Research Article

A New Family of Fourth-Order Optimal Iterative Schemes and Remark on Kung and Traub's Conjecture

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Kung and Traub conjectured that a multipoint iterative scheme without memory based on m evaluations of functions has an optimal convergence order $p = 2^{m-1}$. In the paper, we first prove that the two-step fourth-order optimal iterative schemes of the same class have a common feature including a same term in the error equations, resorting on the conjecture of Kung and Traub. Based on the error equations, we derive a constantly weighting algorithm obtained from the combination of two iterative schemes, which converges faster than the departed ones. Then, a new family of fourth-order optimal iterative schemes is developed by using a new weight function technique, which needs three evaluations of functions and whose convergence order is proved to be $p = 2^{3-1} = 4$.

1. Introduction

The most basic problem in engineering and scientific applications is to find the root of a given nonlinear equation

$$f(x) = 0, \quad (1)$$

where $f \in \mathcal{C}(I, \mathbb{R})$ and $I \subset \mathbb{R}$ is an interval we are interested in, and we suppose that $r \in I$ is a simple solution with $f(r) = 0$ and $f'(r) \neq 0$.

The famous Newton method (NM) for iteratively solving equation (1) is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots, \quad (2)$$

which is quadratically convergent. Due to its simplicity and rapid convergence, the Newton method is still the first choice to solve equation (1).

An extension of the NM to a third-order iterative scheme was made by Halley [1]:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)}. \quad (3)$$

For the engineering design of the vibrating modes of an elastic system, sometimes we may need to know the eigenvalues of a large-size square matrix, which results in a highly nonlinear and high-order polynomial equation. More often, the function $f(x)$ is itself obtained from other nonlinear ordinary differential equations or partial differential equations. In this situation, it is hard to calculate $f''(x)$ when we apply the Halley method to solve the nonlinear problem.

Kung and Traub conjectured that a multipoint iteration without memory based on m evaluations of functions has an optimal convergence order $p = 2^{m-1}$. It means that the upper bound of the efficiency index (E.I.) = $p^{(1/m)}$ is $2^{(1-1/m)} < 2$. For $m = 2$, the NM is one of the second-order optimal iterative schemes; however, with $m = 3$, the Halley method is not the optimal one whose E.I. = 1.44225 is low.

The pioneering work of Newton has inspired a lot of studies to solve nonlinear equations, whereby different fourth-order iterative methods were developed for more quickly and stably

solving nonlinear equations [2–9]. Many methods to construct the two-step fourth-order optimal schemes were based on the operations of $[f(x_n), f'(x_n), f(y_n)]$ where y_n is obtained from the first Newton step [2, 4–8, 10–14]. Recently, Chicharro et al. [9] proposed a new technique to construct the optimal fourth-order iterative schemes based on the weight function technique.

2. Preliminaries

Before deriving the main results in the next section, we begin with some standard terminologies.

Definition 1. Let the iterative sequence $\{x_n\}$ generated from an iterative scheme converge to a simple root r . If there exists a positive integer p and a real number C such that

$$\lim_{n \rightarrow \infty} \frac{x_{n+1} - r}{(x_n - r)^p} = C, \quad (4)$$

then p is the order of convergence and C is the asymptotic error constant.

Let $e_n = x_n - r$ be the error in the n th iterate. Then, the relation

$$e_{n+1} = Ce_n^p + \mathcal{O}(e_n^{p+1}), \quad (5)$$

is called the error equation of an iterative scheme. For example, for the Newton method, the error equation reads as

$$e_{n+1} = c_2 e_n^2 + \mathcal{O}(e_n^3), \quad (6)$$

where

$$c_n := \frac{f^{(n)}(r)}{n! f'(r)}, \quad n = 2, \dots \quad (7)$$

Definition 2 (see [10]). An iterative scheme is said to have the optimal order p , if $p = 2^{m-1}$ where m is the number of evaluations of functions (including derivatives).

Definition 3. The efficiency index (E.I.) of an iterative scheme is defined by $\text{E.I.} = p^{(1/m)}$.

Definition 4. The conjecture of Kung and Traub asserted that a multipoint iteration without memory based on m evaluations of functions has an optimal order $p = 2^{m-1}$ of convergence [11]. It indicates that the upper bound of the efficiency index is $2^{(1-1/m)} < 2$.

Definition 5. The iterative schemes are of the same class, if they are of the same order p and have the same m evaluations of the same functions.

3. Main Results

We begin with the error equation of the NM:

$$e_{n+1} = c_2 e_n^2 - A_3 e_n^3 - A_4 e_n^4 + \dots, \quad (8)$$

where

$$A_3 = 2c_2^2 - 2c_3, \quad (9)$$

$$A_4 = 7c_2 c_3 - 4c_2^3 - 3c_4. \quad (10)$$

Refer the papers, for instance, [6, 12, 13].

Throughout of the paper, we fix the following notation:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (11)$$

which is the first step of many two-step iterative schemes.

