i, y_i)\,(i = 1, 2)$ in the isotropic plane is defined by $d(P_1, P_2) = x_2 - x_1$, and if $x_1 = x_2$, we say that P_1 and P_2 are *parallel*. For two parallel points P_1 and P_2, we define their *span* by $s(P_1, P_2) = y_2 - y_1$. The *angle* between two lines $y = k_i x + l_i\,(i = 1, 2)$ is $k_2 - k_1$, and if $k_1 = k_2$, we say that they are *parallel*. Any isotropic line is perpendicular to any non-isotropic line. Facts about the isotropic plane can be found in [3, 4].

We say that a triangle is *allowable* if none of its sides is isotropic. If we choose the coordinate system in such a way, the circumscribed circle of an allowable triangle ABC has the equation $y = x^2$, and therefore, its vertices are the points $A = (a, a^2)$, $B = (b, b^2)$, and $C = (c, c^2)$, while $a + b + c = 0$, and we say that the triangle ABC is in *standard position* or shorter triangle ABC is a standard triangle. Its sides BC, CA, and AB have equations $y = -ax - bc$, $y = -bx - ca$, and $y = -cx - ab$. In order to prove geometric facts for any allowable triangle, it suffices to prove them for a standard triangle [5].

Denoting $q = bc + ca + ab$, some equalities have been proved in [5], e.g., $a^2 - bc = b^2 - ca = c^2 - ab = -q$ and $a^2 + ab + b^2 = b^2 + bc + c^2 = c^2 + ca + a^2 = -q$.

2. Main Results

Let B_a, C_a; C_b, A_b; and A_c, B_c be pairs of points on the lines BC, CA, and AB, respectively, such that

$$\begin{aligned} d(A_c, B) &= d(C, A_b) = d(B, C),\\ d(B_a, C) &= d(A, B_c) = d(C, A),\\ d(C_b, A) &= d(B, C_a) = d(A, B). \end{aligned} \tag{1}$$

The circles $\mathscr{G}_a$, $\mathscr{G}_b$, and $\mathscr{G}_c$ through three points A, A_b, A_c; B, B_c, B_a; and C, C_a, C_b, respectively, will be called, by analogy with the Euclidean case, Gob's circles of the allowable triangle ABC (Figure 1).

Theorem 1. *Gob's circles $\mathscr{G}_a$, $\mathscr{G}_b$, and $\mathscr{G}_c$ of a standard triangle ABC have equations*

$$\begin{aligned}\mathscr{G}_a \ldots 3y &= x^2 + ax + a^2,\\ \mathscr{G}_b \ldots 3y &= x^2 + bx + b^2,\\ \mathscr{G}_c \ldots 3y &= x^2 + cx + c^2.\end{aligned} \tag{2}$$

Proof. Let A_b and A_c be points on the lines CA and AB with abscissae $2c - b$ and $2b - c$, respectively. Then,

$$\begin{aligned}d(A_c, B) &= b - (2b - c) = c - b = d(B, C),\\ d(C, A_b) &= 2c - b - c = c - b = d(B, C).\end{aligned} \tag{3}$$

From equation $y = -bx - ca$ of the line CA, the ordinate of the point A_b turns out to be

$$y = -b(2c - b) - ca = -q - 2bc, \tag{4}$$

and similarly, the point A_c has the same ordinate $-q - 2bc$; i.e., we have

$$\begin{aligned}A_b &= (2c - b, -q - 2bc),\\ A_c &= (2b - c, -q - 2bc).\end{aligned} \tag{5}$$

The point $A = (a, a^2)$ lies on the circle $\mathscr{G}_a$ and, by (2), the points A_b and A_c from (5) obviously lie on this circle because, for the former one, we get

$$\begin{aligned}&(2c - b)^2 - (b + c)(2c - b) + (b + c)^2\\ &\quad = 3\left(b^2 - bc + c^2\right) = 3(-q - 2bc).\end{aligned} \tag{6}$$

In the isotropic plane, the circle with the equation $y = ux^2 + vx + w$ has radius equal to $1/2u$. Therefore, the Gob's circles, equation (2), have equal radii. □

