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Research Article

Total Roman {2}-Reinforcement of Graphs

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A total Roman {2}-dominating function (TR2DF) on a graph $\Gamma = (V, E)$ is a function $l: V \longrightarrow \{0, 1, 2\}$, satisfying the conditions that (i) for every vertex $y \in V$ with $l(y) = 0$, either y is adjacent to a vertex labeled 2 under l, or y is adjacent to at least two vertices labeled 1; (ii) the subgraph induced by the set of vertices with positive weight has no isolated vertex. The weight of a TR2DF l is the value $\sum_{y\in V} l(y)$. The total Roman {2}-domination number (TR2D-number) of a graph Γ is the minimum weight of a TR2DF on Γ. The total Roman {2}-reinforcement number (TR2R-number) of a graph is the minimum number of edges that have to be added to the graph in order to decrease the TR2D-number. In this manuscript, we study the properties of TR2R-number and we present some sharp upper bounds. In particular, we determine the exact value of TR2R-numbers of some classes of graphs.

1. Introduction

A total dominating set (TDS) D in an isolated-free graph Γ is a set of vertices of Γ such that each vertex in $V(\Gamma)$ is adjacent to a vertex of D. The total domination number $\gamma_t(\Gamma)$ of Γ is the minimum cardinality of a TDS. Sridharan et al. [1] introduced the total reinforcement number $r_t(\Gamma)$ of a graph Γ as the minimum number of edges that have to be added to the graph in order to decrease the total domination number.. Since the domination number of any isolated-free graph Γ is greater than or equal to 2, by convention, Sridharan, Elias, and Subramanian defined $r_t(\Gamma) = 0$ if $\gamma_t(\Gamma) = 2$.

A function $l: V(\Gamma) \longrightarrow \{0, 1, 2\}$ is a R2DF of Γ if each vertex $y \in V(\Gamma)$ labeled 0 under l satisfies $\sum_{z\in N(y)} l(z) \geq 2$, where $N(y) = \{z \in V(\Gamma): yz \in E(\Gamma)\}$. The minimum value $\sum_{z\in V(\Gamma)} l(z)$ of a R2DF l on Γ is called the R2D-number of Γ. The R2DF was introduced in [2] and has been studied by several authors [3–13]. For more details on R2D-number, we refer the reader to the recent book chapter [14] and survey paper [15].

As a new variant of Roman {2}-domination, total Roman {2}-domination was investigated in [16, 17], where it was called total Roman {2}-domination. A TR2DF on a graph Γ is defined as a function $l: V(\Gamma) \longrightarrow \{0, 1, 2\}$, satisfying the conditions: (i) for every vertex $y \in V(\Gamma)$ with $l(y) = 0$, $l(N(y)) \geq 2$, that is, either there is a vertex $w \in N(y)$ with $l(w) = 2$, or at least two vertices $z_1, z_2 \in N(y)$ with $l(z_1) = l(z_2) = 1$; (ii) the subgraph induced by the set of vertices with positive weight under l has no isolated vertex. The weight of a TR2DF l is the value $\omega(l) = l(V(\Gamma))$. The TR2D-number of a graph Γ, denoted by $\gamma_{t\{R2\}}(\Gamma)$, is the minimum weight of a TR2DF on Γ. A TR2DF on Γ with weight $\gamma_{t\{R2\}}(\Gamma)$ is called a $\gamma_{t\{R2\}}(\Gamma)$-*function*. For a sake of simplicity, at TR2DF l on Γ will be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^l, V_1^l, V_2^l) to refer l) of $V(\Gamma)$ induced by l, where $V_i = \{y \in V(\Gamma): l(y) = i\}$, for $i \in \{0, 1, 2\}$. Since labeling 2 to the vertices of any minimum total domination set of a graph Γ introduces a TR2DF of Γ, we have

$$\gamma_{t\{R2\}}(\Gamma) \leq 2\gamma_t(\Gamma). \tag{1}$$

In this paper, we extend the idea of Roman {2}-reinforcement number to TR2R-number as follows: for a graph Γ, a subset M of $E(\overline{\Gamma})$ is a total Roman {2}-reinforcement set (TR2RS) of Γ if $\gamma_{t\{R2\}}(\Gamma + M) < \gamma_{t\{R2\}}(\Gamma)$. The TR2R-number of a graph Γ, denoted by $r_{t\{R2\}}(\Gamma)$, is the minimum cardinality of a TR2RS of Γ. A TR2Rtotal Roman {2}-reinforcement set (TR2RS M of Γ is called a $r_{t\{R2\}}(\Gamma)$-*set* if $|M| = r_{t\{R2\}}(\Gamma)$. Observe that if $\gamma_{t\{R2\}}(\Gamma) = 2$, then addition of edges does not reduce the TR2D-number. We put $r_{t\{R2\}}(\Gamma) = 0$ if $\gamma_{t\{R2\}}(\Gamma) = 2$. Hence, we always suppose that when we discuss $r_{t\{R2\}}(\Gamma)$, all graphs involved satisfy $\gamma_{t\{R2\}}(\Gamma) \geq 3$. If Γ is a graph such that $\gamma_{t\{R2\}} \geq 3$, then there is always a nonempty set $M \subseteq E(\overline{\Gamma})$ such that $\gamma_{t\{R2\}}(\Gamma + M) < \gamma_{t\{R2\}}(\Gamma)$, i.e., $r_{t\{R2\}}(\Gamma) \geq 1$. For instance, if we take $M = E(\overline{\Gamma})$, then we have that $2 = \gamma_{t\{R2\}}(\Gamma + M) < 3 \leq \gamma_{t\{R2\}}(\Gamma)$.

Our purpose in this manuscript is to initiate the study of TR2R-number in graphs. We derive some sharp upper bounds on $r_{t\{R2\}}(\Gamma)$ and we also determine exact values of TR2R-number of paths and complete multipartite graphs.