We summarize some fourth-order optimal iterative schemes which were modified from the NM by Chun [14]:

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(\frac{f(x_n)}{f(x_n) - f(y_n)} \right)^2 \frac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = (2c_2^3 - c_2 c_3) e_n^4 + \dots, \end{cases} \quad (12)$$

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)}{f^2(x_n) - 2f(x_n)f(y_n) + 2f^2(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = (3c_2^3 - c_2 c_3) e_n^4 + \dots, \end{cases} \quad (13)$$

by Chun [4]:

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(1 + 2 \frac{f(y_n)}{f(x_n)} + \frac{f^2(y_n)}{f^2(x_n)} \right) \frac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = (4c_2^3 - c_2 c_3) e_n^4 + \dots, \end{cases} \quad (14)$$

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(1 + 2 \frac{f(y_n)}{f(x_n)} \right) \frac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = (5c_2^3 - c_2 c_3) e_n^4 + \dots, \end{cases} \quad (15)$$

by King [5]:

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = [(1 + 2\gamma)c_2^3 - c_2 c_3] e_n^4 + \dots, \end{cases} \quad (16)$$

where $\gamma \in \mathbb{R}$, by Chun and Ham [2]:

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{4f^2(x_n) + 6f(x_n)f(y_n) + 3f^2(y_n)}{4f^2(x_n) - 2f(x_n)f(y_n) - f^2(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = (3c_2^3 - c_2c_3)e_n^4 + \dots, \end{cases} \quad (17)$$

by Kuo et al. [8]:

$$\begin{cases} x_{n+1} = x_n - \frac{f^2(x_n) + f^2(y_n)}{f'(x_n)[f(x_n) - f(y_n)]}, \\ e_{n+1} = (3c_2^3 - c_2c_3)e_n^4 + \dots, \end{cases} \quad (18)$$

by Ostrowski [15]:

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)[f(x_n) - f(y_n)]}{f'(x_n)[f(x_n) - 2f(y_n)]}, \\ e_{n+1} = (c_2^3 - c_2c_3)e_n^4 + \dots, \end{cases} \quad (19)$$

by Maheshwari et al. [16]:

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left[\frac{f^2(y_n)}{f^2(x_n)} + \frac{f(x_n)}{f(x_n) - f(y_n)} \right], \\ e_{n+1} = (4c_2^3 - c_2c_3)e_n^4 + \dots, \end{cases} \quad (20)$$

and by Ghanbari [12]:

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left(\frac{f(x_n) + 2f(y_n)}{f(x_n) + f(y_n)} \right)^2 \frac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = (6c_2^3 - c_2c_3)e_n^4 + \dots, \end{cases} \quad (21)$$

$$\begin{cases} x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + f(y_n)}{f(x_n) - f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ e_{n+1} = (3c_2^3 - c_2c_3)e_n^4 + \dots. \end{cases} \quad (22)$$

It is interesting that the iterative schemes (12)-(22) are of the same class because they have same convergence order $p = 4$ and operated with the same evaluations on $[f(x_n), f'(x_n), f(y_n)]$. The efficiency index (E.I.) of the above eleven iterative schemes is the same $\sqrt[3]{4} = 1.5874$, and they are of the optimal fourth-order iterative schemes with three evaluations of $[f(x_n), f'(x_n), f(y_n)]$ in the sense of Kung and Traub, such that $p = 2^{m-1} = 4$. They belong to the same class with the error equations having a common type:

$$e_{n+1} = (a_i c_2^3 - c_2 c_3) e_n^4 + \mathcal{O}(e_n^5), \quad (23)$$

where a_i are different constants for different optimal fourth-order iterative schemes, which may be zero. Can we raise the order to five by a suitable combination of these iterative schemes? Later, we will reply to this problem.

Theorem 1. *If the conjecture of Kung and Traub is true, then the two-step optimal fourth-order iterative scheme*

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - H[f(x_n), f'(x_n), f(y_n)], \end{cases} \quad (24)$$

which is based on the evaluations of $[f(x_n), f'(x_n), f(y_n)]$, must have the following form of error equation:

$$e_{n+1} = (a_0 c_2^3 - c_2 c_3) e_n^4 + \mathcal{O}(e_n^5), \quad (25)$$

where a_0 is some constant, which may be zero.

Proof. Suppose that equation (25) is not true, such that we have

$$e_{n+1} = (a_0 c_2^3 - b_0 c_2 c_3) e_n^4 + \mathcal{O}(e_n^5), \quad (26)$$

where $b_0 \neq 1$.

The weighting factors w_1 , w_2 , and w_3 are subjected to

$$w_1 + w_2 + w_3 = 1. \quad (27)$$

Then, we consider the weighting average of the error equations in equation (23) with $i = 1, 2$ and equation (26) to be zero in e_n^4 :

$$w_1(a_1 c_2^3 - c_2 c_3) + w_2(a_2 c_2^3 - c_2 c_3) + w_3(a_0 c_2^3 - b_0 c_2 c_3) = 0, \quad (28)$$

which leads to

$$\begin{aligned} a_1 w_1 + a_2 w_2 + a_0 w_3 &= 0, \\ w_1 + w_2 + b_0 w_3 &= 0. \end{aligned} \quad (29)$$

The determinant of the coefficient matrix of the linear equations (27) and (29) is $(b_0 - 1)(a_2 - a_1) \neq 0$ because $b_0 \neq 1$ and $a_1 \neq a_2$. From equations (27) and (29), we have the unique solution of (w_1, w_2, w_3) . Thus, we can derive a new iterative scheme by a weighting combination of three optimal fourth-order iterative schemes with the solved factors (w_1, w_2, w_3) whose convergence order is raised to five. This contradicts the conjecture of Kung and Traub, who asserted that the optimal order for the iterative scheme with $m = 3$ is $2^{m-1} = 4$ for a multipoint iteration without memory based on m evaluations of functions.