Theorem 2. *The potential axis of the circumscribed circle $\mathscr{K}$ of an allowable triangle ABC, and Gob's circle $\mathscr{G}_a$ is the angle bisector of the angle A. The other intersection of these two circles is the intersection of the same bisector with the perpendicular bisector of the side BC. Analogous statements hold for $\mathscr{K}$ and circles $\mathscr{G}_b$ and $\mathscr{G}_c$ (Figure 2).*

Proof. From the equation $y = x^2$ of the circumscribed circle $\mathscr{K}$ and the equation (2) of the circle $\mathscr{G}_a$, eliminating x^2, we get $2y = ax + a^2$, which is, by [6], the equation of the angle bisector of the angle A. The point $(-(a/2), (a^2/4))$ lies on this bisector, and it also lies on the bisector $x = -(a/2)$ of the side BC. It also lies on the circles $\mathscr{K}$ and $\mathscr{G}_a$. □

Theorem 3. *The tangents of Gob's circles of an allowable triangle at its vertices pass through the dual Feuerbach point of that triangle (Figure 2).*

Proof. The line $y = ax$ passes through the point $\Phi' = (0, 0)$ which is, by [7], the dual Feuerbach point of the triangle ABC. Inserting $y = ax$ into the first equation (2), we get the equation $x^2 - 2ax + a^2 = 0$ with the double solution $x = a$, and this line touches the circle $\mathscr{G}_a$ at the point A. □

Theorem 4. *The intersections of two Gob's circles of an allowable triangle are the intersections of its orthic line with the corresponding altitudes (Figure 1).*

Proof. From the second equation (2), for $x = a$,

$$3y = a^2 + ab + b^2 = -q, \tag{7}$$

i.e., $y = -(q/3)$, and the same result follows from the third equation (2). So, the circles $\mathscr{G}_b$ and $\mathscr{G}_c$ intersect at the point $(a, -(q/3))$, which lies on lines $x = a$ and $y = -(q/3)$. The former one is the altitude of the triangle ABC through the vertex A, and the latter one is, by [5], the orthic line $\mathscr{H}$ of that triangle.

There are some more interesting circles of a triangle which are related to Gob's circles.

From the equation $y = -ax - bc$ of the line BC, for $x = a$, we obtain $y = -a^2 - bc = q - 2bc$, so the foot of the altitude from the vertex A is the first one of three points:

$$\begin{aligned}A_h &= (a, q - 2bc),\\ B_h &= (b, q - 2ca),\\ C_h &= (c, q - 2ab),\end{aligned} \tag{8}$$

and the other two are feet of altitudes from vertices B and C. □

Theorem 5. *If A_h, B_h, and C_h are feet of altitudes from vertices A, B, and C of a standard triangle ABC, then the circles $\mathscr{H}_a$, $\mathscr{H}_b$, and $\mathscr{H}_c$ through three points A, B_h, C_h; B, C_h, A_h; and C, A_h, B_h, respectively, (Figure 3) have equations*

$$\begin{aligned}\mathscr{H}_a \ldots y &= -x^2 + ax + a^2,\\ \mathscr{H}_b \ldots y &= -x^2 + bx + b^2,\\ \mathscr{H}_c \ldots y &= -x^2 + cx + c^2.\end{aligned} \tag{9}$$

Proof. The point $A = (a, a^2)$ obviously lies on the circle $\mathscr{H}_a$ from (9), and so do also points B_h and C_h from (8) because, e.g., for B_h, we get

$$-b^2 + ab + a^2 = -b^2 - ca = q - 2ca. \tag{10}$$

□

Theorem 6. *Using the notation from Theorems 1 and 5, the circumscribed circle of the triangle ABC belongs to three pencils of circles determined by the circle pairs $\mathscr{G}_a$, $\mathscr{H}_a$; $\mathscr{G}_b$,*

