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Research Article

Distribution of αp^2 Modulo One with Prime Variable p of a Special Form

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Let $\mathscr{P}_r$ denote an almost-prime with at most r prime factors, counted according to multiplicity. In this paper, it is proved that, for $\alpha \in (\mathbb{R}/\mathbb{Q})$, $\beta \in \mathbb{R}$, and $0 < \theta < (10/1561)$, there exist infinitely many primes p, such that $\|\alpha p^2 + \beta\| < p^{-\theta}$ and $p + 2 = \mathscr{P}_4$, which constitutes an improvement upon the previous result.

1. Introduction and Main Result

Let $\mathscr{P}_r$ denote an almost-prime with at most r prime factors, counted according to multiplicity. The famous prime twins conjecture states that there exist infinitely many primes p such that $p + 2$ is a prime too. Up to now, this conjecture is still open, but many approximations about this conjecture were established. One of the most interesting results is due to Chen [1], who showed, in 1973, that there exist infinitely many primes p such that $p + 2 = \mathscr{P}_2$.

In 1981, Heath-Brown [2] showed that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is $\mathscr{P}_2$. In 2006, Green and Tao [3] established that there exist infinitely many arithmetic progressions consisting of three different primes $p_1 < p_2 < p_3$ such that $p_j + 2 = \mathscr{P}_2$ for each $j = 1, 2, 3$. Later, in 2008, Green and Tao [4] showed that, for any $k \geqslant 3$, there exist infinitely many arithmetic progressions consisting of k different primes $p_1 < p_2 < \cdots < p_k$ such that $p_j + 2 = \mathscr{P}_2$ for each $j = 1, 2, \ldots, k$.

Suppose that there is a problem including primes and let $r \geqslant 2$ be an integer. Having in mind Chen's result, one may consider the problem with primes p, such that $p + 2 = \mathscr{P}_r$. Many authors investigated several kinds of problems of this type, such as Peneva and Tolev [5], Peneva [6], and Tolev [7–9].

Let α be an irrational real number and $\|x\|$ denote the distance from x to the nearest integer. Earlier work about the distribution of the fractional parts of the sequence $\{\alpha p\}$ was first considered by Vinogradov [10], who showed that, for any real number β, there are infinitely many primes p such that for $\theta = (1/5) - \varepsilon$; then,

$$\|\alpha p + \beta\| < p^{-\theta}, \tag{1}$$

where ε denotes arbitrarily small positive number. After that, the first improvement on (1) was due to Vaughan [11], who obtained $\theta = (1/4)$ in (1) and who also required an additional factor $(\log p)^8$ on the right-hand side of (1). Since then, many authors improved the upper bound of the exponent θ, such as Harman [12, 13], Jia [14, 15], and Heath-Brown and Jia [16]. So far, the best result is given by Matomäki [17] with $\theta = (1/3) - \varepsilon$. Moreover, it seems very natural to consider the sequence $\{\alpha p^k\}$ for $k \geqslant 2$, where p denotes a prime variable. Also, many authors studied the fractional parts of the sequence $\{\alpha p^k\}$ for $k \geqslant 2$, such as Baker and Harman [18], Harman [19], and Wong [20].

In 2010, Todorova and Tolev [21] considered the distribution of αp modulo one with primes of the form specified above and showed that, for $\theta = (1/100)$, there are infinitely many solutions in primes p to (1) such that

$p+2=\mathcal{P}_4$. Later, Matomäki [22] showed that this result actually holds with $p+2=\mathcal{P}_2$ and $\theta=(1/1000)$. After that, Shi [23] continued to improve the result of Matomäki [22] and showed that there are infinitely many solutions in primes p to (1) such that $p+2=\mathcal{P}_2$ and $\theta=(3/200)$.

Moreover, for the case $k=2$, Shi and Wu [24] established the result that there exist infinitely many primes p, which satisfy $\|\alpha p^2+\beta\|<p^{-\theta}$, such that $p+2=\mathcal{P}_4$ and $\theta=(2/375)-\varepsilon$.

In this paper, we shall continue to improve the result of Shi and Wu [24] and establish the following theorem.

Theorem 1. *Suppose that* $\alpha\in(\mathbb{R}/\mathbb{Q})$, $\beta\in\mathbb{R}$, *and* $0<\theta<(10/1561)$. *Then, there exist infinitely many primes p, which satisfy* $p+2=\mathcal{P}_4$, *such that*

$$\|\alpha p^2+\beta\|<p^{-\theta}. \tag{2}$$

Remark 1. According to the work of Shi and Wu [24], our improvement comes from using the methods developed by Tolev [9] with more delicate iterative techniques and various bounds for exponential sums, combining with a version of Lemma 2.2 of [25], while the previous method, in dealing exponential sum, e.g., [24], is based on the traditional pattern of exponential sum estimates.

2. Notation

Let X be a sufficiently large real number. Set

$$\begin{aligned}&\delta=0.307708,\\&\rho=0.23077,\\&\eta=0.076928,\\&\kappa=1.4999676,\\&0<\theta<\frac{10}{1561}.\end{aligned} \tag{3}$$

Also, we put

$$\begin{aligned}z&=X^{\eta},\\y&=X^{\rho},\\D&=X^{\delta},\\\Delta&=\Delta(X)=X^{-\theta},\\H&=\Delta^{-1}\log^2X.\end{aligned} \tag{4}$$

Throughout this paper, we always denote primes by p and q. ε always denotes an arbitrary small positive constant, which may not be the same at different occurrences. As usual, we use $\Omega(n),\varphi(n),\mu(n)$, and $\Lambda(n)$ to denote the number of prime factors of n counted according to multiplicity, Euler's function, Möbius' function, and Mangold's function, respectively. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1m_2\ldots m_k=n$ in natural variables $m_1,\ldots,m_k$. Especially, we write $\tau_2(n)=\tau(n)$. Let $(m_1,m_2,\ldots,m_k)$ and $[m_1,m_2,\ldots,m_k]$ be the greatest common divisor and the least common multiple of $m_1,m_2,\ldots,m_k$, respectively. Also, we use $[x]$ and $\|x\|$, respectively, to denote the integer part of x and the distance from x to the nearest integer. $f(x)\ll g(x)$ means that $f(x)=O(g(x))$; $f(x)\asymp g(x)$ means that $f(x)\ll g(x)\ll f(x)$; $e(x)=e^{2\pi ix}$; $\mathcal{L}=\log X$. $\mathcal{P}_r$ always denotes an almost-prime with at most r prime factors, counted according to multiplicity.

3. Preliminary Lemmas

Lemma 1. *Let* $M\leqslant N<N_1\leqslant M_1$ *and* a_n *be any complex numbers. Then, we have*

$$\left|\sum_{N<n\leqslant N_1}a_n\right|\leqslant\int_{-\infty}^{+\infty}\mathcal{K}(\theta)\left|\sum_{M<m\leqslant M_1}a_me(\theta m)\right|\mathrm{d}\theta, \tag{5}$$

where

$$\mathcal{K}(\theta)=\min\left(M_1-M+1,\frac{1}{\pi|\theta|},\frac{1}{\pi^2\theta^2}\right), \tag{6}$$

which satisfies

$$\int_{-\infty}^{+\infty}\mathcal{K}(\theta)\mathrm{d}\theta\leqslant3\log(2+M_1-M). \tag{7}$$

Proof. See Lemma 2.2 of [25]. □

Lemma 2. *Let* $3\leqslant u<v<w<X$, *and suppose that* $w-(1/2)\in\mathbb{N}$ *and that* $w\geqslant4u^2$, $X\geqslant64w^2u$, *and* $v^3\geqslant32X$. *Assume further that* $f(n)$ *is a complex-valued function. Then, the sum*

$$\sum_{(X/2)<n\leqslant X}\Lambda(n)f(n), \tag{8}$$

can be decomposed into $O(log^{10}X)$ *sums, each of which is either of Type I:*

$$\sum_{M<m\leqslant M_1}a_m\sum_{L<\ell\leqslant L_1}f(m\ell), \tag{9}$$

with $M<M_1\leqslant2M, L<L_1\leqslant2L, L\geqslant w, a_m\ll m^{\varepsilon}, ML\asymp X$, *or of Type II:*

$$\sum_{M<m\leqslant M_1}a_m\sum_{L<\ell\leqslant L_1}b_{\ell}f(m\ell), \tag{10}$$

with $M<M_1\leqslant2M, L<L_1\leqslant2L, u\leqslant L\leqslant v, a_m\ll m^{\varepsilon}, b_{\ell}\ll\ell^{\varepsilon}$, *and* $ML\asymp X$.

Proof. See Lemma 3 of [26]. □

Lemma 3. *For* $P\geqslant1$, *we have*

$$\sum_{1\leqslant n\leqslant P}e(\alpha n)\leqslant\min\left(P,\frac{1}{2\|\alpha\|}\right). \tag{11}$$

Proof. See Lemma 4 of Chapter VI of [27]. □

Lemma 4. *Suppose that Y_1, Y_2, and α are real numbers with $Y_1 \geqslant 1$ and $Y_2 \geqslant 1$ and that $|\alpha - (a/q)| \leqslant q^{-2}$ with $(a,q)=1$. Then, we have*

$$\sum_{n\leqslant Y_1} \min\left(\frac{Y_1Y_2}{n}, \frac{1}{\|\alpha n\|}\right) \ll Y_1Y_2\left(\frac{1}{q}+\frac{1}{Y_2}+\frac{q}{Y_1Y_2}\right)\log(2Y_1q). \tag{12}$$

Proof. See Lemma 2.2 of [28]. □

4. Proof of Theorem 1

As shown in [21], we take a periodic function $\chi(t)$ with period 1 such that

$$\begin{cases} 0<\chi(t)<1, & \text{if } -\Delta<t<\Delta, \\ \chi(t)=0, & \text{if } \Delta\leqslant t\leqslant 1-\Delta, \end{cases} \tag{13}$$

which has a Fourier series,

$$\chi(t) = \Delta + \sum_{|k|>0} g(k)e(kt), \tag{14}$$

with coefficients satisfying

$$\begin{aligned} &g(0)=\Delta, \\ &g(k)\ll\Delta, \quad \text{for all } k, \\ &\sum_{|k|>H} |g(k)| \ll X^{-1}. \end{aligned} \tag{15}$$

The existence of such a function is a consequence of a well-known lemma of Vinogradov. For instance, one can see Chapter I, §2 in [27]. Consider the sum

$$\Gamma \coloneqq \Gamma(X) = \sum_{(X/2)<p\leqslant X\,(p+2,P(z))=1} \chi\left(\alpha p^2+\beta\right)\mathscr{W}_p \log p, \tag{16}$$

where

$$P(z) = \prod_{2<p\leqslant z} p, \tag{17}$$

$$\mathscr{W}_p = 1-\kappa \sum_{z<q\leqslant y q|p+2}\left(1-\frac{\log q}{\log y}\right). \tag{18}$$

