A Hybrid Mean Value Involving Dedekind Sums and the Generalized Kloosterman Sums

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In this paper, we use the mean value theorem of Dirichlet $L$-functions and the properties of Gauss sums and Dedekind sums to study the hybrid mean value problem involving Dedekind sums and the general Kloosterman sums and give an interesting identity for it.

1. Introduction

Let $q$ be a natural number and $h$ be an integer coprime to $q$. The classical Dedekind sums

$$S(h, q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q}\right)\right) \left(\frac{ah}{q}\right),$$

where

$$\left(\frac{x}{q}\right) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \text{ is not an integer}, \\ 0 & \text{if } x \text{ is an integer}, \end{cases}$$

describes the behaviour of the logarithm of the $\eta$-function (see [1, 2]) under modular transformations. There are many papers written on their various properties (see the examples in [3–10] and [11]).

In particular, Zhang and Liu [12] studied the hybrid mean value problems related to Dedekind sums and Kloosterman sums:

$$K(m, n; q) = \sum_{a=1}^{q} e\left(\frac{ma + na}{q}\right),$$

where $q \geq 3$ is an integer, $\sum_{a=1}^{q}$ denotes the summation over all $1 \leq a \leq q$ with $(a, q) = 1$, $e(x) = e^{2\pi ix}$, and $\overline{a}$ denotes the multiplicative inverse of $a \mod q$. They proved the following results:

**Theorem 1.** Let $p$ be an odd prime, then one has the identity

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1; p)|^2 \cdot |K(b, 1; p)|^2 \cdot S(a\overline{b}, p) = \begin{cases} \frac{1}{12} \cdot p^2 \cdot ((p-1)(p-2) - 12 \cdot \overline{h}^2), & \text{if } p \equiv 3 \mod 4, \\
12 \cdot p^2 \cdot (p-1)(p-2), & \text{if } p \equiv 1 \mod 4, \end{cases}$$

for a hybrid mean value problem involving Dedekind sums and the general Kloosterman sums.
where $h_p$ denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{-p})$.

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1; p)|^2 \cdot |K(b, 1; p)|^2 \cdot |S(a\overline{b}, q)|^2 = \frac{1}{24} p^5 + O \left( p^4 \cdot \exp \left( \frac{3 \ln \ln p}{\ln p} \right) \right).
\]  

where $\exp(y) = e^y$.

It is natural that people will ask, for the general Kloosterman sums

\[
K(m, n, \chi; p) = \sum_{a=1}^{p-1} \chi(a) e \left( \frac{ma + n\overline{a}}{p} \right),
\]

what will happen? Whether there exists an identity similar to Theorem 1? Here, $\chi$ denotes any Dirichlet character mod $p$.

**Theorem 4.** Let $p$ be an odd prime with $p \equiv 3 \mod 4$. Then, for any Dirichlet character $\chi$ mod $p$, we have the identity:

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot |S(a\overline{b}, p)| = \begin{cases} 
\frac{1}{12} p^3 (p-1)(p-2), & \text{if } \chi(-1) = 1, \\
\frac{1}{12} p^3 (p-1)(p-2) - \frac{p^3}{\pi^2} |L(1, \chi)|^2, & \text{if } \chi(-1) = -1.
\end{cases}
\]  

It is clear that if $\chi = \chi_0$, then $K(a, 1, \chi; p) = K(a, 1; p)$. Note that $\chi_0(-1) = 1$, from Theorems 3 and 4, we may immediately deduce Theorem 1 in [12], so our results are the generalization of [12].

2. Several Lemmas

In this section, we shall give several simple lemmas, which are necessary to the proofs of our theorems. Hereafter, we shall use many properties of characters sums and Gauss sums, and all of these can be found in reference [13]. First, we have the following.
\[
\sum_{m=1}^{p-1} \chi_1(m) \cdot |K(m, 1, \chi; p)|^2 = \sum_{m=1}^{p-1} \chi_1(m) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \overline{\chi}(b) e\left(\frac{am - mb + \overline{a} - \overline{b}}{p}\right)
\]
\[
= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \overline{\chi}(b) e\left(\frac{mb(a - 1) + \overline{b} \cdot (\overline{a} - 1)}{p}\right) = \tau(\chi_1) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \overline{\chi_1}(b) \overline{\chi_1}(a - 1) e\left(\frac{\overline{b} \cdot (\overline{a} - 1)}{p}\right)
\]
\[
= \tau(\chi_1) \sum_{a=1}^{p-1} \chi(a) \overline{\chi_1}(a - 1) \sum_{b=1}^{p-1} \overline{\chi_1}(b) e\left(\frac{\overline{b} \cdot (\overline{a} - 1)}{p}\right) = \tau^2(\chi_1) \sum_{a=1}^{p-1} \chi(a) \overline{\chi_1}(a - 1) \overline{\chi_1}(\overline{a} - 1)
\]
\[
= \overline{\chi_1}(-1) \tau^2(\chi_1) \sum_{a=1}^{p-1} \chi(a) \overline{\chi_1}(a) \overline{\chi_1}(a - 1) = \overline{\chi_1}(-1) \tau^2(\chi_1) \sum_{a=1}^{p-1} \chi(a + 1) \overline{\chi_1}(a + 1) \overline{\chi_1}(a).
\]

