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Research Article

A Hybrid Mean Value Involving Dedekind Sums and the Generalized Kloosterman Sums

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In this paper, we use the mean value theorem of Dirichlet L-functions and the properties of Gauss sums and Dedekind sums to study the hybrid mean value problem involving Dedekind sums and the general Kloosterman sums and give an interesting identity for it.

1. Introduction

Let q be a natural number and h be an integer coprime to q. The classical Dedekind sums

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right), \tag{1}$$

where

$$((x))=\begin{cases} x-[x]-\dfrac{1}{2}, & \text{if } x \text{ is not an integer},\\ 0, & \text{if } x \text{ is an integer},\end{cases} \tag{2}$$

describes the behaviour of the logarithm of the η-function (see [1, 2]) under modular transformations. There are many papers written on their various properties (see the examples in [3–10] and [11]).

In particular, Zhang and Liu [12] studied the hybrid mean value problems related to Dedekind sums and Kloosterman sums:

$$K(m,n;q)=\sum_{a=1}^{q}{}' e\left(\frac{ma+n\bar{a}}{q}\right), \tag{3}$$

where $q\geq 3$ is an integer, $\sum_{a=1}^{q'}$ denotes the summation over all $1\leq a\leq q$ with $(a,q)=1$, $e(y)=e^{2\pi i y}$, and $\bar{a}$ denotes the multiplicative inverse of $a \bmod q$. They proved the following results:

Theorem 1. *Let p be an odd prime, then one has the identity*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1;p)|^2\cdot|K(b,1;p)|^2\cdot S(a\bar{b},p)=\begin{cases}\dfrac{1}{12}\cdot p^2\cdot\left((p-1)(p-2)-12\cdot h_p^2\right), & \text{if } p\equiv 3 \bmod 4,\\ \dfrac{1}{12}\cdot p^2\cdot(p-1)(p-2), & \text{if } p\equiv 1 \bmod 4,\end{cases} \tag{4}$$

where h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

Theorem 2. *Let p be an odd prime, then one has the asymptotic formula:*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1;p)|^2\cdot|K(b,1;p)|^2\cdot|S(a\overline{b},q)|^2=\frac{1}{24}p^5+O\left(p^4\cdot\exp\left(\frac{3\ln\ \ln p}{\ln p}\right)\right), \tag{5}$$

where $\exp(y)=e^y$.

It is natural that people will ask, for the general Kloosterman sums

$$K(m,n,\chi;p)=\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma+n\overline{a}}{p}\right), \tag{6}$$

what will happen? Whether there exists an identity similar to Theorem1? Here, χ denotes any Dirichlet character mod p.

The main purpose of this paper is to answer these questions. That is, we shall use the mean value theorem of Dirichlet L-functions and the properties of Gauss sums and Dedekind sums to prove the following.

Theorem 3. *Let p be an odd prime with $p\equiv 1 \bmod 4$. Then, for any Dirichlet character $\chi \bmod p$, we have the identity:*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p)=\begin{cases}\frac{1}{12}\cdot p^2(p-1)(p-2), & if\ \chi(-1)=1,\\ \frac{1}{12}\cdot p^2(p-1)(p-2)-\frac{p^3}{\pi^2}\cdot|L(1,\chi)|^2, & if\ \chi(-1)=-1.\end{cases} \tag{7}$$

Theorem 4. *Let p be an odd prime with $p\equiv 3 \bmod 4$. Then, for any Dirichlet character $\chi \bmod p$, we have the identity:*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p)=\begin{cases}\frac{1}{12}\cdot p^2(p-1)(p-2)-p^2\cdot h_p^2, & if\ \chi(-1)=1,\\ \frac{p^2(p-1)(p-2)}{12}-p^2\cdot h_p^2-\frac{2p^3}{\pi^2}\cdot|L(1,\chi)|^2, & if\ \chi(-1)=-1\ and\ \chi\neq\chi_2,\\ \frac{p^2(p-1)(p-2)}{12}+p^2\cdot(p-4)\cdot h_p^2, & if\ \chi=\chi_2,\end{cases} \tag{8}$$

where $\chi_2=(*/p)$ denotes the Legendre symbol and h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

It is clear that if $\chi=\chi_0$, then $K(a,1,\chi;p)=K(a,1;p)$. Note that $\chi_0(-1)=1$, from Theorems 3 and 4, we may immediately deduce Theorem1 in [12], so our results are the generalization of [12].