Let Γ_1 denote the sum of the terms of $\Gamma(X)$ in which $\mathscr{W}_p>0$. Then, we have

$$\Gamma(X)\leqslant\Gamma_1. \tag{19}$$

If we denote by Γ_2 the sum of the terms of Γ_1 in which $\mu(p+2)=0$, it is easy to see that

$$\begin{aligned} 0\leqslant\Gamma_2 &\ll \sum_{q\geqslant z}\ \sum_{n\leqslant X n+2\equiv 0\,(\mathrm{mod}\,q^2)} \log n \ll (\log X)\sum_{z\leqslant q\leqslant\sqrt{X+2}}\left(\frac{X}{q^2}+1\right) \\ &\ll X^{1+\varepsilon}z^{-1}+X^{(1/2)+\varepsilon} \ll X^{1-\eta+\varepsilon}. \end{aligned} \tag{20}$$

By noting the fact that the contribution of the terms (if such terms exist) in Γ_1, for which $X-2<p\leqslant X$, is $O(\log X)$, we deduce that

$$\Gamma\leqslant\Gamma_3+O\left(X^{1-\eta+\varepsilon}\right), \tag{21}$$

where

$$\Gamma_3 = \sum_{\substack{(X/2)<p\leqslant X-2\mathscr{W}_p>0,\mu^2(p+2)=1 \\ (p+2,P(z))=1}} \chi\left(\alpha p^2+\beta\right)\mathscr{W}_p\log p. \tag{22}$$

On the one hand, if we assume that

$$\Gamma(X) \gg \frac{\Delta X}{\log X}, \tag{23}$$

then from (21), we obtain

$$\Gamma_3 \gg \frac{\Delta X}{\log X}, \tag{24}$$

and thus $\Gamma_3>0$. Hence, there exists a prime p, which satisfies

$$\begin{aligned} &\frac{X}{2}<p\leqslant X-2, \quad \mathscr{W}_p>0, \\ &\mu^2(p+2)=1, \quad (p+2,P(z))=1, \end{aligned} \tag{25}$$

and such that

$$\chi\left(\alpha p^2+\beta\right)>0. \tag{26}$$

Combining (13), (25), and (26), we can see that this prime p satisfies

$$\left\|\alpha p^2+\beta\right\|<p^{-\theta}. \tag{27}$$

On the other hand, by the properties of the weights $\mathscr{W}_p$ (for example, one can see Chapter 9 of [29]), it is easy to see that if p satisfies (25), then

$$\begin{aligned} \Omega(p+2) &= \sum_{q>z,\,q|p+2} 1 < \frac{1}{\kappa} + \sum_{q>z,\,q|p+2}\frac{\log q}{\log y} = \frac{1}{\kappa}+\frac{\log(p+2)}{\log y} \\ &\leqslant \frac{1}{\kappa}+\frac{1}{\rho}<5, \end{aligned} \tag{28}$$

which implies $p+2=\mathscr{P}_4$. Therefore, in order to prove Theorem 1, it is sufficient to show that there exists a sequence $\{X_j\}_{j=1}^{\infty}$, which satisfies

$$\begin{aligned} &\lim_{j\longrightarrow\infty} X_j = +\infty, \\ &\Gamma\left(X_j\right) \gg \frac{\Delta\left(X_j\right)X_j}{\log X_j}, \quad j=1,2,3\ldots. \end{aligned} \tag{29}$$

By (16) and (18), we can write Γ as follows:

$$\Gamma = \Psi - \kappa\Phi, \tag{30}$$

where

$$\Psi = \sum_{(X/2)<p\leqslant X\,(p+2,P(z))=1} \chi\left(\alpha p^2+\beta\right)\log p, \tag{31}$$

$$\Phi = \sum_{(X/2)<p\leqslant X\,(p+2,P(z))=1} \chi\left(\alpha p^2+\beta\right)(\log p) \sum_{z<q\leqslant y q|p+2} \left(1-\frac{\log q}{\log y}\right). \tag{32}$$

Next, we shall give lower bound estimate of Ψ and upper bound estimate of Φ by using lower bound linear sieve and upper bound linear sieve, respectively. First, we consider Ψ. Let $\lambda^-(d)$ be the lower bounds for Rosser's weights of level D. Hence, for any positive integer d, there holds

$$\begin{aligned} &|\lambda^-(d)|\leqslant 1,\\ &\lambda^-(d)=0, \quad \text{if } d>D \text{ or } \mu(d)=0, \end{aligned} \tag{33}$$

$$\sum_{d|n}\lambda^-(d)\leqslant \sum_{d|n}\mu(d) = \begin{cases} 1, & \text{if } n=1,\\ 0, & \text{if } n\in\mathbb{N}, n>1. \end{cases} \tag{34}$$

Also, we shall use the fact if $2<s<4$, then there holds

$$\sum_{d|P(z)} \frac{\lambda^-(d)}{\varphi(d)} \geqslant \Pi(z)\left(\frac{2e^{\gamma}\log(s-1)}{s} + O\left((\log X)^{(-1/3)}\right)\right), \tag{35}$$

where

$$\Pi(z) = \prod_{2<p\leqslant z}\left(1-\frac{1}{p-1}\right). \tag{36}$$

Now, we take

$$s = \frac{\log D}{\log z} = \frac{\delta}{\eta} = \frac{76927}{19232} \in (2,4), \tag{37}$$

in (35). By (31) and (34), we obtain

$$\begin{aligned} \Psi &= \sum_{(X/2)<p\leqslant X} \chi\left(\alpha p^2+\beta\right)(\log p) \sum_{d|(p+2,P(z))} \mu(d) \geqslant \sum_{(X/2)<p\leqslant X} \chi\left(\alpha p^2+\beta\right)(\log p) \sum_{d|(p+2,P(z))} \lambda^-(d)\\ &= \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0 \,(\mathrm{mod}\, d)} \chi\left(\alpha p^2+\beta\right)\log p = \Psi_1, \text{ say.} \end{aligned} \tag{38}$$

From (32), we have

$$\begin{aligned} \Psi_1 &= \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, d)} \left(\Delta + \sum_{|k|>0} g(k)e\left(\alpha p^2 k+\beta k\right)\right)\log p\\ &= \Delta \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, d)} \log p + \sum_{d|P(z)} \lambda^-(d) \sum_{|k|>0} g(k)e(\beta k) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, d)} e\left(\alpha p^2 k\right)\log p\\ &= \Delta \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, d)} \log p + \sum_{d|P(z)} \lambda^-(d) \sum_{0<|k|\leqslant H} \left(\Delta^{-1} g(k)e(\beta k)\right) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, d)} e\left(\alpha p^2 k\right)\log p\\ &\quad + \sum_{d|P(z)} \lambda^-(d) \sum_{|k|>H} g(k)e(\beta k) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, d)} e\left(\alpha p^2 k\right)\log p. \end{aligned} \tag{39}$$

By (33) and the fact that $\lambda^-(d)=0$ for $d>D$, we obtain

$$\begin{aligned} &\sum_{d|P(z)} \lambda^-(d) \sum_{|k|>H} g(k)e(\beta k) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, d)} e\left(\alpha p^2 k\right)\log p\\ &\ll \sum_{d|P(z)} |\lambda^-(d)| \sum_{|k|>H} |g(k)| \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, d)} \log p \ll \sum_{d\leqslant D} \frac{1}{\varphi(d)} \ll \log D \ll \log X. \end{aligned} \tag{40}$$

Therefore, we obtain

$$\Psi_1 = \Delta\left(\Psi_2 + \Psi_3\right) + O(\log X), \tag{41}$$

where

$$\begin{aligned}
\Psi_2 &= \sum_{d|P(z)} \lambda^-(d) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0 (\mathrm{mod}\, d)} \log p,\\
\Psi_3 &= \sum_{d|P(z)} \lambda^-(d) \sum_{0<|k|\leqslant H} c(k) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0 (\mathrm{mod}\, d)} e\left(\alpha p^2 k\right)\log p,\\
c(k) &= \Delta^{-1} g(k) e(\beta k) \ll 1.
\end{aligned} \tag{42}$$

For Ψ_2, by Bombieri–Vinogradov's mean value theorem (see Chapter 28 of [30]) and (33), we derive that

$$\Psi_2 = \frac{X}{2} \sum_{d|P(z)} \frac{\lambda^-(d)}{\varphi(d)} + O\left(\frac{X}{\log^2 X}\right). \tag{43}$$

It follows from Mertens' prime number theorem (see [31]) that

$$\Pi(z) \asymp \frac{1}{\log z}. \tag{44}$$

Then, from (35), (43), and (44), we obtain

$$\Psi_2 \geqslant e^{\gamma} X \Pi(z) \frac{\log(s-1)}{s} + O\left(\frac{X}{\log^{(4/3)} X}\right), \tag{45}$$

where s is defined by (37). For Ψ_3, we shall investigate it in Section 5.

Now, we study the sum Φ, which is defined by (32). We rewrite Φ in the following form:

$$\Phi = \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \sum_{\substack{(X/2) < p\leqslant X\, p+2 \equiv 0 (\mathrm{mod}\, d)\\ (p+2, P(z))=1}} \chi\left(\alpha p^2 + \beta\right)\log p. \tag{46}$$

In order to give upper bound estimate of Φ, we shall apply an upper bound linear sieve. Let $\lambda_q^+(d)$ be the upper bounds for Rosser's weights of level (D/q). Hence, for any positive integer d, we have

$$\begin{aligned}
&\left|\lambda_q^+(d)\right| \leqslant 1,\\
&\lambda_q^+(d) = 0, \quad \text{if } d > \frac{D}{q} \text{ or } \mu(d) = 0,
\end{aligned} \tag{47}$$

$$\sum_{d|n} \lambda_q^+(d) \geqslant \sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1,\\ 0, & \text{if } n \in \mathbb{N}, n > 1. \end{cases} \tag{48}$$

Also, we shall use the fact, for $1 < s_1 < 3$, and there holds

$$\sum_{d|P(z)} \frac{\lambda_q^+(d)}{\varphi(d)} \leqslant \Pi(z)\left(\frac{2e^{\gamma}}{s_1} + O\left((\log X)^{(-1/3)}\right)\right). \tag{49}$$

For prime q in the sum Φ, we take

$$s_1 = \frac{\log(D/q)}{\log z}. \tag{50}$$

Then, it is easy to check that $1 < s_1 < 3$, and thus, (49) holds. By (46)–(48), we obtain

$$\begin{aligned}
\Phi &= \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0 (\mathrm{mod}\, d)} \chi\left(\alpha p^2 + \beta\right)(\log p) \sum_{d|(p+2, P(z))} \mu(d)\\
&\leqslant \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0 (\mathrm{mod}\, d)} \chi\left(\alpha p^2 + \beta\right)(\log p) \sum_{d|(p+2, P(z))} \lambda_q^+(d)\\
&= \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \sum_{d|P(z)} \lambda_q^+(d) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0 (\mathrm{mod}\, d)} \chi\left(\alpha p^2 + \beta\right)(\log p)\\
&= \sum_{m\leqslant D} \nu(m) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0 (\mathrm{mod}\, m)} \chi\left(\alpha p^2 + \beta\right)(\log p) =: \Phi_1,
\end{aligned} \tag{51}$$

where

$$\nu(m) = \sum_{\substack{z<q<y\ d|P(z)\\ m=dq}} \left(1-\frac{\log q}{\log y}\right)\lambda_q^{+}(d). \tag{52}$$