(10)

On the other hand, from the properties of Gauss sums, we have

\[
\sum_{a=1}^{p-1} \chi(a + 1) \overline{\chi_1}(a + 1) \overline{\chi_1}(a) = \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{p-1} \overline{\chi_1}(a) \sum_{b=1}^{p-1} \overline{\chi_1}(b) e\left(\frac{b(a + 1)}{p}\right)
\]
\[
= \frac{1}{\tau(\overline{\chi_1})} \sum_{b=1}^{p-1} \overline{\chi_1}(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \overline{\chi_1}(a) e\left(\frac{ba}{p}\right) = \frac{\tau(\overline{\chi_1})}{\tau(\overline{\chi_1})} \sum_{b=1}^{p-1} \overline{\chi_1}(b) \overline{\chi_1}(b) e\left(\frac{b}{p}\right) = \frac{\tau(\overline{\chi_1})}{\tau(\overline{\chi_1})} \tau(\chi_1)
\]

(11)

Combining (10) and (11), we may immediately deduce the identity:

\[
\sum_{m=1}^{p-1} \chi_1(m) \cdot |K(m, 1, \chi; p)|^2 = \frac{\overline{\chi_1}(-1) \cdot \tau^2(\chi_1) \cdot \tau(\overline{\chi_1})}{\tau(\overline{\chi_1})} \sum_{m=1}^{p-1} \overline{\chi}(m) \cdot |K(m, 1, \chi; p)|^2 = \tau^2(\overline{\chi}).
\]

(12)

This proves Lemma 1.

Lemma 2. Let $p$ be a prime with $p \equiv 1 \mod 4$ and $\chi$ be any odd character mod $p$. Then, we have the identity:

\[
\sum_{m=1}^{p-1} \overline{\chi}(m) \cdot |K(m, 1, \chi; p)|^2 = \tau^2(\overline{\chi}).
\]

(13)

Proof. Since $p \equiv 1 \mod 4$ and $\chi$ is an odd character mod $p$, we know $\chi$ is not the Legendre symbol and $\chi^2 \neq \chi_0$. Note that $\chi(-1) = -1$, from (10), we have

\[
\sum_{m=1}^{p-1} \overline{\chi}(m) \cdot |K(m, 1, \chi; p)|^2 = \tau^2(\overline{\chi}) \sum_{a=1}^{p-1} \chi(a) \overline{\chi}(a - 1) \overline{\chi}(\overline{a} - 1)
\]
\[
= \chi(-1) \tau^2(\overline{\chi}) \sum_{a=1}^{p-1} \chi^2(a - 1) = -\tau^2(\overline{\chi}) \left( \sum_{a=0}^{p-1} \chi^2(a) - 1 \right) = \tau^2(\overline{\chi}).
\]

(14)

This proves Lemma 2.
Lemma 3. Let \( p \) be a prime with \( p \equiv 3 \mod 4 \) and \( \chi_2 \) be the Legendre symbol. Then, we have the identity:

\[
\sum_{m=1}^{p-1} \left| \sum_{x=1}^{p-1} \chi(m,1,\chi_2;\chi)p \right|^2 = -(p-2) \cdot \tau^2(\chi_2).
\] (15)

Proof. Note that \( \chi_2 = \chi_2' \), \( \chi_2'' = \chi_0 \), and \( \chi_2(-1) = -1 \), and from the definition of \( K(m,1,\chi_2;\chi)p \) and the properties of Gauss sums, we have

\[
\sum_{m=1}^{p-1} \left| \sum_{x=1}^{p-1} \chi(m,1,\chi_2;\chi)p \right|^2 = \tau(\chi_2) \sum_{m=1}^{p-1} \chi(m)(a-b)(\chi(p-1)) - (p-2) \cdot \tau^2(\chi_2).
\] (16)