We close this section with some useful results.

Observation 1. For any graph Γ with $\gamma_{t\{R2\}}(\Gamma) \geq 3$ and $\gamma_{t\{R2\}}(\Gamma)$-function $l = (V_0, V_1, V_2)$, $pn(y, l) \neq \emptyset$, for every $y \in V_2$.

Proposition 1. *For any graph Γ with $\gamma_{t\{R2\}}(\Gamma) \geq 3$, let M be a $r_{t\{R2\}}(\Gamma)$-set and let l be a $\gamma_{t\{R2\}}(\Gamma + M)$-function. Then, the following hold:*

(i) For any edge $v_1v_2 \in M$, $\{l(v_1), l(v_2)\} \neq \{0\}$.

(ii) $\gamma_{t\{R2\}}(\Gamma) - 2 \leq \gamma_{t\{R2\}}(\Gamma + M) \leq \gamma_{t\{R2\}}(\Gamma) - 1$.

Proof.

(i) Suppose, to the contrary, that there exists an edge $v_1v_2 \in M$ such that $l(v_i) = 0$ for each $i \in \{1, 2\}$. Then, l is a TR2DF on $\Gamma + (M - \{v_1v_2\})$ and so $M - \{v_1v_2\}$ is a TR2DS of Γ, implying that $r_{t\{R2\}}(\Gamma) \leq |M - \{v_1v_2\}| = |M| - 1$, a contradiction. Hence, (i) holds.

(ii) Since M is a $r_{t\{R2\}}(\Gamma)$-set, $\gamma_{t\{R2\}}(\Gamma + M) \leq \gamma_t\{R2\}(\Gamma) - 1$. Now, we prove the lower bound. If $\gamma_{t\{R2\}} \in \{3, 4\}$, then we are done. Now, we assume that $\gamma_{t\{R2\}} \geq 5$ and suppose, to the contrary, that $\gamma_{t\{R2\}}(\Gamma + M) \leq \gamma_{t\{R2\}}(\Gamma) - 3$. Let $v_1v_2 \in M$. By (i), we can suppose that $l(v_2) \geq 1$. If $l(v_1) = 0$, then the function s, defined by $s(v_1) = 1$ and $s(z) = l(z)$ for the other vertices, is a TR2DF on $\Gamma + (M - \{v_1v_2\})$ with $\omega(s) = \omega(l) + 1 \leq \gamma_{t\{R2\}}(\Gamma) - 1$, and so $M - \{v_1v_2\}$ is a TR2-set of Γ, implying that $r_{t\{R2\}}(\Gamma) \leq |M - \{v_1v_2\}| = |M| - 1$, a contradiction. Assume that $l(v_1) \geq 1$. Let u_i be a neighbor of v_i in Γ for $i = \{1, 2\}$. Then, the function s, defined by $s(u_i) = \max\{1, l(u_i)\}$ for $i = \{1, 2\}$ and $s(x) = l(x)$ for the other vertices, is a TR2DF on $\Gamma + (M - \{v_1v_2\})$ with $\omega(s) = \omega(l) + 2 \leq \gamma_{t\{R2\}}(\Gamma) - 1$, and so $M - \{v_1v_2\}$ is a TR2-set of Γ, implying that $r_{t\{R2\}}(\Gamma) \leq |M - \{v_1v_2\}| = |M| - 1$, a contradiction. So (ii) also holds. □

Proposition 2 (see [16, 17]). *Let Γ be a nontrivial connected graph of order p. Then, $\gamma_{t\{R2\}}(\Gamma) = p$ if and only if $\Gamma \in \{K_2, P_3\}$ or any vertex of Γ is either a leaf or a weak stem.*

Theorem 1 (see [18]). *If Γ is a connected graph of order $p \geq 3$ and $\Delta(\Gamma) \leq p - 2$, then $\gamma_t(\Gamma) \leq p - \Delta(\Gamma)$.*

2. Bounds

The aim of this section is to obtain basic properties of the TR2R-number in graphs. Let Γ be a graph and $l = (V_0, V_1, V_2)$ be a TR2DF. For each $y \in V_1 \cup V_2$, the l-private neighborhood of y, denoted by $pn(y, l)$, consists of all vertices $z \in N(y) \cap V_0$ such that $\sum_{w \in N(z)} l(w) = 2$.

Theorem 2. *Let Γ be a graph with $\gamma_{t\{R2\}}(\Gamma) \geq 3$ and let $l = (V_0, V_1, V_2)$ be a $\gamma_{t\{R2\}}(\Gamma)$-function. Then, the following assertions hold.*

(i) For any $y \in V_2$, $r_{t\{R2\}}(\Gamma) \leq |pn(y, l)|$. This bound is sharp for double star $DS_{r,s}$ $(s \geq r \geq 2)$.

(ii) If $|V_1| + |V_2| \geq 3$, then for each $y \in V_1$,

$$r_{t\{R2\}}(\Gamma) \leq |pn(y, l)| + \lceil \frac{|I(l)|}{2} \rceil + 1, \tag{2}$$

when $I(l)$ is the set of all isolate vertices of induced subgraph $\Gamma[V_1 \cup V_2 - \{y\}]$.

Proof.

(i) Let $v \in V_2$. By definition, there exists $u \in N(v) \cap (V_1 \cup V_2)$. Let $M = \{ux | x \in pn\ (v, l)\}$. Notice that $M \neq \emptyset$ by Observation 1. Then, the function s defined by $s(v) = 1$ and $s(y) = l(y)$, for the other vertices, is a TR2DF of $\Gamma + M$ and so $r_{t\{R2\}}(\Gamma) \leq |M| = |pn(v, l)|$.