If $m\leqslant z$, then $\nu(m)=0$. If $z<m\leqslant D$, by (17) and (52), we know that the representation $m=dq$ with $z<q<y$ and $d|P(z)$ is unique. Thus, it is easy to see that

$$|\nu(m)|\leqslant 1. \tag{53}$$

From (14), we obtain

$$\begin{aligned}
\Phi_1 &= \sum_{m\leqslant D} \nu(m) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} \left(\Delta + \sum_{|k|>0} g(k)e\left(\alpha p^2 k+\beta k\right)\right)\log p\\
&= \Delta \sum_{m\leqslant D} \nu(m) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} e\left(\alpha p^2 k\right)\log p + \sum_{m\leqslant D} \nu(m) \sum_{|k|>0} g(k)e(\beta k)\\
&\quad \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} e\left(\alpha p^2 k\right)\log p = \Delta \sum_{m\leqslant D} \nu(m) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} \log p + \Delta \sum_{m\leqslant D} \nu(m)\\
&\quad \sum_{0<|k|\leqslant H} \left(\Delta^{-1} g(k)e(\beta k)\right) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} e\left(\alpha p^2 k\right)\log p\\
&\quad + \sum_{m\leqslant D} \nu(m) \sum_{|k|>H} g(k)e(\beta k) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} e\left(\alpha p^2 k\right)\log p.
\end{aligned} \tag{54}$$

By (15) and (53), we obtain

$$\begin{aligned}
&\sum_{m\leqslant D} \nu(m) \sum_{|k|>H} g(k)e(\beta k) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} e\left(\alpha p^2 k\right)\log p\\
&\ll \sum_{|k|>H} |g(k)| \sum_{m\leqslant D} \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} \log p \ll \sum_{m\leqslant D} \frac{1}{\varphi(m)} \ll \log X.
\end{aligned} \tag{55}$$

Thus, we derive that

$$\Phi_1 = \Delta\left(\Phi_2+\Phi_3\right) + O(\log X), \tag{56}$$

where

$$\begin{aligned}
\Phi_2 &= \sum_{m\leqslant D} \nu(m) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} \log p,\\
\Phi_3 &= \sum_{m\leqslant D} \nu(m) \sum_{0<|k|\leqslant H} c(k) \sum_{(X/2)<p\leqslant X\, p+2\equiv 0\,(\mathrm{mod}\, m)} e\left(\alpha p^2 k\right)\log p,\\
c(k) &= \Delta^{-1} g(k)e(\beta k) \ll 1.
\end{aligned} \tag{57}$$

By Bombieri–Vinogradov's mean value theorem and (53), we have

$$\Phi_2 = \frac{X}{2} \sum_{m \leqslant D} \frac{\nu(m)}{\varphi(m)} + O\left(\frac{X}{\log^2 X}\right). \tag{58}$$

Using (49) and (52), we obtain

$$\begin{aligned}\sum_{m \leqslant D} \frac{\nu(m)}{\varphi(m)} &= \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \sum_{d|P(z)} \frac{\lambda_q^{+}(d)}{\varphi(q\, d)} = \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \sum_{d|P(z)} \frac{\lambda_q^{+}(d)}{\varphi(d)} \\ &\leqslant \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \Pi(z) \left(2e^{\gamma} \left(\frac{\log(D/q)}{\log z}\right)^{-1} + O\left((\log X)^{(-1/3)}\right)\right).\end{aligned} \tag{59}$$

Therefore, by (44), (58), and (59), we have

$$\begin{aligned}\Phi_2 \leqslant e^{\gamma} X \Pi(z) &\sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \left(\frac{\log(D/q)}{\log z}\right)^{-1} \\ &+ O\left(\frac{X}{(\log X)^{(4/3)-\varepsilon}}\right).\end{aligned} \tag{60}$$

Now, we find a lower bound for the sum Γ. From (30), (38), (41), (45), (51), (56), and (60), we derive that

$$\Gamma \geqslant e^{\gamma} \Delta X \Pi(z) \mathfrak{S} + O\left(\frac{\Delta X}{(\log X)^{(4/3)-\varepsilon}}\right) + O\left(\Delta \left|\Psi_3 - \kappa \Phi_3\right|\right), \tag{61}$$

where

$$\begin{aligned}\mathfrak{S} &= \frac{\log(s-1)}{s} - \kappa \sum_{z<q<y} \left(1 - \frac{\log q}{\log y}\right) \frac{1}{q-1} \left(\frac{\log(D/q)}{\log z}\right)^{-1}, \\ s &= \frac{\log D}{\log z}.\end{aligned} \tag{62}$$

Moreover, by partial summation and the prime number theorem, it is easy to show that

$$\mathfrak{S} = \mathfrak{S}_0 + O\left(\frac{1}{\log X}\right), \tag{63}$$

where

$$\mathfrak{S}_0 = \frac{\log(s-1)}{s} - \kappa\eta \int_{\eta}^{\rho} \left(\frac{1}{u} - \frac{1}{\rho}\right) \frac{1}{\delta - u} du. \tag{64}$$

According to simple numerical calculation, we know that

$$\mathfrak{S}_0 \geqslant 0.000032113949. \tag{65}$$

From (44), (61), and (63), we obtain

$$\Gamma \geqslant e^{\gamma} \Delta X \Pi(z) \mathfrak{S}_0 + O\left(\frac{\Delta X}{(\log X)^{(4/3)-\varepsilon}}\right) + O\left(\Delta \left|\Psi_3 - \kappa \Phi_3\right|\right). \tag{66}$$

We shall illustrate that if X runs over a suitable sequence, which tends to infinity, then the second error term in (66) can be absorbed. Hence, we need the following lemma.

Lemma 5. *Suppose that $\alpha \in (\mathbb{R}/\mathbb{Q})$ and δ, θ, D, and H are defined in (3) and (4). Let $\xi(d)$ and $c(k)$ be complex numbers defined for $d \leqslant D$ and $0 < |k| \leqslant H$, respectively, which satisfy*

$$\begin{aligned}\xi(d) &\ll 1, \\ c(k) &\ll 1.\end{aligned} \tag{67}$$

Then, there exists a sequence $\{X_j\}_{j=1}^{\infty}$ satisfying $\lim_{j \longrightarrow \infty} X_j = +\infty$, such that the sum $S(X)$ is defined by

$$S(X) = \sum_{d \leqslant D} \xi(d) \sum_{1 \leqslant |k| \leqslant H} c(k) \sum_{(X/2) < p \leqslant X\, p+2 \equiv 0 \, (\mathrm{mod}\, d)} (\log p) e\left(\alpha p^2 k\right), \tag{68}$$

which satisfies

$$S\left(X_j\right) \ll \frac{X_j}{\log^2 X_j}, \quad j = 1, 2, 3, \ldots. \tag{69}$$

The proof of Lemma 5 will be given in Section 5. From (42) and (57), we know that $\Psi_3 - \kappa \Phi_3$ can be represented as a sum of type (68) with

$$\xi(d) = \lambda^{*}(d) - \kappa \nu(d), \tag{70}$$

where

$$\lambda^{*}(d) = \begin{cases} \lambda^{-}(d), & \text{if } d | P(z), \\ 0, & \text{otherwise.} \end{cases} \tag{71}$$

According to Lemma 5 and (66), there exists a sequence $\{X_j\}_{j=1}^{\infty}$, which tends to infinity, such that

$$\Gamma\left(X_j\right) \geqslant e^{\gamma} \Delta X_j \Pi(z) \mathfrak{S}_0 + O\left(\frac{\Delta X_j}{\left(\log X_j\right)^{(4/3)-\varepsilon}}\right). \tag{72}$$

From (44) and (72), we know that there exists a positive constant $c > 0$ such that

$$\Gamma\left(X_j\right) \geqslant \frac{c\Delta\left(X_j\right)X_j}{\log X_j} > 0, \quad j = 1, 2, 3, \ldots. \tag{73}$$

This completes the proof of Theorem 1.

5. Proof of Lemma 5

In this section, we shall prove Lemma 5. Since $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, by Dirichlet's approximation theorem, there exist infinitely many integers A and natural numbers Q with $(A, Q) = 1$ such that

$$\left|\alpha - \frac{A}{Q}\right| < \frac{1}{Q^2}. \tag{74}$$

For each such Q, we choose X in a suitable way, i.e., as in (144). In this way, we construct our sequence $\left\{X_j\right\}_{j=1}^{\infty}$.

First, we have

$$S(X) = W + O\left(HX^{(1/2)+\varepsilon}\right), \tag{75}$$

where

$$W = \sum_{(X/2)<n\leqslant X} \Lambda(n) \sum_{1\leqslant|k|\leqslant H} c(k)e\left(\alpha n^2 k\right) \sum_{d\leqslant D,\, d|n+22|d} \xi(d). \tag{76}$$

According to Lemma 2, by taking $u = 2^{-7}X^{(\delta/2)}$, $v = 2^7 X^{(1/3)}$, and $w = X^{(1/2)-(\delta/4)}$, it is easy to see that the sum W can be decompose into $O(\log^{10} X)$ sums, each of which is either of Type I,

$$S_I = \sum_{M<m\leqslant M_1} a_m \sum_{\substack{L<\ell\leqslant L_1 \\ (X/2)<m\ell\leqslant X}} \sum_{1\leqslant|k|\leqslant H} c(k)e\left(\alpha m^2\ell^2 k\right) \sum_{d\leqslant D,\, d|m\ell+22|d} \xi(d), \tag{77}$$

with $M_1 \leqslant 2M$, $L_1 \leqslant 2L$, $L \geqslant w$, $a_m \ll m^{\varepsilon}$, and $ML \asymp X$, or of Type II

$$S_{II} = \sum_{M<m\leqslant M_1} a_m \sum_{\substack{L<\ell\leqslant L_1 \\ (X/2)<m\ell\leqslant X}} b_\ell \sum_{1\leqslant|k|\leqslant H} c(k)e\left(\alpha m^2\ell^2 k\right) \sum_{d\leqslant D,\, d|m\ell+22|d} \xi(d), \tag{78}$$

with $M_1 \leqslant 2M$, $L_1 \leqslant 2L$, $u \leqslant L \leqslant v$, $a_m \ll m^{\varepsilon}$, $b_\ell \ll \ell^{\varepsilon}$, and $ML \asymp X$.

Next, we shall deal with the sums of Type I and Type II in the following sections, respectively.