3. Proof of the Theorems

In this section, we will complete the proof of our theorems. First we prove Theorem 3. From Lemma 4 and the definition of \( S(a,\chi)p \), we have

\[
S(a,\chi)p = \frac{1}{\chi^{2}} \cdot \prod_{p-1}^{\chi} \sum_{d|\chi} \chi(a)|L(1,\chi)|^2,
\] (18)

and (with \( a = 1 \))

\[
\sum_{\chi \mod d}^{\chi(\chi) \equiv 1} |L(1,\chi)|^2 = \frac{\pi^2}{\chi^{2}} \cdot \frac{(\chi-1)^2}{\chi(\chi-1)^{2}} = \frac{12}{\chi^{2}} \cdot \frac{(p-1)^2(\chi-2)}{\chi(\chi-1)^{2}}.
\] (19)

Since \( p \equiv 1 \mod 4 \), we know the Legendre symbol \((1/p) = \chi_2\) is an even character mod \( p \). Note that, for any nonprincipal character \( \chi \mod p \), \( |\tau(\chi)| = \sqrt{p} \). So, if \( \chi \) is an even character mod \( p \), then from Lemma 1, (18), and (19), we have

\[
S(a,\chi)p = \frac{1}{\chi^{2}} \cdot \prod_{p-1}^{\chi} \sum_{d|\chi} \chi(a)|L(1,\chi)|^2
\]

If \( \chi \) is an odd character mod \( p \), then note that the identity:

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a,1,\chi;\chi)p| \cdot |K(b,1,\chi;\chi)p| \cdot S(\alpha\beta,p) = \frac{p^{\chi-2}}{\chi(\chi-1)^{2}} \sum_{a=1}^{p-1} |\chi(a)|L(1,\chi)|^2
\]

\[
= \frac{p^{\chi-2}}{\chi(\chi-1)^{2}} \sum_{a=1}^{p-1} \frac{\tau^2(\chi_1) \cdot \tau(\chi_1)}{\tau(\chi_1)} \cdot |L(1,\chi)|^2
\]

\[
= \frac{1}{12} \cdot p^2 (\chi-2)(\chi-2).
\] (20)
\[ \sum_{x_1 \mod p, \ x_1 \neq \chi_0} f(x_1) = \sum_{x_1 \mod p, \ x_1 \neq \chi_0} f(x_1) + f(\overline{\chi}). \]  

(21)

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a,1,\chi; p)|^2 \cdot |K(b,1,\chi; p)|^2 \cdot S(a\overline{\chi}, p) = \frac{p \cdot \pi^2}{p - 1} \sum_{x_1 \mod p, \ x_1 \neq \chi_0} \left| \sum_{a=1}^{p-1} x_1(a) |K(a,1,\chi; p)|^2 \right|^2 \cdot \left| L(1,\chi) \right|^2
\]

\[
= \frac{p \cdot \pi^2}{p - 1} \sum_{x_1 \mod p, \ x_1 \neq \chi_0} \left| \frac{\tau(x_1)}{\tau(\overline{\chi} x_1)} \right|^2 \cdot \left| L(1,\chi) \right|^2 + \frac{p \cdot \pi^2}{p - 1} \left| L(1,\chi) \right|^2 \cdot \left| L(1,\overline{\chi}) \right|^2
\]

\[
= \frac{p^4 \cdot \pi^2}{p - 1} \sum_{x_1 \mod p, \ x_1 \neq \chi_0} \left| L(1,\chi) \right|^2 - \frac{p \cdot \pi^2}{p - 1} \left| L(1,\chi) \right|^2 + \frac{p^3 \cdot \pi^2}{p - 1} \left| L(1,\chi) \right|^2 = \frac{1}{12} \cdot p^2 (p - 1) (p - 2) - \frac{p^3}{\pi^2} \cdot \left| L(1,\chi) \right|^2,
\]

where \( \sum_{x_1 \mod p, \ x_1 \neq \chi_0} \) denotes the summation over all odd \( x_1 \) with \( x_1 \neq \overline{\chi} \).

Combining (20) and (22), we may immediately deduce the identity:

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a,1,\chi; p)|^2 \cdot |K(b,1,\chi; p)|^2 \cdot S(a\overline{\chi}, p) = \begin{cases} 
\frac{1}{12} \cdot p^2 (p - 1) (p - 2), & \text{if } \chi(-1) = 1, \\
\frac{1}{12} \cdot p^2 (p - 1) (p - 2) - \frac{p^3}{\pi^2} \cdot \left| L(1,\chi) \right|^2, & \text{if } \chi(-1) = -1.
\end{cases}
\]

(23)

This proves Theorem 3.