(ii) Let $v \in V_1$. By definition, there exists $w \in N(v) \cap (V_1 \cup V_2)$. Let $pn(v, l) = \{w_1, w_2, \ldots, w_t\}$ if $pn(v, l) \neq \emptyset$ and $I(l) = \{z_1, z_2, \ldots, z_s\}$ when $I(l) \neq \emptyset$. Since $|V_1| + |V_2| \geq 3$, for any vertex $x \in pn(v, l) \cup I(l)$, there is a vertex $x' \in V_1 \cup V_2$ such that $xx' \notin E(\Gamma)$. Let $M = \{xx' | x \in pn(v, l)\} \cup \{z_{2i-1}z\ 2i | 1 \leq i \leq (t - 1/2)\} \cup \{z_t z_t'\}$ when t is odd and $M = \{xx' | x \in pn(v, l)\} \cup \{z_{2i-1}z_{2i} | 1 \leq i \leq (t/2)\}$ if t is even. If v joins at least two vertices in $V_1 \cup V_2$, then the function s defined by $s(v) = 0$, $s(y) = l(y)$, for the other vertices, is a TR2DF of $\Gamma + M$. If v joins only one $V_1 \cup V_2$ and $v' \in V_1 \cup V_2 - N(v)$, then the function s, defined before, is a TR2DF of $\Gamma + (M \cup \{vv'\})$. Hence,

$$r_{t\{R2\}}(\Gamma) \leq |M| + 1 \leq |pn(v, l)| + \lceil \frac{|I(l)|}{2} \rceil + 1. \tag{3}$$

□

Proposition 3. *Let Γ be a graph. Then, $r_{t\{R2\}}(\Gamma) \leq \Delta(\Gamma) + 1$.*

Proof. If $\gamma_{t\{R2\}}(\Gamma) = 2$, then $r_{t\{R2\}}(\Gamma) = 0 < \Delta + 1$. Hence, assume that $\gamma_{t\{R2\}}(\Gamma) \geq 3$ and let $l = (V_0, V_1, V_2)$ be a $\gamma_{t\{R2\}}(\Gamma)$-function. If $V_2 \neq \emptyset$ and $v \in V_2$, then by Theorem 2(i), we have $r_{t\{R2\}}(\Gamma) \leq |pn(v, l)| \leq \Delta - 1$. Assume that

$V_2 = \varnothing$. Then, we have $\gamma_{t\{R2\}}(\Gamma) = |V_1| \geq 3$. In this case, the result holds by Theorem 2(ii). □

Proposition 4. *Let Γ be a graph of order p with $\gamma_{t\{R2\}}(\Gamma) \geq 4$. Then,*

$$r_{t\{R2\}}(\Gamma) \leq p - \Delta - 1. \quad (4)$$

Proof. Let v be a vertex of degree Δ. Since $\gamma_{t\{R2\}}(\Gamma) \geq 4$, we have $|V(\Gamma) \setminus N_\Gamma[v]| = p - \Delta(\Gamma) - 1 \geq 1$. Let $u \in N(v)$ and $V(\Gamma) \setminus N_\Gamma[v] = \{u_1, \ldots, u_t\}$. Then, the function s defined by $s(v) = 2, s(u) = 1$ and $s(x) = 0$, for the other vertices, is a TR2DF on $\Gamma + \{vu_i | 1 \leq i \leq p - \Delta(\Gamma) - 1\}$ with $\omega(s) = 3$. Thus, M is a TR2-set of Γ and so $r_{t\{R2\}}(\Gamma) \leq |M| = p - \Delta - 1$, establishing the desired upper bound. □

Next result is an immediate consequence of Propositions 3 and 4.

Corollary 1. *For any graph Γ of order p with $\gamma_{t\{R2\}}(\Gamma) \geq 4$,*

$$r_{t\{R2\}}(\Gamma) \leq \lfloor \frac{p}{2} \rfloor. \quad (5)$$

This bound is sharp for P_5.

Proof. If $\Delta \leq \lfloor p/2 \rfloor - 1$, then Proposition 3 yields $r_{t\{R2\}}(\Gamma) \leq \Delta + 1 \leq \lfloor p/2 \rfloor$. If $\Delta \geq \lfloor p/2 \rfloor$, then by Proposition 4, we obtain $r_{t\{R2\}}(\Gamma) \leq p - \Delta - 1 \leq p - \lfloor p/2 \rfloor - 1 = \lfloor p/2 \rfloor$. □

Let Γ_k be the graph obtained from two copies of $K_2 \vee \overline{K_k}$ $(k \geq 4)$ by joining exactly two vertices of degree 2 to obtain a connected graph (for $k = 4$, see Figure 1). Clearly, $\Delta(\Gamma_k) = k + 1$ and $\gamma_{t\{R2\}}(\Gamma_k) = 4$, and it is not hard to see that $r_{t\{R2\}}(\Gamma_k) = \Delta(\Gamma_k) + 1$. This shows that the bounds established in Theorem 2(ii), Propositions 3 and 4, and Corollary 1 are sharp.