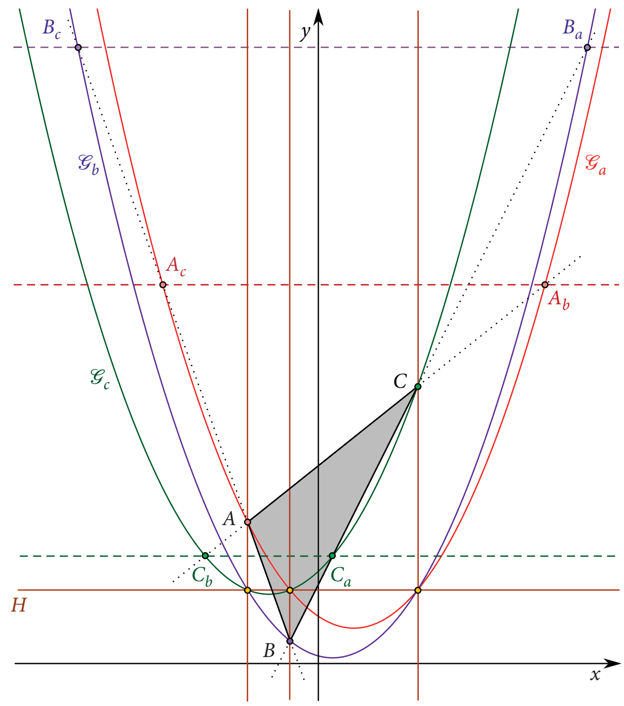

FIGURE 1: Gob's circles $\mathscr{G}_a$, $\mathscr{G}_b$, and $\mathscr{G}_c$ of an allowable triangle ABC. Visualization of the statement of Theorem 4.

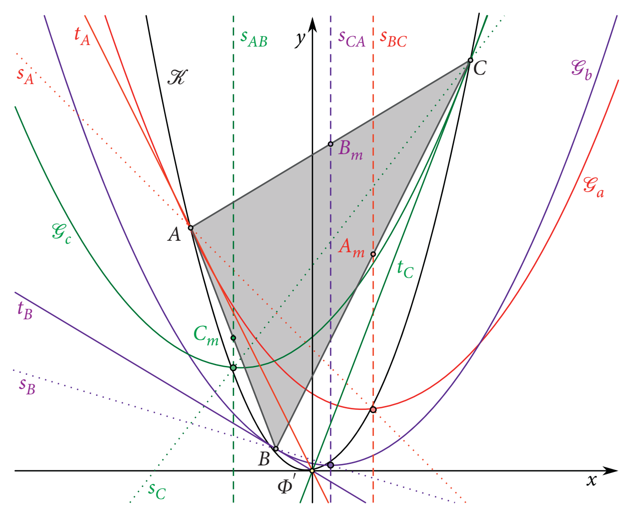

FIGURE 2: Visualization of statements of theorems 2 and 3.

$\mathscr{H}_b$; and $\mathscr{G}_c$, $\mathscr{H}_c$, whose potential axes are the angle bisectors of angles A, B, and C (Figure 3).

Proof. By Theorem 2, it is enough to prove, e.g., that the angle bisector of angle A having the equation $2y = ax + a^2$ is the potential axis of the circle $\mathscr{H}_a$ from (9) and the circumscribed circle $y = x^2$, and this is obtained by adding the equations of these two circles.

The circles $\mathscr{G}_a$ and $\mathscr{H}_a$ pass through the point A and through points A_b, A_c and B_h, C_h, respectively; hence, it is interesting to consider circles through the point A and through points B_h, A_c and C_h, A_b. Let us denote these circles by $\mathscr{I}_{bc}$ and $\mathscr{I}_{cb}$. There are four more analogous circles $\mathscr{I}_{ca}$, $\mathscr{I}_{ac}$, $\mathscr{I}_{ab}$, and $\mathscr{I}_{ba}$ (Figure 3). □

Theorem 7. *In a standard triangle ABC, the circles $\mathscr{I}_{bc}$, $\mathscr{I}_{cb}$, $\mathscr{I}_{ca}$, $\mathscr{I}_{ac}$, $\mathscr{I}_{ab}$, and $\mathscr{I}_{ba}$ have the following equations:*

$$\begin{aligned}
&\mathscr{I}_{bc} \dots y = x^2 - (b - c)x + a(b - c),\\
&\mathscr{I}_{cb} \dots y = x^2 + (b - c)x - a(b - c),\\
&\mathscr{I}_{ca} \dots y = x^2 - (c - a)x + b(c - a),\\
&\mathscr{I}_{ac} \dots y = x^2 + (c - a)x - b(c - a),\\
&\mathscr{I}_{ab} \dots y = x^2 - (a - b)x + c(a - b),\\
&\mathscr{I}_{ba} \dots y = x^2 + (a - b)x - c(a - b).
\end{aligned} \tag{11}$$