5.1. The Estimate of Type II Sums. In this section, we shall deal with the estimate of the sums of Type II. First, we have

$$\begin{aligned} S_{II} &= \sum_{1\leqslant|k|\leqslant H} c(k) \sum_{M<m\leqslant M_1} a_m \sum_{\substack{L<\ell\leqslant L_1 \\ (X/2)<m\ell\leqslant X}} b_\ell e\left(\alpha m^2\ell^2 k\right) \sum_{d\leqslant D,\, d|m\ell+22|d} \xi(d) \\ &\ll X^{\varepsilon} \sum_{1\leqslant|k|\leqslant H} \sum_{M<m\leqslant M_1} \left| \sum_{\substack{L<\ell\leqslant L_1 \\ (X/2)<m\ell\leqslant X}} b_\ell e\left(\alpha m^2\ell^2 k\right) \sum_{d\leqslant D,\, d|m\ell+22|d} \xi(d) \right|. \end{aligned} \tag{79}$$

By Cauchy's inequality, we obtain

$$\begin{aligned}
|S_{II}|^2 &\ll X^{\varepsilon} HM \sum_{1\leqslant |k|\leqslant H} \sum_{M<m\leqslant M_1} \left| \sum_{L<\ell\leqslant L_1\,(X/2)<m\ell\leqslant X} b_{\ell} e\left(\alpha m^2\ell^2 k\right) \sum_{d\leqslant D d|m\ell+22|d} \xi(d) \right|^2 \\
&= X^{\varepsilon} HM \sum_{1\leqslant |k|\leqslant H} \sum_{M<m\leqslant M_1} \sum_{L<\ell\leqslant L_1\,(X/2)<m\ell\leqslant X} \sum_{\substack{d_1,d_2\leqslant Dm\,\ell_1+2\equiv 0\,(\mathrm{mod}\,d_1) \\ m\ell_2+2\equiv 0\,(\mathrm{mod}\,d_2)\,(d_1 d_1,2)=1}} b_{\ell_1}\overline{b_{\ell_2}}\xi(d_1)\overline{\xi(d_2)} e\left(\alpha m^2 k\left(\ell_1^2-\ell_2^2\right)\right) \\
&\ll X^{\varepsilon} HM \sum_{1\leqslant |k|\leqslant H} \sum_{L<\ell_1,\ell_2\leqslant L_1} \sum_{\substack{d_1,d_2\leqslant D\,(d_1 d_2,2) \\ =1\,(\ell_1,d_1)=(\ell_2,d_2)=1}} \left| \sum_{\substack{M'<m\leqslant M_1' m\ell_1+2\equiv 0\,(\mathrm{mod}\,d_1) \\ m\ell_2+2\equiv 0\,(\mathrm{mod}\,d_2)}} e\left(\alpha m^2 k\left(\ell_1^2-\ell_2^2\right)\right) \right| \\
&\ll X^{\varepsilon} HM \sum_{1\leqslant |k|\leqslant H} \sum_{L<\ell_1,\ell_2\leqslant L_1} \sum_{\substack{d_1,d_2\leqslant D\,(d_1 d_2,2) \\ =1\,(\ell_1,d_1)=(\ell_2,d_2)=1}} |\mathscr{V}|,
\end{aligned} \tag{80}$$

where

$$\begin{aligned}
\mathscr{V} &= \sum_{\substack{M'<m\leqslant M_1' m\ell_1+2\equiv 0\,(\mathrm{mod}\,d_1) \\ m\ell_2+2\equiv 0\,(\mathrm{mod}\,d_2)}} e\left(\alpha m^2 k\left(\ell_1^2-\ell_2^2\right)\right), \\
M' &= \max\left(M, \frac{X}{2\ell_1}, \frac{X}{2\ell_2}\right), \\
M_1' &= \min\left(M_1, \frac{X}{\ell_1}, \frac{X}{\ell_2}\right).
\end{aligned} \tag{81}$$

If the system of the congruence,

$$\begin{cases} m\ell_1 + 2 \equiv 0 \ (\mathrm{mod}\, d_1), \\ m\ell_2 + 2 \equiv 0 \ (\mathrm{mod}\, d_2), \end{cases} \tag{82}$$

has no solution, then $\mathscr{V} = 0$. Assume that (82) has a solution. Then, there exists an $f_0 = f_0(\ell_1, \ell_2, d_1, d_2)$ such that (82) is equivalent to $m \equiv f_0 (\mathrm{mod}[d_1, d_2])$. In this case, we have

$$\begin{aligned}
|\mathscr{V}| &= \left| \sum_{M'<m\leqslant M_1' m\equiv f_0\,(\mathrm{mod}\,[d_1,d_2])} e\left(\alpha m^2 k\left(\ell_1^2-\ell_2^2\right)\right) \right| = \left| \sum_{\frac{M'-f_0}{[d_1,d_2]}<r\leqslant \frac{M_1'-f_0}{[d_1,d_2]}} e\left(\alpha\left(f_0 + r[d_1,d_2]\right)^2 k\left(\ell_1^2-\ell_2^2\right)\right) \right| \\
&= \left| \sum_{\frac{M'-f_0}{[d_1,d_2]}<r\leqslant \frac{M_1'-f_0}{[d_1,d_2]}} e\left(\alpha\left(r^2[d_1,d_2]^2 + 2f_0 r[d_1,d_2]\right)k\left(\ell_1^2-\ell_2^2\right)\right) \right| = \left| \sum_{R<r\leqslant R_1} e\left(\alpha\left(r^2[d_1,d_2]^2 + 2f_0 r[d_1,d_2]\right)k\left(\ell_1^2-\ell_2^2\right)\right) \right|,
\end{aligned} \tag{83}$$

where

$$\begin{aligned}
R &= \frac{M'-f_0}{[d_1,d_2]}, \\
R_1 &= \frac{M_1'-f_0}{[d_1,d_2]}.
\end{aligned} \tag{84}$$

The contribution of $\mathscr{V}$ with $\ell_1 = \ell_2$ to $|S_{II}|^2$ is

$$\begin{aligned}
&\ll X^{\varepsilon} HM^2 \sum_{1\leqslant k\leqslant H} \sum_{L<\ell_1,\ell_2\leqslant L_1 \ell_1=\ell_2} \sum_{d_1,d_2\leqslant D} \frac{1}{[d_1,d_2]} \\
&\ll X^{\varepsilon} H^2 M^2 L \sum_{d_1,d_2\leqslant D} \frac{1}{[d_1,d_2]} \ll X^{\varepsilon} H^2 M^2 L \sum_{h\leqslant D^2} \frac{\tau^2(h)}{h} \\
&\ll X^{\varepsilon} H^2 M^2 L \ll X^{1+\varepsilon} H^2 M.
\end{aligned} \tag{85}$$

Therefore, we have

$$\left|S_{II}\right|^2 \ll X^{1+\varepsilon}H^2M + X^{\varepsilon}HM \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}\ \sum_{\substack{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1\\ \ell_1\neq\ell_2}} |\mathscr{V}|. \tag{86}$$

Moreover, by Cauchy's inequality again, we obtain

$$\begin{aligned}
\left|S_{II}\right|^4 &\ll X^{2+\varepsilon}H^4M^2 + X^{\varepsilon}H^3M^2 \sum_{1\leqslant k\leqslant H}\left(\sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}\ \sum_{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1\ell_1\neq\ell_2} |\mathscr{V}|\right)^2\\
&\ll X^{2+\varepsilon}H^4M^2 + X^{\varepsilon}H^3M^2 \sum_{1\leqslant k\leqslant H}\left(\sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}\frac{1}{[d_1,d_2]}\right)\times \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}[d_1,d_2]\left(\sum_{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1\ell_1\neq\ell_2} |\mathscr{V}|\right)^2\\
&\ll X^{2+\varepsilon}H^4M^2 + X^{\varepsilon}H^3M^2L^2\left(\sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}\frac{1}{[d_1,d_2]}\right)\times \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}[d_1,d_2]\ \sum_{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1\ell_1\neq\ell_2} |\mathscr{V}|^2\\
&\ll X^{2+\varepsilon}H^4M^2 + X^{\varepsilon}H^3M^2L^2 \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}[d_1,d_2]\ \sum_{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1\ell_1\neq\ell_2} |\mathscr{V}|^2\\
&\ll X^{2+\varepsilon}H^4M^2 + X^{\varepsilon}H^3M^2L^2\cdot\Sigma_0,
\end{aligned} \tag{87}$$

where

$$\Sigma_0 = \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}[d_1,d_2]\ \sum_{\substack{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1\\ \ell_1\neq\ell_2}} \times \sum_{R<r_1,r_2\leqslant R_1} e\left(\alpha\left(\left(r_1^2-r_2^2\right)[d_1,d_2]^2 + 2f_0\left(r_1-r_2\right)[d_1,d_2]\right)k\left(\ell_1^2-\ell_2^2\right)\right). \tag{88}$$

For Σ_0, we have

$$\begin{aligned}\Sigma_0 = & \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}[d_1,d_2] \\ & \sum_{\substack{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1 \\ \ell_1\neq\ell_2}} \times \sum_{s_1,s_2}\left(\sum_{R<r_1,r_2\leqslant R_1 r_1-r_2=s_1 r_1+r_2=s_2} 1\right) e\left(\alpha\left(s_1s_2[d_1,d_2]^2+2f_0s_1[d_1,d_2]\right)k\left(\ell_1^2-\ell_2^2\right)\right) \\ = & \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}[d_1,d_2] \\ & \sum_{\substack{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1 \\ \ell_1\neq\ell_2}} \times \sum_{\substack{s_1,s_2:\ s_1\equiv s_2\ (\mathrm{mod}\ 2)2R<s_2+s_1 \\ \leqslant 2R_1 2R<s_2-s_1\leqslant 2R_1}} e\left(\alpha\left(s_1s_2[d_1,d_2]^2+2f_0s_1[d_1,d_2]\right)k\left(\ell_1^2-\ell_2^2\right)\right) \\ = & \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1}[d_1,d_2] \sum_{\substack{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1 \\ \ell_1\neq\ell_2}}\ \sum_{|s_1|\leqslant 2R_1-2R} e\left(2\alpha f_0s_1[d_1,d_2]k\left(\ell_1^2-\ell_2^2\right)\right) \\ & \times \sum_{\substack{s_2:\ s_2\equiv s_1\ (\mathrm{mod}\ 2)2R<s_2+s_1 \\ \leqslant 2R_1 2R<s_2-s_1\leqslant 2R_1}} e\left(\alpha s_1s_2[d_1,d_2]^2k\left(\ell_1^2-\ell_2^2\right)\right).\end{aligned} \tag{89}$$