Theorem 3. *Let Γ be a graph of order $p \geq 3$. If $r_{t\{R2\}}(\Gamma) \neq 0$, then*

$$r_{t\{R2\}}(\Gamma) \leq p - \Delta(\Gamma) - \lfloor \frac{\gamma_{t\{R2\}}}{2} \rfloor + 1. \quad (6)$$

Proof. It is enough to show that $\gamma_{t\{R2\}}(\Gamma) \leq 2p - 2\Delta(\Gamma) - 2r_{t\{R2\}}(\Gamma) + 2$. Since $\Delta(\Gamma) \leq p - 1 - r_{t\{R2\}}(\Gamma)$ (see Proposition 4), we add $r_{t\{R2\}}(\Gamma) - 1$ edges incident with a vertex of maximum degree and call such that a set of edges F. Clearly, $\gamma_{t\{R2\}}(\Gamma) = \gamma_{t\{R2\}}(\Gamma + F)$. Clearly, $\Delta(\Gamma + F) \leq n - 2$ and by Theorem 1, $\gamma_t(\Gamma + F) \leq p - \Delta(\Gamma + F)$. Using inequality (1), we obtain

$$\begin{aligned} \gamma_{t\{R2\}}(\Gamma) &= \gamma_{t\{R2\}}(\Gamma + F) \\ &\leq 2\gamma_t(\Gamma + F) \\ &\leq 2p - 2\Delta(\Gamma + F) \\ &\leq 2p - 2\Delta(\Gamma) - 2r_{t\{R2\}}(\Gamma) + 2. \end{aligned} \quad (7)$$

□

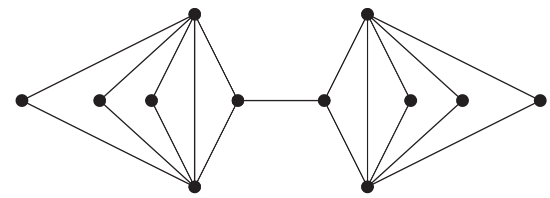

Figure 1: The graph Γ_4.

Next, we consider graphs Γ with $\gamma_{t\{R2\}}(\Gamma) \in \{2\gamma_t(\Gamma), 2\gamma_t(\Gamma) - 1\}$.

Proposition 5. *Let Γ be a graph with $\gamma_{t\{R2\}}(\Gamma) \in \{2\gamma_t(\Gamma), 2\gamma_t(\Gamma) - 1\}$. Then,*

$$r_{r\{R2\}}(\Gamma) \leq r_t(\Gamma). \quad (8)$$

Proof. Let F be a $r_t(\Gamma)$-set. By inequality (1), we have $\gamma_{t\{R2\}}(\Gamma + F) \leq 2\gamma_t(\Gamma + F) \leq 2\gamma_t(\Gamma) - 2 < \gamma_{t\{R2\}}(\Gamma)$ and this implies that $r_{t\{R2\}}(\Gamma) \leq |F| = r_t(\Gamma)$. □

3. Graphs Γ with Small $r_{t\{R2\}}$

In this section, we provide sufficient conditions for the graph to have small $r_{t\{R2\}}$. We first give a characterization of graphs Γ with $r_{t\{R2\}}(\Gamma) = 1$.

Lemma 1. *Let Γ be a connected graph of order $p \geq 3$ with $\gamma_{t\{R2\}}(\Gamma) = p$. Then, $r_{t\{R2\}}(\Gamma) = 1$.*

Proof. Let $\gamma_{t\{R2\}}(\Gamma) = p$. If Γ is the $P_3 = xyz$, then $\Gamma + xz$ is a complete graph and so $\gamma_{t\{R2\}}(\Gamma + xz) = 2 < \gamma_{t\{R2\}}(\Gamma)$. Assume that Γ is the corona graph $\Gamma = H \circ K_1$ of some connected graphs H of order greater than or equal to two. Let y_1 and y_2 be two leaves in Γ. Then, $\Gamma + y_1y_2$ is not a corona graph and it follows from Proposition 2 that $\gamma_{t\{R2\}}(\Gamma + y_1y_2) \leq p - 1 < \gamma_{t\{R2\}}(\Gamma)$. Thus, $r_{t\{R2\}}(\Gamma) = 1$. □

Theorem 4. *Let Γ be a connected graph of order $p \geq 3$. Then, $r_{t\{R2\}}(\Gamma) = 1$ if and only if $\gamma_{t\{R2\}}(\Gamma) = p$ or there exists a function $l = (V_0, V_1, V_2)$ of weight less that $\gamma_{t\{R2\}}(\Gamma)$ for which one of the following conditions holds.*

(i) For each $z \in V_0$, $\sum_{u \in N(z)} l(u) \geq 2$, $|V_1 \cup V_2| \geq 2$, and the induced subgraph $\Gamma[V_1 \cup V_2]$ has at most two isolated vertices.

(ii) $\Gamma[V_1 \cup V_2]$ has no isolated vertices and $\sum_{u \in N(z)} l(u) \geq 2$ for each vertex $z \in V_0$ except one, say w, such that $\sum_{u \in N(w)} l(u) \leq 1$ if $V_2 \neq \varnothing$ or $\sum_{u \in N(w)} l(u) = 1$ if $V_2 = \varnothing$.

Proof. According to Lemma 1, we can suppose that $\gamma_{t\{R2\}}(\Gamma) \leq p - 1$. Let there exist a function $l = (V_0, V_1, V_2)$ of weight less that $\gamma_{t\{R2\}}(\Gamma)$, satisfying (i) or (ii). Since $\omega(l) \leq p - 2$, we have $V_1 \cup V_2 \neq \varnothing$. First, let l satisfy (ii) and let $z \in V_0$ be a vertex for which $\sum_{u \in N(z)} l(u) \leq 1$. Suppose $u \in V_1 \cup V_2$ is a vertex for which $uz \notin E(\Gamma)$ and $l(u)$ is as large as possible. Clearly, l is a TR2DF of $\Gamma + \{uz\}$ and this implies that $r_{t\{R2\}}(\Gamma) = 1$. Next, assume that l satisfies (i). If $\Gamma[V_1 \cup V_2]$ has two isolated vertices u, v, then l is a TR2DF of $\Gamma + \{uv\}$ and if $\Gamma[V_1 \cup V_2]$ has exactly one isolate vertex, say u, then l is a TR2DF of $\Gamma + \{uv\}$, where $v \in (V_1 \cup V_2) - \{u\}$. This implies that $r_{t\{R2\}}(\Gamma) = 1$.