2. Several Lemmas

In this section, we shall give several simple lemmas, which are necessary to the proofs of our theorems. Hereafter, we shall use many properties of character sums and Gauss sums, and all of these can be found in reference [13]. First, we have the following.

Lemma 1. *Let $p>3$ be a prime, χ be any fixed Dirichlet character $\bmod p$. Then, for any nonprincipal character $\chi_1 \bmod p$ with $\chi\chi_1\neq\chi_0$, we have the identity:*

$$\sum_{m=1}^{p-1}\chi_1(m)\cdot|K(m,1,\chi;p)|^2=\frac{\overline{\chi}_1(-1)\cdot\tau^2(\chi_1)\cdot\tau\left(\overline{\chi}_1^2\right)\cdot\tau(\overline{\chi}\chi_1)}{\tau(\overline{\chi}\overline{\chi}_1)}, \tag{9}$$

where χ_0 denotes the principal character $\bmod p$, $\tau(\chi)$ denotes the Gauss sums defined as $\tau(\chi)=\sum_{a=1}^{p-1}\chi(a)e(a/p)$, and $\overline{\chi}$ denotes the complex conjugate of χ.

Proof. From the definition of Kloosterman sums and the properties of Gauss sums, we have

$$\sum_{m=1}^{p-1}\chi_1(m)\cdot|K(m,1,\chi;p)|^2=\sum_{m=1}^{p-1}\chi_1(m)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\overline{\chi}(b)e\left(\frac{am-mb+\overline{a}-\overline{b}}{p}\right)$$
$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\sum_{m=1}^{p-1}\chi_1(m)e\left(\frac{mb(a-1)+\overline{b}\cdot(\overline{a}-1)}{p}\right)=\tau(\chi_1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\overline{\chi}_1(b)\overline{\chi}_1(a-1)e\left(\frac{\overline{b}\cdot(\overline{a}-1)}{p}\right)$$
$$=\tau(\chi_1)\sum_{a=1}^{p-1}\chi(a)\overline{\chi}_1(a-1)\sum_{b=1}^{p-1}\overline{\chi}_1(b)e\left(\frac{\overline{b}\cdot(\overline{a}-1)}{p}\right)=\tau^2(\chi_1)\sum_{a=1}^{p-1}\chi(a)\overline{\chi}_1(a-1)\overline{\chi}_1(\overline{a}-1)$$
$$=\overline{\chi}_1(-1)\tau^2(\chi_1)\sum_{a=1}^{p-1}\chi(a)\chi_1(a)\overline{\chi}_1^2(a-1)=\overline{\chi}_1(-1)\tau^2(\chi_1)\sum_{a=1}^{p-1}\chi(a+1)\chi_1(a+1)\overline{\chi}_1^2(a). \tag{10}$$

On the other hand, from the properties of Gauss sums, we have

$$\sum_{a=1}^{p-1}\chi(a+1)\chi_1(a+1)\overline{\chi}_1^2(a)=\frac{1}{\tau(\overline{\chi\chi_1})}\sum_{a=1}^{p-1}\overline{\chi}_1^2(a)\sum_{b=1}^{p-1}\overline{\chi\chi_1}(b)e\left(\frac{b(a+1)}{p}\right)$$
$$=\frac{1}{\tau(\overline{\chi\chi_1})}\sum_{b=1}^{p-1}\overline{\chi\chi_1}(b)e\left(\frac{b}{p}\right)\sum_{a=1}^{p-1}\overline{\chi}_1^2(a)e\left(\frac{ba}{p}\right)=\frac{\tau(\overline{\chi}_1^2)}{\tau(\overline{\chi\chi_1})}\sum_{b=1}^{p-1}\overline{\chi}(b)\chi_1(b)e\left(\frac{b}{p}\right)=\frac{\tau(\overline{\chi}_1^2)\cdot\tau(\overline{\chi}\chi_1)}{\tau(\overline{\chi\chi_1})}. \tag{11}$$

Combining (10) and (11), we may immediately deduce the identity:

$$\sum_{m=1}^{p-1}\chi_1(m)\cdot|K(m,1,\chi;p)|^2=\frac{\overline{\chi}_1(-1)\cdot\tau^2(\chi_1)\cdot\tau(\overline{\chi}_1^2)\cdot\tau(\overline{\chi}\chi_1)}{\tau(\overline{\chi\chi_1})}. \tag{12}$$

This proves Lemma 1.