Set

$$D_0 = X^{(50/3)\theta}. \tag{90}$$

Then, we divide Σ_0 into two parts

$$\Sigma_0 = \Sigma_1 + \Sigma_2, \tag{91}$$

where Σ_1 denotes the part of Σ_0 which satisfies $[d_1,d_2]\leqslant D_0$, while Σ_2 denotes the remaining part of Σ_0 which satisfies $[d_1,d_2]>D_0$. We set $s_2 = s_1+2t$ in Σ_1 and Σ_2 and derive that

$$\Sigma_1 \leqslant \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D_0\ (d_1d_2),2=1}[d_1,d_2] \sum_{\substack{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1 \\ \ell_1\neq\ell_2}} \times \sum_{|s_1|\leqslant 2R_1-2R}\left|\sum_{R'<t\leqslant R_1'} e\left(2\alpha s_1t[d_1,d_2]^2k\left(\ell_1^2-\ell_2^2\right)\right)\right|, \tag{92}$$

$$\Sigma_2 \leqslant \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant D\ (d_1d_2,2)=1\ [d_1,d_2]>D_0}[d_1,d_2] \sum_{\substack{L<\ell_1,\ell_2\leqslant L_1\ (\ell_1,d_1)=(\ell_2,d_2)=1 \\ \ell_1\neq\ell_2}} \times \sum_{|s_1|\leqslant 2R_1-2R}\left|\sum_{R'<t\leqslant R_1'} e\left(2\alpha s_1t[d_1,d_2]^2k\left(\ell_1^2-\ell_2^2\right)\right)\right|, \tag{93}$$

where

$$\begin{aligned}R' &= \max\left(R-s_1,R\right), \\ R_1' &= \min\left(R_1-s_1,R_1\right).\end{aligned} \tag{94}$$

First, we consider the upper bound for Σ_1. Let $\Sigma_1^{(1)}$ and $\Sigma_1^{(2)}$ denote the contribution of the right-hand side of (92) for $s_1\neq 0$ and $s_1=0$, respectively. Trivially, there holds

$$\Sigma_1^{(2)} \ll \mathrm{HML}^2 \sum_{d_1,d_2 \leqslant D_0\,(d_1 d_2,2)=1} 1 \ll \mathrm{HML}^2 D_0^2 \ll D_0^2 \mathrm{HXL}. \tag{95}$$

For $\Sigma_1^{(1)}$, by Lemma 3, we have

$$\begin{aligned}
\Sigma_1^{(1)} &\ll \sum_{1\leqslant k\leqslant H} \sum_{d_1,d_2\leqslant D_0} [d_1,d_2] \sum_{L<\ell_1,\ell_2\leqslant L_1 \ell_1 \neq \ell_2} \times \sum_{0<|s|\leqslant (2M/[d_1,d_2])} \min\left(\frac{M}{[d_1,d_2]}, \frac{1}{\left\|2\alpha s[d_1,d_2]^2 k\left(\ell_1^2-\ell_2^2\right)\right\|}\right) \\
&\ll \sum_{1\leqslant k\leqslant H} \sum_{h\leqslant D_0^2} h\left(\sum_{d_1,d_2\leqslant D_0\,[d_1,d_2]=h} 1\right) \sum_{L<\ell_1,\ell_2\leqslant L_1 \ell_1 \neq \ell_2} \sum_{0<|s|\leqslant (2M/h)} \min\left(\frac{M}{h}, \frac{1}{\left\|2\alpha s h^2 k\left(\ell_1^2-\ell_2^2\right)\right\|}\right) \\
&\ll D_0^2 \sum_{1\leqslant k\leqslant H} \sum_{h\leqslant D_0^2} \sum_{L<\ell_1,\ell_2\leqslant L_1 \ell_1 \neq \ell_2} \sum_{0<|s|\leqslant 2M} \min\left(M, \frac{1}{\left\|2\alpha s h^2 k\left(\ell_1^2-\ell_2^2\right)\right\|}\right) \\
&\ll D_0^2 \sum_{1\leqslant k\leqslant H} \sum_{h\leqslant D_0^2} \sum_{t_1,t_2} \left(\sum_{L<\ell_1,\ell_2\leqslant L_1 \ell_1-\ell_2=t_1 \ell_1+\ell_2=t_2 \ell_1 \neq \ell_2} 1\right) \sum_{0<|s|\leqslant 2M} \min\left(M, \frac{1}{\left\|2\alpha s h^2 k t_1 t_2\right\|}\right) \\
&\ll D_0^2 \sum_{1\leqslant k\leqslant H} \sum_{h\leqslant D_0^2} \sum_{1\leqslant |t_1|\leqslant L 1\leqslant t_2\leqslant 4L} \sum_{0<|s|\leqslant 2M} \min\left(M, \frac{1}{\left\|2\alpha s h^2 k t_1 t_2\right\|}\right) \\
&\ll D_0^2 \sum_{1\leqslant k\leqslant H} \sum_{h\leqslant D_0^2} \sum_{1\leqslant t_1,t_2\leqslant 4L} \sum_{1\leqslant |s|\leqslant 2M} \min\left(M, \frac{1}{\left\|2\alpha s h^2 k t_1 t_2\right\|}\right) \\
&\ll D_0^2 \sum_{1\leqslant m\leqslant 64 D_0^4 \mathrm{HML}^2} \tau_7(m) \min\left(M, \frac{1}{\|\alpha m\|}\right).
\end{aligned} \tag{96}$$

By Lemma 4, we have

$$\begin{aligned}
\sum_{1\leqslant m\leqslant 64 D_0^4 \mathrm{HML}^2} \tau_7(m) \min\left(M, \frac{1}{\|\alpha m\|}\right) &\ll X^{\varepsilon} \sum_{1\leqslant m\leqslant 64 D_0^4 \mathrm{HML}^2} \min\left(\frac{64 D_0^4 H M^2 L^2}{m}, \frac{1}{\|\alpha m\|}\right) \\
&\ll X^{\varepsilon} D_0^4 H M^2 L^2 \left(\frac{1}{Q} + \frac{1}{M} + \frac{Q}{D_0^4 H M^2 L^2}\right) \ll X^{\varepsilon}\left(\frac{H X^2 D_0^4}{Q} + H X L D_0^4 + Q\right).
\end{aligned} \tag{97}$$

Combining (92), (93), (96), and (97) and by noting the fact that $ML{\asymp}X$, we obtain

$$\Sigma_1 \ll X^{\varepsilon}\left(H X^2 D_0^6 Q^{-1} + H X L D_0^6 + Q D_0^2\right). \tag{98}$$

Now, we consider the estimate of Σ_2. According to (93), by a splitting argument, we have

$$\Sigma_2 \ll \mathscr{L} \max_{D_0 \ll T \ll D^2} \left(T \Sigma_2^{(1)}\right), \tag{99}$$

where

$$\Sigma_2^{(1)} = \Sigma_2^{(1)}(T) = \sum_{1\leqslant k\leqslant H} \sum_{d_1,d_2\leqslant D\,(d_1 d_2,2)=1 T<[d_1,d_2]\leqslant 2T} \sum_{L<\ell_1,\ell_2\leqslant L_1\,(\ell_1,d_1)=(\ell_2,d_2)=1 \ell_1 \neq \ell_2} \times \sum_{|s_1|\leqslant 2R_1-2R} \left| \sum_{R' < t\leqslant R_1'} e\left(2\alpha s_1 t [d_1,d_2]^2 k\left(\ell_1^2-\ell_2^2\right)\right) \right|. \tag{100}$$

By Lemma 1, we have

$$\begin{aligned}\Sigma_2^{(1)} = \Sigma_2^{(1)}(T) &\leqslant \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2\leqslant\ DT\ <[d_1,d_2]}\\ &\leqslant 2T \sum_{L<\ell_1,\ell_2\leqslant L_1 \ell_1\neq\ell_2}\ \sum_{|s|\leqslant(2M/T)} \times \int_{-\infty}^{+\infty}\mathscr{K}(\theta)\left|\sum_{(M/4T)<t\leqslant(4M/T)} e\left(2\alpha st[d_1,d_2]^2k\left(\ell_1^2-\ell_2^2\right)+\theta t\right)\right|\mathrm{d}\theta\\ &=: \int_{-\infty}^{+\infty}\mathscr{K}(\theta)\cdot\Sigma_2^{(2)}(\theta,T)\mathrm{d}\theta,\end{aligned} \tag{101}$$

where

$$\begin{aligned}\mathscr{K}(\theta) &= \min\left(\frac{15M}{4T}+1,\frac{1}{\pi|\theta|},\frac{1}{\pi^2\theta^2}\right),\\ \Sigma_2^{(2)}(\theta,T) &= \sum_{1\leqslant k\leqslant H}\ \sum_{d_1,d_2}\leqslant\ DT\ <[d_1,d_2]\leqslant 2T \sum_{L<\ell_1,\ell_2\leqslant L_1 \ell_1\neq\ell_2}\ \sum_{|s|\leqslant(2M/T)}\times\left|\sum_{(M/4T)<t\leqslant(4M/T)} e\left(2\alpha st[d_1,d_2]^2k\left(\ell_1^2-\ell_2^2\right)+\theta t\right)\right|.\end{aligned} \tag{102}$$

According to (7) and (101), it is easy to see that

$$\Sigma_2^{(1)} \ll \mathscr{L}\max_{0\leqslant\theta\leqslant 1}\Sigma_2^{(2)}(\theta,T). \tag{103}$$

For $\Sigma_2^{(2)}(\theta,T)$, we have

$$\begin{aligned}\Sigma_2^{(2)}(\theta,T) &= \sum_{1\leqslant k\leqslant H}\ \sum_{T<h\leqslant 2T}\left(\sum_{d_1,d_2\leqslant\ D[d_1,d_2]=h}1\right)\sum_{L<\ell_1,\ell_2\leqslant L_1\ell_1\neq\ell_2}\ \sum_{|s|\leqslant(2M/T)}\times\left|\sum_{(M/4T)<t\leqslant(4M/T)} e\left(2\alpha sth^2k\left(\ell_1^2-\ell_2^2\right)+\theta t\right)\right|\\ &\ll \sum_{1\leqslant k\leqslant H}\ \sum_{T<h\leqslant 2T}\tau^2(h)\sum_{L<\ell_1,\ell_2\leqslant L_1\ell_1\neq\ell_2}\ \sum_{|s|\leqslant(2M/T)}\left|\sum_{(M/4T)<t\leqslant(4M/T)} e\left(2\alpha sth^2k\left(\ell_1^2-\ell_2^2\right)+\theta t\right)\right|\\ &= \sum_{1\leqslant k\leqslant H}\ \sum_{T<h\leqslant 2T}\tau^2(h)\sum_{t_1,t_2}\left(\sum_{L<\ell_1,\ell_2\leqslant L_1\ell_1-\ell_2=t_1\ell_1+\ell_2=t_2\ell_1\neq\ell_2}1\right)\times\sum_{|s|\leqslant(2M/T)}\left|\sum_{(M/4T)<t\leqslant(4M/T)} e\left(2\alpha sth^2kt_1t_2+\theta t\right)\right|\\ &\ll \sum_{1\leqslant k\leqslant H}\ \sum_{T<h\leqslant 2T}\tau^2(h)\sum_{1\leqslant|t_1|,|t_2|\leqslant 4L}\ \sum_{|s|\leqslant(2M/T)}\left|\sum_{(M/4T)<t\leqslant(4M/T)} e\left(2\alpha sth^2kt_1t_2+\theta t\right)\right|\\ &\ll \mathscr{L}^3HML^2+\sum_{1\leqslant k\leqslant H}\ \sum_{T<h\leqslant 2T}\tau^2(h)\sum_{1\leqslant|t_1|,|t_2|}\leqslant 4L\sum_{1\leqslant|s|}\leqslant\frac{2M}{T}\times\left|\sum_{(M/4T)<t\leqslant(4M/T)} e\left(2\alpha sth^2kt_1t_2+\theta t\right)\right|\\ &\ll \mathscr{L}^3HML^2+\sum_{1\leqslant k\leqslant H}\ \sum_{T<h\leqslant 2T}\tau^2(h)\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\tau_3(|m|)\times\left|\sum_{(M/4T)<t\leqslant(4M/T)} e\left(2\alpha th^2km+\theta t\right)\right|\\ &= \mathscr{L}^3HML^2+\Sigma_2^{(3)},\ \text{say}.\end{aligned} \tag{104}$$