Conversely, assume that $r_{t\{R2\}}(\Gamma) = 1$, and let $M = \{e = zy\}$ be a $r_{t\{R2\}}(\Gamma)$-set. If $\gamma_{t\{R2\}}(\Gamma) = p$, then we are done. Hence, we assume that $\gamma_{t\{R2\}}(\Gamma) \le p - 1$. Let l be a $\gamma_{t\{R2\}}(\Gamma + e)$-function. If $z, y \in V_1 \cup V_2$, then l satisfies item (i) and if $z \in V_0$ or $y \in V_0$, then l satisfies item (ii). □

Next, we consider connected graphs with minimum degree 1.

Proposition 6. *Let Γ be a connected graph of order $p \ge 3$. If Γ has a stem v, then $r_{t\{R2\}}(\Gamma) \le deg(v)$. This bound is sharp for P_5.*

Proof. If $p = 3$, then $\Gamma = P_3$ and the result follows from Proposition 2 and Lemma 1. Assume that $p \ge 4$. Let u be a leaf-neighbor of v and l be a $\gamma_{t\{R2\}}(\Gamma)$-function such that $l(u)$ is as small as possible. If $l(u) = 0$, then $l(v) = 2$, and by Theorem 2, we have $r_{t\{R2\}}(\Gamma) \le |pn(v, l)| \le \deg(v) - 1$. Assume that $l(u) \ge 1$. Then, by the choice of l, we must have $l(u) = l(v) = 1$. Since $n \ge 4$ and u is a leaf, we have $|V_1 \cup V_2| \ge 3$. Let $w \in (V_1 \cup V_2) \setminus \{u, v\}$. Then, the function s, defined by $s(u) = 0$ and $s(y) = l(y)$ for the other vertices, is a TR2DF of $\Gamma + \{uw, vw\}$ of weight less than $\gamma_{t\{R2\}}(\Gamma)$. Hence, $r_{t\{R2\}}(\Gamma) \le 2 \le \deg(v)$. □

Proposition 7. *If Γ is a connected graph containing a path $v_1v_2v_3v_4v_5$ in which $d(v_i) = 2$ for $i \in \{2, 3, 4\}$, then $r_{t\{R2\}}(\Gamma) \le 3$.*

Proof. If v_1 or v_5 is a leaf, then the desired result follows by Proposition 6. Hence, we suppose $\deg(v_1) \ge 2$ and $\deg(v_5) \ge 2$. Let $l = (V_0, V_1, V_2)$ be a $\gamma_{t\{R2\}}(\Gamma)$-function. If $l(v_3) = 2$, then to totally dominate v_3, we can suppose without loss of generality that $l(v_2) \ge 1$ and the function $s: V(\Gamma + \{v_2v_4\}) \longrightarrow [2]$, defined by $s(v_3) = 1$ and $s(y) = l(y)$ for the other vertices, is a TR2DF of $\Gamma + \{v_2v_4\}$ of weight less than $\gamma_{t\{R2\}}(\Gamma)$. Assume that $l(v_3) \le 1$. Likewise, we can suppose that $l(v_2), l(v_4) \le 1$. We consider two cases.

Case 1. $l(v_3) = 1$.

To totally dominate v_3, we can suppose that $l(v_2) \ge 1$. If $l(v_4) = 1$, then the function $s: V(\Gamma + \{v_2v_4\}) \longrightarrow [2]$, defined by $s(v_3) = 0$ and $s(y) = l(y)$ for the other vertices, is a TR2DF of $\Gamma + \{v_2v_4\}$ of weight less than $\gamma_{t\{R2\}}(\Gamma)$. Suppose that $l(v_4) = 0$. Then, to Roman {2}-dominate v_4, we must have $l(v_5) \ge 1$ and the function $s: V(\Gamma + \{v_2v_4, v_3v_5, v_2v_5\}) \longrightarrow [2]$, defined by $s(v_3) = 0$ and $s(y) = l(y)$ for the other vertices, is a TR2DF of $\Gamma + \{v_2v_4, v_2v_5, v_3v_5\}$ of weight less than $\gamma_{t\{R2\}}(\Gamma)$.

Case 2. $l(v_3) = 0$.

Since $l(v_2), l(v_4) \le 1$, to Roman {2}-dominate v_3, we must have $l(v_2) = l(v_4) = 1$ and to totally dominate v_2, v_4, we must have $l(v_1) \ge 1$ and $l(v_5) \ge 1$, respectively. Define $s: V(\Gamma + \{v_2v_4, v_1v_4, v_3v_5\}) \longrightarrow [2]$ by $s(v_2) = 0$ and $s(y) = l(y)$ for the other vertices. Clearly, s is a TR2DF of $\Gamma + \{v_2v_4, v_1v_4, v_3v_5\}$ of weight less that of $\gamma_{t\{R2\}}(\Gamma)$. All in all, we have $r_{t\{R2\}}(\Gamma) \le 3$. □

4. Exact Values

In this section, we determine the TR2R-number of paths and complete multipartite graphs. The TR2D-number of paths is determined in [17].

Proposition 8 (see [17]). *For any positive integer $p \ge 2$,*

(i) $\gamma_{t\{R2\}}(P_p) = \begin{cases} 2\lceil p/3 \rceil + 1 & \text{if } p \equiv 0 \,(\text{mod} 3) \\ 2\lceil p/3 \rceil & \text{otherwise.} \end{cases}$

(ii) $\gamma_{t\{R2\}}(C_p) = \lceil 2p/3 \rceil$.

Theorem 5. *For any integer $p \ge 3$,*

$$r_{t\{R2\}}(P_p) = \begin{cases} 1, & \text{if } p \equiv 0, 1 \,(\text{mod} 3), \\ 2, & \text{otherwise.} \end{cases} \tag{9}$$

Proof. Let $P_p = v_1, v_2, \ldots, v_p$. If $p \equiv 0, 1\,(\text{mod} 3)$, then by Proposition 8, we have $\gamma_{t\{R2\}}(P_n + v_1v_p) < \gamma_{t\{R2\}}(P_p)$ and hence $r_{t\{R2\}}(P_p) = 1$.