Lemma 2. *Let p be a prime with $p\equiv 1 \bmod 4$ and χ be any odd character mod p. Then, we have the identity:*

$$\sum_{m=1}^{p-1}\overline{\chi}(m)\cdot|K(m,1,\chi;p)|^2=\tau^2(\overline{\chi}). \tag{13}$$

Proof. Since $p\equiv 1 \bmod 4$ and χ is an odd character mod p, we know χ is not the Legendre symbol and $\chi^2\neq\chi_0$. Note that $\chi(-1)=-1$, from (10), we have

$$\sum_{m=1}^{p-1}\overline{\chi}(m)\cdot|K(m,1,\chi;p)|^2=\tau^2(\overline{\chi})\sum_{a=1}^{p-1}\chi(a)\chi(a-1)\chi(\overline{a}-1)$$
$$=\chi(-1)\tau^2(\overline{\chi})\sum_{a=1}^{p-1}\chi^2(a-1)=-\tau^2(\overline{\chi})\left(\sum_{a=0}^{p-1}\chi^2(a)-1\right)=\tau^2(\overline{\chi}). \tag{14}$$

This proves Lemma 2.

Lemma 3. *Let p be a prime with $p \equiv 3 \bmod 4$ and χ_2 be the Legendre symbol. Then, we have the identity:*

$$\sum_{m=1}^{p-1} \chi_2(m)\left|K(m,1,\chi_2;p)\right|^2 = -(p-2)\cdot\tau^2(\chi_2). \tag{15}$$

Proof. Note that $\chi_2 = \overline{\chi}_2$, $\chi_2^2 = \chi_0$, and $\chi_2(-1) = -1$, and from the definition of $K(m,1,\chi_2;p)$ and the properties of Gauss sums, we have

$$\begin{aligned}\sum_{m=1}^{p-1} \chi_2(m)\left|K(m,1,\chi_2;p)\right|^2 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(ab)\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{m(a-b)+\overline{a}-\overline{b}}{p}\right)\\ &= \tau(\chi_2)\sum_{a=1}^{p-1}\chi_2(a)\sum_{b=1}^{p-1}\chi_2(b(a-1))e\left(\frac{\overline{b}(\overline{a}-1)}{p}\right) = \tau^2(\chi_2)\sum_{a=1}^{p-1}\chi_2(a)\chi_2(a-1)\chi_2(\overline{a}-1) = -(p-2)\cdot\tau^2(\chi_2).\end{aligned} \tag{16}$$

This proves Lemma 3.

Lemma 4. *Let $q > 2$ be an integer, then for any integer a with $(a,q) = 1$, we have the identity:*

$$S(a,q) = \frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)\left|L(1,\chi)\right|^2, \tag{17}$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to the character $\chi \bmod d$.

Proof. See Lemma 2 of [7].

3. Proof of the Theorems

In this section, we will complete the proof of our theorems. First we prove Theorem 3. From Lemma 4 and the definition of $S(a,p)$, we have

$$S(a,p) = \frac{1}{\pi^2}\cdot\frac{p}{p-1}\cdot\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(a)\left|L(1,\chi)\right|^2, \tag{18}$$

and (with $a = 1$)

$$\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2 = \frac{\pi^2\cdot(p-1)}{p}\cdot S(1,p) = \frac{\pi^2\cdot(p-1)}{p}\cdot\sum_{a=1}^{p-1}\left(\frac{a}{p}-\frac{1}{2}\right)^2 = \frac{\pi^2}{12}\cdot\frac{(p-1)^2(p-2)}{p^2}. \tag{19}$$

Since $p \equiv 1 \bmod 4$, we know the Legendre symbol $(*/p) = \chi_2$ is an even character $\bmod\, p$. Note that, for any nonprincipal character $\chi \bmod p$, $|\tau(\chi)| = \sqrt{p}$. So, if χ is an even character $\bmod\, p$, then from Lemma 1, (18), and (19), we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p) &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod d\\ \chi_1(-1)=-1}}\left|\sum_{a=1}^{p-1}\chi_1(a)|K(a,1,\chi;p)|^2\right|^2\cdot\left|L(1,\chi_1)\right|^2\\ &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod d\\ \chi_1(-1)=-1}}\left|\frac{\tau^2(\chi_1)\cdot\tau(\overline{\chi}_1^2)\cdot\tau(\overline{\chi}\chi_1)}{\tau(\overline{\chi}\overline{\chi}_1)}\right|^2\cdot\left|L(1,\chi_1)\right|^2 = \frac{p^4\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod d\\ \chi_1(-1)=-1}}\left|L(1,\chi_1)\right|^2\\ &= \frac{1}{12}\cdot p^2(p-1)(p-2).\end{aligned} \tag{20}$$