It follows from Cauchy's inequality that

$$\begin{aligned}
\left(\Sigma_2^{(3)}\right)^2 &\ll H\sum_{1\leqslant k\leqslant H}\left(\sum_{T<h\leqslant 2T}\tau^2(h)\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\tau_3(|m|)\left|\sum_{(M/4T)<t\leqslant(4M/T)}e\left(2\alpha th^2km+\theta t\right)\right|\right)^2\\
&\ll H\sum_{1\leqslant k\leqslant H}\left(\sum_{T<h\leqslant 2T}\tau^4(h)\right)\sum_{T<h\leqslant 2T}\left(\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\tau_3(|m|)\times\left|\sum_{(M/4T)<t\leqslant(4M/T)}e\left(2\alpha th^2km+\theta t\right)\right|\right)^2\\
&\ll H\left(\sum_{T<h\leqslant 2T}\tau^4(h)\right)\left(\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\tau_3^2(|m|)\right)\sum_{1\leqslant k\leqslant H}\sum_{T<h\leqslant 2T}\times\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\left|\sum_{(M/4T)<t\leqslant(4M/T)}e\left(2\alpha th^2km+\theta t\right)\right|^2\\
&\ll \mathscr{L}^{23}HML^2\sum_{1\leqslant k\leqslant H}\sum_{T<h\leqslant 2T}\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\left|\sum_{(M/4T)<t\leqslant(4M/T)}\left(2\alpha th^2km+\theta t\right)e\right|^2\\
&= \mathscr{L}^{23}HML^2\cdot\Sigma_2^{(4)},\ \text{say.}
\end{aligned}\tag{105}$$

For $\Sigma_2^{(4)}$, we have

$$\begin{aligned}
\Sigma_2^{(4)} &= \sum_{1\leqslant k\leqslant H}\sum_{T<h\leqslant 2T}\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\sum_{(M/4T)<t_1,t_2\leqslant(4M/T)}e\left(\left(2\alpha h^2km+\theta\right)\left(t_1-t_2\right)\right)\\
&\ll \sum_{1\leqslant k\leqslant H}\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\sum_{(M/4T)<t_1,t_2\leqslant(4M/T)}\left|\sum_{T<h\leqslant 2T}e\left(\left(2\alpha h^2km\right)\left(t_1-t_2\right)\right)\right|\\
&\ll \frac{HM^2L^2}{T}+\frac{M}{T}\sum_{1\leqslant k\leqslant H}\sum_{1\leqslant|m|\leqslant(32ML^2/T)}\sum_{1\leqslant|n|\leqslant(4M/T)}\left|\sum_{T<h\leqslant 2T}e\left(2\alpha h^2kmn\right)\right|\\
&\ll \frac{HM^2L^2}{T}+\frac{M}{T}\sum_{1\leqslant k\leqslant H}\sum_{1\leqslant|s|\leqslant(256M^2L^2/T^2)}\tau_3(|s|)\left|\sum_{T<h\leqslant 2T}e\left(\alpha h^2ks\right)\right|\\
&= \frac{HM^2L^2}{T}+\frac{M}{T}\cdot\Sigma_2^{(5)},\quad\text{say.}
\end{aligned}\tag{106}$$

By Cauchy's inequality, we deduce that

$$\begin{aligned}
\left(\Sigma_2^{(5)}\right)^2 &\ll H\sum_{1\leqslant k\leqslant H}\left(\sum_{1\leqslant s\leqslant(256M^2L^2/T^2)}\tau_3(s)\left|\sum_{T<h\leqslant 2T}e\left(\alpha h^2ks\right)\right|\right)^2\\
&\ll H\left(\sum_{1\leqslant s\leqslant(256M^2L^2/T^2)}\tau_3^2(s)\right)\sum_{1\leqslant k\leqslant H}\sum_{1\leqslant s\leqslant(256M^2L^2/T^2)}\left|\sum_{T<h\leqslant 2T}e\left(\alpha h^2ks\right)\right|^2\\
&\ll \frac{\mathscr{L}^8HM^2L^2}{T^2}\sum_{1\leqslant k\leqslant H}\sum_{1\leqslant s\leqslant(256M^2L^2/T^2)}\sum_{T<h_1,h_2\leqslant 2T}e\left(\alpha ks\left(h_1^2-h_2^2\right)\right)\\
&\ll \frac{\mathscr{L}^8H^2M^4L^4}{T^3}+\frac{\mathscr{L}^8HM^2L^2}{T^2}\sum_{1\leqslant k\leqslant H}\sum_{1\leqslant s\leqslant(256M^2L^2/T^2)}\sum_{T<h_1,h_2\leqslant 2T h_1\neq h_2}e\left(\alpha ks\left(h_1^2-h_2^2\right)\right)\\
&= \frac{\mathscr{L}^8H^2M^4L^4}{T^3}+\frac{\mathscr{L}^8HM^2L^2}{T^2}\cdot\Sigma_2^{(6)},\ \text{say.}
\end{aligned}\tag{107}$$

For $\Sigma_2^{(6)}$, from Lemma 3, we have

$$\begin{aligned}\Sigma_2^{(6)} &= \sum_{1\leqslant k\leqslant H}\sum_{t_1,t_2}\left(\sum_{T<h_1,h_2\leqslant 2T h_1-h_2=t_1 h_1+h_2=t_2 h_1\neq h_2} 1\right)\sum_{1\leqslant s\leqslant (256M^2L^2/T^2)} e(\alpha k s t_1 t_2)\\ &\ll \sum_{1\leqslant k\leqslant H}\sum_{1\leqslant t_1,t_2\leqslant 4T}\left|\sum_{1\leqslant s\leqslant (256M^2L^2/T^2)} e(\alpha k s t_1 t_2)\right| \ll \sum_{1\leqslant k\leqslant H}\sum_{1\leqslant t_1,t_2\leqslant 4T}\min\left(\frac{M^2L^2}{T^2},\frac{1}{\|\alpha k t_1 t_2\|}\right)\\ &\ll \sum_{1\leqslant n\leqslant 16HT^2}\tau_3(n)\min\left(\frac{M^2L^2}{T^2},\frac{1}{\|\alpha n\|}\right).\end{aligned} \tag{108}$$

It follows from Lemma 4 that

$$\begin{aligned}\sum_{1\leqslant n\leqslant 16HT^2}\tau_3(n)\min\left(\frac{M^2L^2}{T^2},\frac{1}{\|\alpha n\|}\right) &\ll X^{\varepsilon}\sum_{1\leqslant n\leqslant 16HT^2}\min\left(\frac{16HM^2L^2}{n},\frac{1}{\|\alpha n\|}\right)\\ &\ll X^{\varepsilon}HM^2L^2\left(\frac{1}{Q}+\frac{T^2}{M^2L^2}+\frac{Q}{HM^2L^2}\right) \ll X^{\varepsilon}\left(HX^2Q^{-1}+HT^2+Q\right).\end{aligned} \tag{109}$$

From (107), (108), and (109), we obtain

$$\Sigma_2^{(5)} \ll X^{\varepsilon}\left(\frac{HX^2}{T^{(3/2)}}+\frac{HX^2}{TQ^{(1/2)}}+HX+\frac{H^{(1/2)}XQ^{(1/2)}}{T}\right). \tag{110}$$

Putting (110) into (106), we obtain

$$\Sigma_2^{(4)} \ll X^{\varepsilon}\left(\frac{HX^2}{T}+\frac{HX^2M}{T^{(5/2)}}+\frac{HX^2M}{T^2Q^{1/2}}+\frac{HXM}{T}+\frac{H^{(1/2)}XQ^{(1/2)}M}{T^2}\right). \tag{111}$$

Combining (105) and (111), one has

$$\Sigma_2^{(3)} \ll X^{\varepsilon}\left(\frac{HX^{(3/2)}L^{(1/2)}}{T^{(1/2)}}+\frac{HX^2}{T^{(5/4)}}+\frac{H^{(1/2)}X^2}{TQ^{(1/4)}}+\frac{H^{(3/4)}X^{(3/2)}Q^{(1/4)}}{T}\right). \tag{112}$$

Inserting (112) into (104), we derive that

$$\Sigma_2^{(2)}(\theta,T) \ll X^{\varepsilon}\left(HXL+\frac{HX^{(3/2)}L^{(1/2)}}{T^{(1/2)}}+\frac{HX^2}{T^{(5/4)}}+\frac{H^{(1/2)}X^2}{TQ^{1/4}}+\frac{H^{(3/4)}X^{(3/2)}Q^{(1/4)}}{T}\right), \tag{113}$$

which combines (99) and (103) to obtain

$$\begin{aligned}\Sigma_2 &\ll X^{\varepsilon} \max_{D_0 \ll T \ll D^2}\left(HXLT + HX^{(3/2)}L^{(1/2)}T^{(1/2)} + HX^2T^{-(1/4)} + H^{(1/2)}X^2Q^{-(1/4)} + H^{(3/4)}X^{(3/2)}Q^{(1/4)}\right)\\ &\ll X^{\varepsilon}\left(HXLD^2 + HX^{(3/2)}L^{(1/2)}D + HX^2D_0^{-(1/4)} + H^{(1/2)}X^2Q^{-(1/4)} + H^{(3/4)}X^{(3/2)}Q^{(1/4)}\right).\end{aligned} \tag{114}$$