Suppose next that $p \equiv 2\,(\text{mod} 3)$. Clearly, the function l defined by $l(v_1) = l(v_i) = 0$ for $i \equiv 0\,(\text{mod} 3)$, and $l(x) = 1$ otherwise, is a TR2DF on $P_p + \{v_1v_p, v_2v_p\}$ with $\omega(l) = 2\lceil p/3 \rceil - 1 = \gamma_{t\{R2\}}(P_p) - 1$ implying that $r_{t\{R2\}}(P_p) \le 2$. Now, we show that $r_{t\{R2\}}(P_p) \ge 2$. In order to this, we show that there is no function $l' = (V_0, V_1, V_2)$ of weight less than $\gamma_{t\{R2\}}(\Gamma)$, satisfying one of the conditions (i) and (ii) of Theorem 4.

Suppose, to the contrary, that there exists a function $l' = (V_0, V_1, V_2)$ of weight less than $\gamma_{t\{R2\}}(\Gamma)$, satisfying one of the conditions (i) and (ii) of Theorem 4. Choose such a function l' such that $|V_2|$ is as few as possible. Let $p = 3r + 2$ for some positive integers r. First, let l' satisfy (ii). We note that $V_2 = \varnothing$; otherwise, if $v_i \in V_2$, then by total, we can suppose that $l'(v_{i+1}) \ge 1$ and the function s defined by $s(v_{i-1}) = s(v_i) = 1$, and $s(y) = l'(y)$ otherwise, is a function on P_p of weight less than $\gamma_{t\{R2\}}(\Gamma)$, satisfying (ii), a contradiction with the choice of l'. Then, we must have $|V_1| \le 2r + 1$ and so $|V_0| \ge r + 1$. Assume without loss of generality that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} \subseteq V_0$ such that $i_1 < \cdots < i_r$ and $\sum_{u \in N(v_{i_j})} l'(u) \ge 2$ for each $1 \le j \le r$. Then, we must have $l'(v_{i_j - 1}) = l'(v_{i_j + 1}) = 1$. Since $\Gamma[V_1]$ has no isolated vertex, we have $l'(v_{i_j - 2}) = l'(v_{i_j + 2}) = 1$ for each $1 \le j \le r$. It follows that $|V_1| \ge 2r + 2$ which is a contradiction.

Now, let l' satisfy (i). Using the above argument, we can see that each vertex in V_2 is isolated in $\Gamma[V_1 \cup V_2]$. Thus, $|V_2| \le 2$. First, let $|V_2| = 2$. Since $|V_1| + 2|V_2| = |V_1| + 4 \le 2r + 1$, we have $|V_1| \le 2r - 3$ and this implies that $|V_0| \ge r + 3$. Clearly, four vertices in V_0 are dominated by vertices with weight 2. Assume without loss of generality that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{r-1}}\} \subseteq V_0$ in which $i_1 < \cdots < i_{r-1}$ and that v_j has no neighbor in V_2. As mentioned above, we can see that

$|V_1| \geq 2r$ which leads to the contradiction $|V_1| + 2|V_2| \geq 2r + 4$.

Next, assume that $|V_2| = 1$. Let $l'(v_i) = 2$. We claim that $i \leq p - 2$. If $i = p$, then the function s defined by $s(v_p) = s(v_{p-1}) = 1$ and $s(y) = l'(y)$ is a TR2DF of P_p of weight less than $\gamma_{t\{R2\}}(\Gamma)$ which is a contradiction. If $i = p - 1$, then we must have $l'(v_p) = l'(v_{p-2}) = 0$ and $l'(v_{p-3}) \geq 1$ (note that $|V_2| = 1$) and the function s, defined before, is a TR2DF of P_p of weight less than $\gamma_{t\{R2\}}(\Gamma)$, a contradiction again. Thus, $i \leq p - 2$. Similarly, we can see that $i \geq 3$. Since $|V_2| = 1$, we must have $l'(v_{i-2}) = l'(v_{i+2}) = 1$. Then, the function s, defined by $s(v_{i-1}) = s(v_{i+1}) = 1, s(v_i) = 0$ and $s(y) = l'(y)$ otherwise, is a function on P_p of weight less than $\gamma_{t\{R2\}}(\Gamma)$, a contradiction.

Finally, let $|V_2| = 0$. We distinguish two situations.

(1) $\Gamma[V_1]$ has exactly one isolated vertex, say v_i. If $i = 1$ (the case $i = p$ is similar), then $l'(v_2) = 0$ and the function s, defined by $s(v_1) = 0, s(v_2) = 1$ and $s(y) = l'(y)$ otherwise, is a function of weight less than $\gamma_{t\{R2\}}(\Gamma)$ which satisfies (ii) and the result follows as above. Thus, $2 \leq i \leq p - 1$. Then, $l'(v_{i-1}) = l'(v_{i+1}) = 0$. To Roman {2}-dominate v_{i-1} and v_{i+1}, we must have $l'(v_{i-2}) = l'(v_{i+2}) = 1$. On the other hand, since v_i is the only isolated vertex of $\Gamma[V_1]$, we have $l'(v_{i-3}) = l'(v_{i+3}) = 1$. Let P^1 be the path obtained from P_p by deleting the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}$ and adding the edge $v_{i-3}v_{i+3}$. Clearly, the restriction of l' on P^1 is a TR2DF of P^1, and by Proposition 8, we have $\omega(l') \geq 3 + (2r - 1) = 2r + 2$ which is a contradiction.