If χ is an odd character $\bmod\, p$, then note that the identity:

$$\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}} f(\chi_1) = \sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1\\ \chi_1\chi\neq\chi_0}} f(\chi_1) + f(\overline{\chi}), \tag{21}$$

from (18), (19), Lemmas 1 and 2, and the method of proving (20), we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p) &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\left|\sum_{a=1}^{p-1}\chi_1(a)|K(a,1,\chi;p)|^2\right|^2\cdot\left|L(1,\chi_1)\right|^2\\
&= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1\\ \chi_1\chi\neq\chi_0}}\left|\sum_{a=1}^{p-1}\chi_1(a)|K(a,1,\chi;p)|^2\right|^2\cdot\left|L(1,\chi_1)\right|^2 + \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1\\ \chi_1\chi=\chi_0}}\left|\sum_{a=1}^{p-1}\chi_1(a)|K(a,1,\chi;p)|^2\right|^2\cdot\left|L(1,\chi_1)\right|^2\\
&= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1\\ \chi_1\chi\neq\chi_0}}\left|\frac{\tau^2(\chi_1)\cdot\tau(\overline{\chi}_1^2)\cdot\tau(\overline{\chi}\chi_1)}{\tau(\overline{\chi}\overline{\chi}_1)}\right|^2\cdot\left|L(1,\chi_1)\right|^2 + \frac{p\cdot\pi^{-2}}{p-1}\left|\tau^2(\overline{\chi})\right|^2\cdot|L(1,\overline{\chi})|^2\\
&= \frac{p^4\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\left|L(1,\chi_1)\right|^2 - \frac{p^4\cdot\pi^{-2}}{p-1}|L(1,\chi)|^2 + \frac{p^3\cdot\pi^{-2}}{p-1}|L(1,\chi)|^2 = \frac{1}{12}\cdot p^2(p-1)(p-2) - \frac{p^3}{\pi^2}\cdot|L(1,\chi)|^2,\end{aligned} \tag{22}$$

where $\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1\\ \chi_1\chi\neq\chi_0}}$ denotes the summation over all odd characters χ_1 with $\chi_1\neq\overline{\chi}$.

Combining (20) and (22), we may immediately deduce the identity:

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p) = \begin{cases}\dfrac{1}{12}\cdot p^2(p-1)(p-2), & if\ \chi(-1)=1,\\[2ex] \dfrac{1}{12}\cdot p^2(p-1)(p-2) - \dfrac{p^3}{\pi^2}\cdot|L(1,\chi)|^2, & if\ \chi(-1)=-1.\end{cases} \tag{23}$$

This proves Theorem 3.

Now, we prove Theorem 4. Since $p\equiv 3 \bmod 4$, we know the Legendre symbol χ_2 is an odd character $\bmod p$ and $|\tau(\overline{\chi}_2^2)|=1$. If χ is an even character $\bmod p$, then note that $L(1,\chi_2)=(\pi\cdot h_p/\sqrt{p})$ (see reference [14]), from (18), (19), Lemma 1 and the properties of Gauss sums, we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p) &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\left|\frac{\tau^2(\chi_1)\cdot\tau(\overline{\chi}_1^2)\cdot\tau(\overline{\chi}\chi_1)}{\tau(\overline{\chi\chi}_1)}\right|^2\cdot\left|L(1,\chi_1)\right|^2\\ &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1\\ \chi_1\neq\chi_2}}\left|\frac{\tau^2(\chi_1)\cdot\tau(\overline{\chi}_1^2)\cdot\tau(\overline{\chi}\chi_1)}{\tau(\overline{\chi\chi}_1)}\right|^2\cdot\left|L(1,\chi_1)\right|^2+\frac{p^3\cdot\pi^{-2}}{p-1}\cdot\left|L(1,\chi_2)\right|^2\\ &= \frac{p^4\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\left|L(1,\chi_1)\right|^2+\frac{p^3\cdot\pi^{-2}}{p-1}\cdot\left|L(1,\chi_2)\right|^2-\frac{p^4\cdot\pi^{-2}}{p-1}\cdot\left|L(1,\chi_2)\right|^2\\ &= \frac{1}{12}\cdot p^2(p-1)(p-2)-p^2\cdot h_p^2,\end{aligned} \tag{24}$$