From (91), (98), and (114), we obtain

$$\Sigma_0 \ll X^{\varepsilon}\left(D_0^6HX^2Q^{-1} + D_0^6HXL + D_0^2Q + HXLD^2 + HX^{(3/2)}L^{(1/2)}D + HX^2D_0^{-(1/4)} + H^{(1/2)}X^2Q^{-(1/4)} + H^{(3/4)}X^{(3/2)}Q^{(1/4)}\right), \tag{115}$$

which combines (87) yields

$$\begin{aligned} S_{II} &\ll X^{\varepsilon}\left(\begin{array}{c} HXL^{-(1/2)} + D_0^{(3/2)}HXQ^{-(1/4)} + D_0^{(3/2)}HX^{(3/4)}L^{(1/4)} + D_0^{(1/2)}Q^{(1/4)}H^{(3/4)}X^{(1/2)} + \\ HX^{(3/4)}L^{(1/4)}D^{(1/2)} + HX^{(7/8)}L^{(1/8)}D^{(1/4)} + HXD_0^{-(1/16)} + H^{(7/8)}XQ^{-(1/16)} + H^{(15/16)}X^{(7/8)}Q^{(1/16)}\end{array}\right)\\ &\ll X^{\varepsilon}\left(\begin{array}{c} HXu^{-(1/2)} + D_0^{(3/2)}HXQ^{-(1/4)} + D_0^{(3/2)}HX^{(3/4)}v^{(1/4)} + D_0^{(1/2)}Q^{(1/4)}H^{(3/4)}X^{(1/2)} + HX^{(3/4)}v^{(1/4)}D^{(1/2)}\\ +HX^{(7/8)}v^{(1/8)}D^{(1/4)} + HXD_0^{-(1/16)} + H^{(7/8)}XQ^{-(1/16)} + H^{(15/16)}X^{(7/8)}Q^{(1/16)}\end{array}\right).\end{aligned} \tag{116}$$

5.2. The Estimate of Type I Sums. In this section, we shall deal with the estimate of the sums of Type I. First, we have

$$S_I = \sum_{1\leqslant |k|\leqslant H} c(k) \sum_{d\leqslant D\,(d,2)=1} \xi(d) \sum_{M<m\leqslant M_1} a_m \sum_{L' < \ell\leqslant L_1' m\ell+2 \equiv 0\,(\mathrm{mod}\, d)} e\left(\alpha m^2\ell^2 k\right), \tag{117}$$

where

$$\begin{aligned} L' &= \max\left(L, \frac{X}{2m}\right),\\ L_1' &= \min\left(L_1, \frac{X}{m}\right).\end{aligned} \tag{118}$$

By a splitting argument, there holds

$$S_I \ll X^{\varepsilon}\cdot \max_{1\ll T\ll D} \Sigma_3, \tag{119}$$

where

$$\Sigma_3 = \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1}\ \sum_{M<m\leqslant M_1\,(m,d)=1} \left| \sum_{L' < \ell\leqslant L_1' m\ell+2 \equiv 0\,(\mathrm{mod}\, d)} e\left(\alpha m^2\ell^2 k\right)\right|. \tag{120}$$

For $(m,d)=1$, there exists $\overline{m}$, which satisfies $0\leqslant \overline{m}\leqslant d-1$, such that $m\overline{m}\equiv 1\,(\mathrm{mod}\, d)$. Therefore, the equation $m\ell+2\equiv 0\,(\mathrm{mod}\, d)$ is equivalent to $\ell\equiv -2\overline{m}$ $(\mathrm{mod}\, d)$, i.e., $\ell = -2\overline{m} + dr$ for some $r\in\mathbb{Z}$. Then, it follows from Cauchy's inequality that

$$
\begin{aligned}
(\Sigma_3)^2 &\ll HMT \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1}\ \sum_{M<m\leqslant M_1\,(m,d)=1}\left|\sum_{L'<\ell\leqslant L_1' m\ell+2\equiv 0\,(\mathrm{mod}\,d)} e\left(\alpha m^2\ell^2 k\right)\right|^2\\
&\ll HMT \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)}\ \sum_{M<m\leqslant M_1\,(m,d)}\left|\sum_{\left(L'+2\overline{m}/d\right)<r\leqslant\left(L_1'+2\overline{m}/d\right)} e\left(\alpha m^2(-2\overline{m}+dr)^2 k\right)\right|^2\\
&= HMT \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1}\ \sum_{M<m\leqslant M_1\,(m,d)=1}\left|\sum_{\left(L'+2\overline{m}/d\right)<r\leqslant\left(L_1'+2\overline{m}/d\right)} e\left(\alpha m^2\left(d^2r^2-4\overline{m}dr\right)k\right)\right|^2\\
&= HMT \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1} = 1 \sum_{M<m\leqslant M_1\,(m,d)=1} = 1\times \sum_{\left(L'+2\overline{m}/d\right)<r\leqslant\left(L_1'+2\overline{m}/d\right)} e\left(\alpha m^2\left(d^2\left(r_1^2-r_2^2\right)-4\overline{m}d\left(r_1-r_2\right)\right)k\right).
\end{aligned}
\tag{121}
$$

Set

$$
\begin{aligned}
R &= \frac{L'+2\overline{m}}{d},\\
R_1 &= \frac{L_1'+2\overline{m}}{d}.
\end{aligned}
\tag{122}
$$

Then, we have

$$
\begin{aligned}
(\Sigma_3)^2 &\ll X^{\varepsilon}H^2M^2LT + HMT\left|\sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)}\ \sum_{M<m\leqslant M_1\,(m,d)=1}\times \sum_{R<r_1,r_2\leqslant R_1 r_1\neq r_2} e\left(\alpha m^2\left(d^2\left(r_1^2-r_2^2\right)-4\overline{m}d\left(r_1-r_2\right)\right)k\right)\right|\\
&\ll X^{\varepsilon}H^2M^2LT + HMT\cdot\left|\Sigma_3^{(1)}\right|,
\end{aligned}
\tag{123}
$$

where

$$
\Sigma_3^{(1)} = \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1}\ \sum_{M<m\leqslant M_1\,(m,d)=1}\ \sum_{R<r_1,r_2\leqslant R_1 r_1\neq r_2} e\left(\alpha m^2\left(d^2\left(r_1^2-r_2^2\right)-4\overline{m}d\left(r_1-r_2\right)\right)k\right).
\tag{124}
$$

For $\Sigma_3^{(1)}$, we have

$$
\begin{aligned}
\Sigma_3^{(1)} &= \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1}\ \sum_{M<m\leqslant M_1\,(m,d)=1}\ \sum_{s_1,s_2}\left(\sum_{R<r_1,r_2\leqslant R_1 r_1-r_2=s_1 r_1+r_2=s_2 r_1\neq r_2} 1\right) e\left(\alpha m^2\left(d^2 s_1 s_2-4\overline{m}ds_1\right)k\right)\\
&= \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1}\ \sum_{M<m\leqslant M_1\,(m,d)=1}\ \sum_{s_1,s_2 2R<s_1+s_2\leqslant 2R_1 2R<s_2-s_1\leqslant 2R_1 s_1\equiv s_2\,(\mathrm{mod}\,2), s_1\neq 0} e\left(\alpha m^2\left(d^2 s_1 s_2-4\overline{m}ds_1\right)k\right)\\
&\ll \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1}\ \sum_{M<m\leqslant M_1\,(m,d)=1}\ \sum_{1\leqslant\left|s_1\right|\leqslant(4L/T)}\left|\sum_{s_2:\ s_2\equiv s_1\,(\mathrm{mod}\,2) 2R-s_1<s_2\leqslant 2R_1-s_1 2R+s_1<s_2\leqslant 2R_1+s_1} e\left(\alpha m^2 d^2 s_1 s_2 k\right)\right|\\
&\ll \sum_{1\leqslant k\leqslant H}\ \sum_{T<d\leqslant 2T\,(d,2)=1}\ \sum_{M<m\leqslant M_1\,(m,d)=1}\ \sum_{1\leqslant\left|s_1\right|\leqslant(4L/T)}\left|\sum_{R-s_1<t\leqslant R_1-s_1 R<t\leqslant R_1} e\left(2\alpha m^2 d^2 s_1 t k\right)\right|.
\end{aligned}
\tag{125}
$$

Next, we will discuss the estimate of the right-hand side of (125) in two cases.

Case 1. Suppose that $MT \leqslant D_0$, and under this condition, there holds $1 \ll M, T \ll D_0$. By Lemma 4, we have

$$\begin{aligned}
\Sigma_3^{(1)} &\ll \sum_{1\leqslant k\leqslant H}\sum_{T<d\leqslant 2T}\sum_{M<m\leqslant M_1}\sum_{1\leqslant s\leqslant 8L}\min\left(L,\frac{1}{\|2\alpha m^2d^2sk\|}\right)\\
&\ll \sum_{1\leqslant k\leqslant H}\sum_{T<d\leqslant 2T}\sum_{M<m\leqslant M_1}\sum_{1\leqslant s\leqslant 8L}\min\left(\frac{256HM^2T^2L^2}{2m^2d^2sk},\frac{1}{\|2\alpha m^2d^2sk\|}\right)\\
&\ll \sum_{1\leqslant n\leqslant 256HM^2T^2L}\tau_7(n)\min\left(\frac{HM^2T^2L^2}{n},\frac{1}{\|\alpha n\|}\right)\ll X^{\varepsilon}HX^2T^2\left(\frac{1}{Q}+\frac{1}{L}+\frac{Q}{HX^2T^2}\right)\\
&\ll X^{\varepsilon}\left(\frac{HX^2D_0^2}{Q}+HX(MT)T+Q\right)\ll X^{\varepsilon}\left(\frac{HX^2D_0^2}{Q}+HXD_0^2+Q\right).
\end{aligned} \tag{126}$$

From (119), (123), and (126), we derive that, under the condition $MT \leqslant D_0$, there holds

$$S_I \ll X^{\varepsilon}\left(HX^{(1/2)}D_0^{(1/2)}+HXD_0^{(3/2)}Q^{-(1/2)}+HX^{(1/2)}D_0^{(3/2)}+H^{(1/2)}D_0^{(1/2)}Q^{(1/2)}\right). \tag{127}$$

Case 2. Now, we suppose that $MT > D_0$. Set

$$\begin{aligned}
R' &= \max(R, R-s_1),\\
R_1' &= \min(R_1, R_1-s_1).
\end{aligned} \tag{128}$$

Applying Lemma 1 to (125), we have

$$\begin{aligned}
\Sigma_3^{(1)} &\ll \sum_{1\leqslant k\leqslant H}\sum_{T<d\leqslant 2T}\sum_{M<m\leqslant M_1}\sum_{1\leqslant |s_1|\leqslant (4L/T)}\left|\sum_{R'<t\leqslant R_1'}e\left(2\alpha m^2d^2s_1tk\right)\right|\\
&\ll \sum_{1\leqslant k\leqslant H}\sum_{T<d\leqslant 2T}\sum_{M<m\leqslant M_1}\sum_{1\leqslant |s_1|\leqslant (4L/T)}\int_{-\infty}^{+\infty}\mathscr{K}(\theta)_1\left|\sum_{(L/4T)<t\leqslant (4L/T)}e\left(2\alpha m^2d^2s_1tk+\theta t\right)\right|\mathrm{d}\theta\\
&=\int_{-\infty}^{+\infty}\mathscr{K}_1(\theta)\cdot\Sigma_3^{(2)}(\theta,T)\mathrm{d}\theta,
\end{aligned} \tag{129}$$

where

$$\begin{aligned}
\mathscr{K}_1(\theta) &= \min\left(\frac{15L}{4T}+1,\frac{1}{\pi|\theta|},\frac{1}{\pi^2\theta^2}\right),\\
\Sigma_3^{(2)}(\theta,T) &= \sum_{1\leqslant k\leqslant H}\sum_{T<d\leqslant 2T}\sum_{M<m\leqslant M_1}\sum_{1\leqslant |s_1|\leqslant (4L/T)}\left|\sum_{(L/4T)<t\leqslant (4L/T)}e\left(2\alpha m^2d^2s_1tk+\theta t\right)\right|.
\end{aligned} \tag{130}$$