(2) $\Gamma[V_1]$ has exactly two isolated vertices. Let v_i, v_j be the isolated vertices of $\Gamma[V_1]$ such that $j > i$. Note that if $i = 1$, then $l'(v_2) = 0$, and to Roman {2}-dominate v_2, we must have $l'(v_3) = 1$. Now, if $l'(v_4) = 0$, then we must have $l'(v_5) = l'(v_6) = 1$ and then the function s, defined by $s(v_2) = 1$ and $s(y) = l'(y)$ otherwise, is a TR2DF of $P_p - v_1$ implying that $\omega(s) = \omega(l') = 2r + 2$ which is a contradiction. Assume that $l'(v_4) = 1$. Then, the function l', defined by $l'(v_3) = 0, l'(v_2) = 1$ and $s(y) = l'(x)$ otherwise, is a function of weight $\omega(l')$, satisfying the condition (i) of Theorem 4. Thus, we can suppose that $i \geq 2$. Similarly, we can suppose that $j \leq p - 1$. Then, we must have $l'(v_{i-1}) = l'(v_{i+1}) = l'(v_{j-1}) = l'(v_{j+1}) = 0$. To Roman {2}-dominate $v_{i-1}, v_{j-1}, v_{i+1}$ and v_{j+1}, we must have $l'(v_{i-2}) = l'(v_{i+2}) = l'(v_{j-2}) = l'(v_{j+2}) = 1$, and to totally dominate v_{i-2} and v_{j+2}, we have $l'(v_{i-3}) = l'(v_{j+3}) = 1$. If $i = j - 2$, then the restriction of l' on P^2, obtained from P_p by deleting the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_j, v_{j+1}, v_{j+2}$ and adding the edge $v_{i-3}v_{j+3}$, is a TR2DF of P^2, and by Proposition 8, we have $\omega(l') \geq 4 + (2r - 2) = 2r + 2$ which is a contradiction. If $i = j - 5$, then $l'(v_{i+2}) = l'(v_{i+3}) = 1$, and the restriction of l' on P^3, obtained from P_p by deleting the vertices $v_{i-2}, \ldots, v_{j+2}$ and adding the edge $v_{i-3}v_{j+3}$, is a TR2DF of P^3, and by Proposition 8, we have $\omega(l') \geq 6 + (2r - 4) = 2r + 2$, a contradiction again. Let $i < j - 5$. Then, we must have $l'(v_{i+2}) = l'(v_{i+3}) = l'(v_{j-2}) = l'(v_{j-3}) = 1$ and the restriction of l' on P^4, obtained from P_p by deleting the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, v_{j-2}, v_{j-1}, v_j, v_{j+1}, v_{j+2}$ and adding the edges $v_{i-3}v_{i+3}, v_{j-3}v_{j+3}$, is a TR2DF of P^4, and by Proposition 8, we have $\omega(l') \geq 6 + (2r - 4) = 2r + 2$ which is a contradiction.

Thus, there is no function $l = (V_0, V_1, V_2)$ of weight less than $\gamma_{t\{R2\}}(\Gamma)$, satisfying one of the conditions (i) and (ii) of Theorem 4. So, $r_{t\{R2\}}(P_p) = 2$. □

The proof of the following result is straightforward and then omitted.

Proposition 9. *For any positive integers $p_1 \leq p_2 \leq \cdots \leq p_t$ with $t \geq 2$,*

$$\gamma_{t\{R2\}}\left(K_{p_1,p_2,\ldots,p_t}\right) = \begin{cases} 2, & \text{if } p_1 = p_2 = 1, \\ 3, & \text{if } p_1 = 1 \text{ and } p_2 \geq 2, \\ 3, & \text{if } p_1 \geq 2 \text{ and } t \geq 3, \\ 3, & \text{if } p_1 = 2 \text{ and } t = 2, \\ 4, & \text{if } p_1 \geq 3 \text{ and } t = 2. \end{cases} \tag{10}$$

Next, we determine the TR2R-number of complete multipartite graphs.

Theorem 6. *For any positive integers $1 \leq p_1 \leq p_2 \leq \cdots \leq p_t$ with $t \geq 2$,*

$$r_{t\{R2\}}\left(K_{p_1,p_2,\ldots,p_t}\right) = \begin{cases} 0, & \text{if } p_1 = p_2 = 1, \\ 1, & \text{if } p_i = 2 \text{ for some } i, \\ p_2 - 1, & \text{if } p_1 = 1, p_2 \geq 2, \\ p_1 - 1, & \text{if } p_1 \geq 3 \text{ and } t = 2, \\ 2p_1 - 3, & \text{if } p_1 \geq 3 \text{ and } t \geq 3. \end{cases} \tag{11}$$