where h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

If χ is an odd nonreal character mod p, then from (18), (19), Lemmas 1 and 2, and the method of proving (22) and (24), we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p) &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\left|\sum_{a=1}^{p-1}\chi_1(a)|K(a,1,\chi;p)|^2\right|^2\cdot\left|L(1,\chi_1)\right|^2\\ &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1\\ \chi_1\chi\neq\chi_0}}\left|\frac{\tau^2(\chi_1)\cdot\tau(\overline{\chi}_1^2)\cdot\tau(\overline{\chi}\chi_1)}{\tau(\overline{\chi\chi}_1)}\right|^2\cdot\left|L(1,\chi_1)\right|^2+\frac{p\cdot\pi^{-2}}{p-1}\left|\sum_{a=1}^{p-1}\overline{\chi}(a)|K(a,1,\chi;p)|^2\right|^2\cdot\left|L(1,\overline{\chi})\right|^2\\ &= \frac{p^4\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\left|L(1,\chi_1)\right|^2+\frac{p^3\cdot\pi^{-2}}{p-1}\cdot|L(1,\overline{\chi})|^2+\frac{p^3\cdot\pi^{-2}}{p-1}\cdot|L(1,\chi)|^2+\frac{p^3\cdot\pi^{-2}}{p-1}\cdot\left|L(1,\chi_2)\right|^2\\ &\quad-\frac{2p^4\cdot\pi^{-2}}{p-1}\cdot|L(1,\chi)|^2-\frac{p^4\cdot\pi^{-2}}{p-1}\cdot\left|L(1,\chi_2)\right|^2=\frac{1}{12}\cdot p^2(p-1)(p-2)\\ &\quad-p^2\cdot h_p^2-\frac{2p^3}{\pi^2}\cdot|L(1,\chi)|^2.\end{aligned} \tag{25}$$

If $\chi=\chi_2$ is the Legendre symbol, then from (18), (19), Lemmas 1 and 3, and the method of proving (25), we have

$$
\begin{aligned}
\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p) &= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\left|\sum_{a=1}^{p-1}\chi_1(a)|K(a,1,\chi;p)|^2\right|^2\cdot\left|L(1,\chi_1)\right|^2\\
&= \frac{p\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1\\ \chi_1\neq\chi_2}}\left|\frac{\tau^2(\chi_1)\cdot\tau(\overline{\chi}_1^2)\cdot\tau(\overline{\chi}_2\chi_1)}{\tau(\overline{\chi}_2\overline{\chi}_1)}\right|^2\cdot\left|L(1,\chi_1)\right|^2+\frac{p\cdot\pi^{-2}}{p-1}\left|\sum_{a=1}^{p-1}\overline{\chi}_2(a)\left|K(a,1,\chi_2;p)\right|^2\right|^2\cdot\left|L(1,\overline{\chi}_2)\right|^2\\
&= \frac{p^4\cdot\pi^{-2}}{p-1}\sum_{\substack{\chi_1 \bmod p\\ \chi_1(-1)=-1}}\left|L(1,\chi_1)\right|^2-\frac{p^4\cdot\pi^{-2}}{p-1}\cdot\left|L(1,\chi_2)\right|^2+\frac{p^3\cdot\pi^{-2}}{p-1}\cdot(p-2)^2\cdot\left|L(1,\chi_2)\right|^2\\
&= \frac{p^2(p-1)(p-2)}{12}+p^2\cdot(p-4)\cdot h_p^2.
\end{aligned}
\tag{26}
$$

Combining (24), (25), and (26), we can deduce the identity:

$$
\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}|K(a,1,\chi;p)|^2\cdot|K(b,1,\chi;p)|^2\cdot S(a\overline{b},p)=\begin{cases}\dfrac{1}{12}\cdot p^2(p-1)(p-2)-p^2\cdot h_p^2, & \text{if } \chi(-1)=1,\\[2ex] \dfrac{p^2(p-1)(p-2)}{12}-p^2\cdot h_p^2-\dfrac{2p^3}{\pi^2}\cdot|L(1,\chi)|^2, & \text{if } \chi(-1)=-1 \text{ and } \chi\neq\chi_2,\\[2ex] \dfrac{p^2(p-1)(p-2)}{12}+p^2\cdot(p-4)\cdot h_p^2, & \text{if } \chi=\chi_2.\end{cases}
\tag{27}
$$

This completes the proof of Theorem 4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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