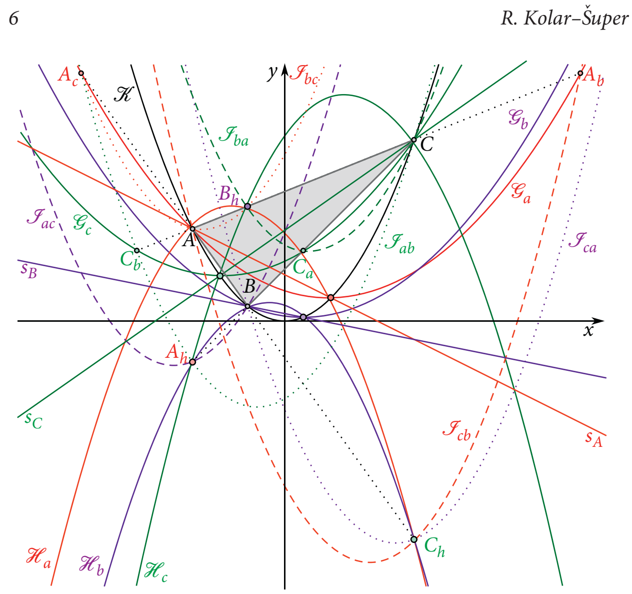

FIGURE 3: Visualization of statements of Theorems 5–7

Proof. The circle $\mathcal{J}_{bc}$ with the first equation (11) obviously passes through the point $A = (a, a^2)$, and it also passes through the points B_h from (8) and A_c from (5) (Figure 3) because we get

$$\begin{aligned} b^2 - (b-c)b + a(b-c) &= bc - ca + ab = q - 2ca, \\ (2b-c)^2 - (b-c)(2b-c) + a(b-c) &= 2b^2 - bc - \left(b^2 - c^2\right) = b^2 - bc + c^2 = -q - 2bc. \end{aligned} \tag{12}$$

The equation of the circle $\mathcal{J}_{cb}$ follows from the equation of $\mathcal{J}_{bc}$ by substituting $b \leftrightarrow c$, and the remaining two equations (11) are obtained by cyclically permuting $a \longrightarrow b \longrightarrow c \longrightarrow a$. □

Theorem 8. *Using the notation from Theorem 7, the midpoint of the line segment joining the intersections of any isotropic line with two circles $\mathcal{J}_{bc}$ and $\mathcal{J}_{cb}$ lies on the circle circumscribed to the triangle ABC; i.e., these two circles are inverse one another with respect to the circumscribed circle. The same holds true for the other two circle pairs: $(\mathcal{J}_{ca}, \mathcal{J}_{ac})$ and $(\mathcal{J}_{ab}, \mathcal{J}_{ba})$.*

Proof. The equation $y = x^2$ of the circumscribed circle is the arithmetic mean of the first two equations (11). According to [8], the second statement of the theorem follows. □

Theorem 9. *Using the notation from Theorem 7, the centroid of three points at which an isotropic line intersecting three circles $\mathcal{J}_{bc}$, $\mathcal{J}_{ca}$, and $\mathcal{J}_{ab}$ lies on the circle circumscribed to the triangle ABC. The same is also true for circles $\mathcal{J}_{cb}$, $\mathcal{J}_{ba}$, and $\mathcal{J}_{ac}$.*

Proof. The equation $y = x^2$ is the arithmetic mean of the first, third, and fifth equations in (11). □

3. Conclusion

In this work, Gob's circles of a triangle in the isotropic plane are introduced, and their connections with some other well-known concepts related to the triangle in the isotropic plane are studied. Some of the most interesting statements are the following: the potential axes of the circumscribed circle and Gob's circles are the angle bisectors of the given triangle, the tangents of Gob's circles of a triangle at its vertices pass through the dual Feuerbach point of the triangle, and the intersections of two Gob's circles of a triangle are the intersections of its orthic line with the corresponding altitudes. The approach used in this paper is introduced and developed in [5].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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