Proof. Let $\Gamma = K_{p_1,p_2,\ldots,p_t}$. If $p_1 = p_2 = 1$, then $\gamma_{t\{R2\}}(\Gamma) = 2$, and by definition, we have $r_{t\{R2\}}(\Gamma) = 0$. If $p_i = 2$ for some i and $\{x, y\}$ is a partite set of size 2, then the function s, defined by $s(x) = s(y) = 1$ and $s(z) = 0$ otherwise, is a TR2DF of $\Gamma + xy$ of weight 2 and this implies that $r_{t\{R2\}}(\Gamma) = 1$. Hence, we assume that $p_2 \geq 3$. First, let $p_1 = 1$. By Proposition 9, we have $\gamma_{t\{R2\}}(\Gamma) = 3$. Let M be a $r_{t\{R2\}}(\Gamma)$-set. Then, we have $\gamma_{t\{R2\}}(\Gamma + M) = 2$. Let l be a $\gamma_{t\{R2\}}(\Gamma + M)$-function and $l(x) = l(y) = 1$. If x, y belong to a partite set $\{x, y, x_1, \ldots, x_{p_i-2}\}$, then to dominate $x_1, \ldots, x_{p_i-2}$, we must have $xx_i, yx_i \in M$, and to totally dominate x, y, we must have $xy \in M$. This implies that $|M| \geq 2(p_i - 2) + 1 = 2p_i - 3$. If x, y belong to different partite sets, say $x \in \{x, x_1, \ldots, x_{p_i-1}\}$ and $y \in \{y, y_1, \ldots, y_{p_j-1}\}$ where $j \geq i$, then clearly we must have $xx_s, yy_\ell \in F$ for each s and each ℓ, yielding $|M| \geq p_i + p_j - 2 \geq p_j - 1$. In either case, $|M| \geq p_2 - 1$. Let x be the vertex of the partite set of size 1 and y be a vertex of a partite set $\{y, y_1, \ldots, y_{p_2-1}\}$. Clearly, the function s, defined by $s(x) = s(y) = 1$ and $s(z) = 0$ otherwise, is a TR2DF of $\Gamma + \{yy_s | 1 \le s \le p_2 - 1\}$ of weight 2 and this implies that $r_{t\{R2\}}(\Gamma) = p_2 - 1$.

Now, let $p_1 \ge 3$. Assume that $t \ge 3$ and let M be a $r_{t\{R2\}}(\Gamma)$-set. Then, $\gamma_{t\{R2\}}(\Gamma + M) = 2$. Let l be a $\gamma_{t\{R2\}}(\Gamma + M)$-function and let $l(x) = l(y) = 1$. As mentioned above, we can see that $|M| \ge 2p_1 - 3$. If $X = \{x_1, \ldots, x_{p_1}\}$ is a partite set, then the function s, defined by $s(x_1) = s(x_2) = 1$ and $s(z) = 0$ otherwise, is a TR2DF of $\Gamma + \{x_1x_2, x_1x_s, x_2x_s | 2 \le s \le p_1\}$ of weight 2 and so $r_{t\{R2\}}(\Gamma) = 2p_1 - 3$.

Assume now that $t = 2$ and let $X_1 = \{x_1^1, \ldots, x_{p_1}^1\}$ and $X_2 = \{x_1^2, \ldots, x_{p_2}^2\}$ be the partite sets of Γ. Let M be a $r_{t\{R2\}}(\Gamma)$-set and let $l = (V_0, V_1, V_2)$ be a $\gamma_{t\{R2\}}(\Gamma + M)$-function. Then, we have $\gamma_{t\{R2\}}(\Gamma + M) = 3$. We consider the following situations:

(i) $|V_2| = |V_1| = 1$.

Let $V_2 = \{x\}$ and $V_1 = \{y\}$. If x, y belong to a partite set X_i, then we can suppose that $x = x_1^i, y = x_2^i$. Then, to dominate the vertices in $X_i - \{x, y\}$ and to totally dominate the vertices x, y, we must have $x_1^i x_s^i \in M$ for $2 \le s \le p_i$ and so $|M| \ge p_i - 1 \ge p_1 - 1$. If x, y belong to different partite sets, then we can suppose that $x = x_1^1$ and $y = x_1^2$. Then, to dominate the vertices $X_1 - \{x\}$, we must have $xx_s^1 \in M$ for $2 \le s \le p_1$ and so $|M| \ge p_1 - 1$. On the other side, the function s, defined by $s(x_1^1) = 2, s(x_2^1) = 1$ and $s(z) = 0$ otherwise, is a TR2DF of $\Gamma + \{x_1^1 x_s^1 | 2 \le s \le p_1\}$ of weight 3 and so $r_{t\{R2\}}(\Gamma) = p_1 - 1$.

(ii) $|V_2| = 0$ and $|V_1| = 3$.

If $V_1 \subseteq X_i$ and $V_1 = \{x_1^i, x_2^i, x_3^i\}$, then to dominate the vertices $x_s^i (s \ge 4)$, M must contain two edges between x_s^i and V_1 for each $4 \le s \le p_1$ and to totally dominate the vertices of V_1, M must contain two edges between x_1^i, x_2^i, x_3^i. It follows that $|M| \ge 2(n_i - 3) + 2 = p_1 - 1$.

If V_1 intersects both X_1, X_2, then $|V_1 \cap X_i| = 2$ for some i, and as mentioned above, we can see that $|M| \ge p_i - 1 \ge p_1 - 1$. On the other side, the function s, defined by $s(x_1^1) = 2, s(x_2^1) = 1$ and $s(z) = 0$ otherwise, is a TR2DF of $\Gamma + \{x_1^1 x_s^1 | 2 \le s \le p_1\}$ of weight 3 and so $r_{t\{R2\}}(\Gamma) = p_1 - 1$. □

Proposition 10. *For any integer* $p \ge 3$, $r_{t\{R2\}}(C_p) = 1$ *if* $p \equiv 1, 2 (\mathrm{mod} 3)$ *and* $r_{t\{R2\}}(C_p) \le 3$ *if* $p \equiv 0 (\mathrm{mod} 3)$.

Proof. Let $C_p = y_1y_2, \ldots, y_py_1$. If $p \equiv 1, 2 (\mathrm{mod} 3)$, then the function l, defined by $l(y_1) = l(y_{3i}) = 0$ for $1 \le i \le \lfloor p/3 \rfloor$, is a TR2DF on $C_p + \{y_2y_p\}$ with $\omega(l) = \lceil 2p/3 \rceil - 1 = \gamma_{t\{R2\}}(C_p) - 1$. Thus, the set $\{y_2y_p\}$ is a TR2-set of C_p and so $r_{t\{R2\}}(C_p) = 1$.

Suppose next that $p \equiv 0 (\mathrm{mod} 3)$. Using Proposition 3, we have $r_{t\{R2\}}(C_p) \le \Delta + 1 = 3$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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