Now, we prove Theorem 4. Since \( p \equiv 3 \mod 4 \), we know the Legendre symbol \( \chi_2 \) is an odd character mod \( p \) and \( |\tau(\overline{\chi_2})| = 1 \). If \( \chi \) is an even character mod \( p \), then note that \( L(1,\chi) = (\pi \cdot h_{\chi}/\sqrt{p}) \) (see reference [14]), from (18), (19), Lemma 1 and the properties of Gauss sums, we have
\[
\sum_{x_1 \mod p \atop x_1 \neq 0} \sum_{x_2 \mod p \atop x_2 \neq 0} [K(a, 1, x; p)]^2 \cdot |K(b, 1, x; p)|^2 \cdot S(ab, p) = \frac{p \cdot \pi^2}{p - 1} \sum_{x_1 \mod p \atop x_1 \neq 0} \left| \tau(x_1) \cdot \tau(x) \right|^2 \cdot |L(1, x_1)|^2 \\
= \frac{p \cdot \pi^2}{p - 1} \sum_{x_1 \mod p \atop x_1 \neq 0} \left| \frac{\tau(x_1)}{\tau(x)} \right|^2 \cdot |L(1, x_1)|^2 + \frac{p^3 \cdot \pi^2}{p - 1} \cdot |L(1, x_2)|^2 \\
= \frac{1}{12} \cdot p^2 \cdot (p - 1) (p - 2) - p^2 \cdot h_p^2,
\]
where \( h_p \) denotes the class number of the quadratic field \( \mathbb{Q}(\sqrt{-p}) \).

If \( \chi \) is an odd nonreal character mod \( p \), then from (18), (19), Lemmas 1 and 2, and the method of proving (22) and (24), we have

\[
\sum_{x_1 \mod p \atop x_1 \neq 0} \sum_{x_2 \mod p \atop x_2 \neq 0} [K(a, 1, x; p)]^2 \cdot |K(b, 1, x; p)|^2 \cdot S(ab, p) = \frac{p \cdot \pi^2}{p - 1} \sum_{x_1 \mod p \atop x_1 \neq 0} \left| \tau(x_1) \cdot \tau(x) \right|^2 \cdot |L(1, x_1)|^2 \\
= \frac{p \cdot \pi^2}{p - 1} \sum_{x_1 \mod p \atop x_1 \neq 0} \left| \frac{\tau(x_1)}{\tau(x)} \right|^2 \cdot |L(1, x_1)|^2 + \frac{p^3 \cdot \pi^2}{p - 1} \cdot |L(1, x_2)|^2 \\
= \frac{1}{12} \cdot p^2 \cdot (p - 1) (p - 2) - p^2 \cdot h_p^2\]

If \( \chi = \chi_2 \) is the Legendre symbol, then from (18), (19), Lemmas 1 and 3, and the method of proving (25), we have
\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(ab, p) = \frac{p \cdot \pi^2}{p-1} \sum_{\chi_{1} \mod p} \left| \sum_{k=1}^{p-1} \chi_{1}(k)K(a, 1, \chi; p) \right|^2 \cdot |L(1, \chi)|^2
\]

\[
= \frac{p \cdot \pi^2}{p-1} \sum_{\chi \mod p \atop \chi(-1)=-1} \left| \frac{\chi_1}{\tau(\chi)} \cdot \frac{\tau(\chi_2)}{\chi(\tau)} \right|^2 \cdot |L(1, \chi)|^2 \cdot \left| L(1, \chi_{1}) \right|^2
\]

\[
= \frac{p^4 \cdot \pi^2}{p-1} \sum_{\chi \mod p \atop \chi(-1)=-1} |L(1, \chi)|^2 \cdot \frac{p^3 \cdot \pi^2}{p-1} \cdot |L(1, \chi)|^2 + \frac{p^3 \cdot \pi^2}{p-1} \cdot (p-2) \cdot |L(1, \chi)|^2
\]

\[
= \frac{p^2 (p-1)(p-2)}{12} + p^2 \cdot (p-4) \cdot h_p^2
\]

(26)

Combining (24), (25), and (26), we can deduce the identity:

\[
\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} |K(a, 1, \chi; p)|^2 \cdot |K(b, 1, \chi; p)|^2 \cdot S(ab, p) =
\begin{cases}
\frac{1}{12} \cdot p^2 (p-1)(p-2) - p^2 \cdot h_p^2, & \text{if } \chi(-1) = 1, \\
\frac{p^2 (p-1)(p-2)}{12} - p^2 \cdot h_p^2 - \frac{2p^3}{\pi^2} \cdot |L(1, \chi)|^2, & \text{if } \chi(-1) = -1 \text{ and } \chi \neq \chi_2,
\end{cases}
\]

\[
\begin{cases}
p^2 (p-1)(p-2) + 2p^2 \cdot (p-4) \cdot h_p^2, & \text{if } \chi = \chi_2.
\end{cases}
\]

(27)

This completes the proof of Theorem 4.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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