According to (7) and (129), it is easy to see that

$$\Sigma_3^{(1)} \ll \mathscr{L} \cdot \max_{0\leqslant\theta\leqslant 1} \Sigma_3^{(2)}(\theta, T). \tag{131}$$

For $\Sigma_3^{(2)}(\theta, T)$, we have

$$\Sigma_3^{(2)}(\theta, T) \ll \sum_{1\leqslant k\leqslant H} \sum_{MT<h\leqslant 4MT} \tau(h) \sum_{1\leqslant |s_1|\leqslant (4L/T)} \left| \sum_{(L/4T)<t\leqslant (4L/T)} e\left(2\alpha h^2 stk + \theta t\right)\right|. \tag{132}$$

It follows from Cauchy's inequality that

$$\begin{aligned}
\left(\Sigma_3^{(2)}(\theta, T)\right)^2 &\ll H\left(\sum_{MT<h\leqslant 4MT} \tau^2(h)\right)\left(\sum_{1\leqslant |s|\leqslant (4L/T)} 1\right) \sum_{1\leqslant k\leqslant H} \sum_{MT<h\leqslant 4MT} \times \sum_{1\leqslant |s|\leqslant (4L/T)} \left| \sum_{(L/4T)<t_1,t_2\leqslant (4L/T)} e\left(2\alpha h^2 stk + \theta t\right)\right|^2 \\
&\ll X^{\varepsilon} HML \sum_{1\leqslant k\leqslant H} \sum_{MT<h\leqslant 4MT} \sum_{1\leqslant |s|\leqslant (4L/T)} \sum_{(L/4T)<t_1,t_2\leqslant (4L/T)} e\left(\left(2\alpha h^2 sk + \theta\right)\left(t_1 - t_2\right)\right) \\
&\ll X^{\varepsilon} HML \sum_{1\leqslant k\leqslant H} \sum_{1\leqslant |s|\leqslant (4L/T)} \sum_{(L/4T)<t_1,t_2\leqslant (4L/T)} \left| \sum_{MT<h\leqslant 4MT} e\left(2\alpha h^2 sk\left(t_1 - t_2\right)\right)\right| \\
&\ll \frac{X^{\varepsilon} H^2 M^2 L^3}{T} + X^{\varepsilon} HML \cdot \Sigma_3^{(3)},
\end{aligned} \tag{133}$$

where

$$\Sigma_3^{(3)} = \sum_{1\leqslant k\leqslant H} \sum_{1\leqslant s\leqslant (4L/T)} \sum_{(L/4T)<t_1,t_2\leqslant (4L/T)} \left| \sum_{MT<h\leqslant 4MT} e\left(2\alpha h^2 sk\left(t_1 - t_2\right)\right)\right|. \tag{134}$$

For $\Sigma_3^{(3)}$, we have

$$\begin{aligned}
\Sigma_3^{(3)} &= \sum_{1\leqslant k\leqslant H} \sum_{1\leqslant s\leqslant (4L/T)} \sum_{1\leqslant |r_1|\leqslant (4L/T)} \sum_{1\leqslant r_2\leqslant (8L/T)} \left(\sum_{\substack{(4L/T)<t_1,t_2\leqslant (4L/T) \\ t_1-t_2=r_1,\, t_1+t_2=r_2}} 1 \right) \left| \sum_{MT<h\leqslant 4MT} e\left(2\alpha h^2 skr_1\right)\right| \\
&\ll \frac{L}{T} \sum_{1\leqslant k\leqslant H} \sum_{1\leqslant s\leqslant (4L/T)} \sum_{1\leqslant s\leqslant (4L/T)} \left| \sum_{MT<h\leqslant 4MT} e\left(2\alpha h^2 skr_1\right)\right| \\
&\ll \frac{L}{T} \sum_{1\leqslant n\leqslant (32HL^2/T^2)} \tau_4(n) \left| \sum_{MT<h\leqslant 4MT} e\left(\alpha h^2 n\right)\right|.
\end{aligned} \tag{135}$$

Therefore, by Cauchy's inequality, one has

$$\begin{aligned}\left(\Sigma_3^{(3)}\right)^2 &\ll \frac{L^2}{T^2}\left(\sum_{1\leqslant n\leqslant (32HL^2/T^2)} \tau_4^2(n)\right)\left(\sum_{1\leqslant n\leqslant (32HL^2/T^2)}\left|\sum_{MT<h\leqslant 4MT} e\left(\alpha h^2 n\right)\right|^2\right)\\ &\ll \frac{X^{\varepsilon}HL^4}{T^4}\sum_{1\leqslant n\leqslant (32HL^2/T^2)}\ \sum_{MT<h_1,h_2\leqslant 4MT} e\left(\alpha\left(h_1^2-h_2^2\right)n\right) \ll \frac{X^{\varepsilon}H^2ML^6}{T^5}+\frac{X^{\varepsilon}HL^4}{T^4}\cdot\Sigma_3^{(4)},\end{aligned} \tag{136}$$

where

$$\Sigma_3^{(4)} = \sum_{MT<h_1,h_2\leqslant 4MT h_1\neq h_2}\left|\sum_{1\leqslant n\leqslant (32HL^2/T^2)} e\left(\alpha\left(h_1^2-h_2^2\right)n\right)\right|. \tag{137}$$

For $\Sigma_3^{(4)}$, by Lemma 3, we have

$$\begin{aligned}\Sigma_3^{(4)} &= \sum_{1\leqslant |t_1|,|t_2|\leqslant 8MT}\left(\sum_{MT<h_1,h_2\leqslant 4MT h_1-h_2=t_1 h_1+h_2=t_2} 1\right)\left|\sum_{1\leqslant n\leqslant (32HL^2/T^2)} e\left(\alpha t_1 t_2 n\right)\right|\\ &\ll \sum_{1\leqslant t_1,t_2\leqslant 8MT}\left|\sum_{1\leqslant n\leqslant (32HL^2/T^2)} e\left(\alpha t_1 t_2 n\right)\right| \ll \sum_{1\leqslant t_1,t_2\leqslant 8MT}\min\left(\frac{HL^2}{T^2},\frac{1}{\|\alpha t_1 t_2\|}\right)\\ &\ll \sum_{1\leqslant t\leqslant 64M^2T^2}\tau(t)\min\left(\frac{HL^2}{T^2},\frac{1}{\|\alpha t\|}\right).\end{aligned} \tag{138}$$

It follows from Lemma 4 that

$$\begin{aligned}\sum_{1\leqslant t\leqslant 64M^2T^2}\tau(t)\min\left(\frac{HL^2}{T^2},\frac{1}{\|\alpha t\|}\right) &\ll X^{\varepsilon}\sum_{1\leqslant t\leqslant 64M^2T^2}\min\left(\frac{64HM^2L^2}{t},\frac{1}{\|\alpha t\|}\right)\\ &\ll X^{\varepsilon}HM^2L^2\left(\frac{1}{Q}+\frac{T^2}{HL^2}+\frac{Q}{HM^2L^2}\right) \ll X^{\varepsilon}\left(\frac{HX^2}{Q}+M^2T^2+Q\right).\end{aligned} \tag{139}$$

From (136), (138), and (139), we derive that

$$\Sigma_3^{(3)} \ll X^{\varepsilon}\left(\frac{HX^{(1/2)}L^{(5/2)}}{T^{(5/2)}}+\frac{HXL^2}{T^2Q^{(1/2)}}+\frac{H^{(1/2)}XL}{T}+\frac{H^{(1/2)}L^2Q^{(1/2)}}{T^2}\right), \tag{140}$$

which combines (133) yields

$$\Sigma_3^{(2)}(\theta,T) \ll X^{\varepsilon}\left(\frac{HXL^{(1/2)}}{T^{(1/2)}}+\frac{HX^{(3/4)}L^{(5/4)}}{T^{(5/4)}}+\frac{HXL}{TQ^{(1/4)}}+\frac{H^{(3/4)}X^{(1/2)}LQ^{(1/4)}}{T}\right). \tag{141}$$

From (123), (131), and (141), we obtain

$$\Sigma_3 \ll X^{\varepsilon}\left(HXL^{-(1/2)}T^{(1/2)} + HXL^{-(1/4)}T^{(1/4)} + HX^{(7/8)}L^{(1/8)}T^{-(1/8)} + HXQ^{-(1/8)} + H^{(7/8)}Q^{(1/8)}X^{(3/4)}\right), \tag{142}$$

from which and (139), we derive that, under the condition $MT > D_0$, and there holds

$$\begin{aligned} S_I &\ll X^{\varepsilon} \max_{1\ll T\ll D}\left(\frac{HXT^{(1/2)}}{L^{(1/2)}} + \frac{HXT^{(1/4)}}{L^{(1/4)}} + \frac{HX^{(7/8)}L^{(1/8)}}{T^{(1/8)}} + \frac{HX}{Q^{(1/8)}} + H^{(7/8)}Q^{(1/8)}X^{(3/4)}\right) \\ &\ll X^{\varepsilon}\left(\frac{HXD^{(1/2)}}{w^{(1/2)}} + \frac{HXD^{(1/4)}}{w^{(1/4)}} + \frac{HX^{(7/8)}L^{(1/8)}M^{(1/8)}}{(MT)^{(1/8)}} + \frac{HX}{Q^{(1/8)}} + H^{(7/8)}Q^{(1/8)}X^{(3/4)}\right) \\ &\ll X^{\varepsilon}\left(HXw^{-(1/2)}D^{(1/2)} + HXw^{-(1/4)}D^{(1/4)} + HXD_0^{-(1/8)} + HXQ^{-(1/8)} + H^{(7/8)}Q^{(1/8)}X^{(3/4)}\right). \end{aligned} \tag{143}$$

5.3. Proof of Lemma 5. From (116), (127), and (143), by taking

$$Q = X^{(4138/15)\theta}, \tag{144}$$

then we deduce that, under conditions (3) and (4), there holds

$$S_I \ll X^{1-\varpi} \text{ and } S_{II} \ll X^{1-\varpi}, \tag{145}$$

for some $\varpi > 0$. This completes the proof of Lemma 5.

Data Availability

The data used to support the findings of the study available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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