Obviously, Theorem 1 demonstrates that we cannot raise the convergence order to five by a weighting combination of any three optimal fourth-order convergence iterative schemes. $\square$

Theorem 2. The following two-step iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - H(\eta_n) \frac{f(y_n)}{f'(x_n)}, \quad (30)$$

for solving $f(x) = 0$ has fourth-order convergence, where y_n is computed by equation (11), and H is a weight function in terms of

$$\eta_n := \frac{f(y_n)}{f(x_n)}, \quad (31)$$

with

$$\begin{aligned} H(0) &= 1, \\ H'(0) &= 2. \end{aligned} \quad (32)$$

The corresponding error equation is

$$e_{n+1} = \left[\left(5 - \frac{H''(0)}{2} \right) c_2^3 - c_2 c_3 \right] e_n^4 + \mathcal{O}(e_n^5). \quad (33)$$

Proof. For the proof of the convergence, we let r be a simple solution of $f(x) = 0$, i.e., $f(r) = 0$ and $f'(r) \neq 0$. We suppose that x_n is sufficiently close to the exact solution r , such that

$$e_n = x_n - r \quad (34)$$

is a small quantity, and it follows that

$$e_{n+1} = e_n + x_{n+1} - x_n. \quad (35)$$

By using the Taylor series, we have

$$f(x_n) = f'(r) [e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \dots], \quad (36)$$

$$f'(x_n) = f'(r) [1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \dots]. \quad (37)$$

It immediately leads to

$$\frac{f(x_n)}{f'(x_n)} = \frac{e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \dots}{1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \dots + \dots} = e_n - c_2 e_n^2 + A_3 e_n^3 + A_4 e_n^4 + \dots. \quad (38)$$

From equations (11) and (38), we have

$$y_n = r + c_2 e_n^2 - A_3 e_n^3 - A_4 e_n^4 + \dots, \quad (39)$$

$$f(y_n) = f'(r) [c_2 e_n^2 - A_3 e_n^3 - (A_4 - c_2^3) e_n^4 + \dots]. \quad (40)$$

From equations (40), (37), and (36), it follows that

$$\frac{f(y_n)}{f'(x_n)} = \frac{c_2 e_n^2 - A_3 e_n^3 - (A_4 - c_2^3) e_n^4 + \dots}{1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \dots} = c_2 e_n^2 + (2c_3 - 4c_2^2) e_n^3 + (13c_2^3 - 14c_2 c_3 + 3c_4) e_n^4 + \dots, \quad (41)$$

$$\frac{f(y_n)}{f(x_n)} = \frac{c_2 e_n^2 - A_3 e_n^3 - (A_4 - c_2^3) e_n^4 + \dots}{e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \dots} = c_2 e_n + (2c_3 - 3c_2^2) e_n^2 + (3c_4 - 10c_2 c_3 + 8c_2^3) e_n^3 + \dots. \quad (42)$$

From equations (31) and (42), we have

$$\eta_n = c_2 e_n + (2c_3 - 3c_2^2) e_n^2 + (3c_4 - 10c_2 c_3 + 8c_2^3) e_n^3 + \dots. \quad (43)$$

Because the least order of the term $(f(y_n)/f'(x_n))$ as shown in equation (41) is two, we only need to expand $H(\eta_n)$ around zero to the second-order by using equation (43) and

$$H(\eta_n) = H(0) + H'(0)\eta_n + \frac{H''(0)}{2}\eta_n^2 + \dots = H(0) + c_2 H'(0)e_n + \left[\frac{c_2^2}{2} [H''(0) - 6H'(0)] + 2c_3 H'(0) \right] e_n^2 + \dots. \quad (44)$$

Inserting equations (11), (39), (44), and (41) into equation (30), we have

$$e_{n+1} = c_2 e_n^2 - A_3 e_n^3 - A_4 e_n^4 - \left(H(0) + c_2 H'(0) e_n + \left[\frac{c_2^2}{2} [H''(0) - 6H'(0)] + 2c_3 H'(0) \right] e_n^2 \right) \times (c_2 e_n^2 + [2c_3 - 4c_2^2] e_n^3 + [13c_2^3 - 14c_2 c_3 + 3c_4] e_n^4) + \dots \quad (45)$$

Through some manipulations, we can derive

$$e_{n+1} = [c_2 - c_2 H(0)] e_n^2 - [2c_2^2 - 4c_2^2 H(0) + H'(0) c_2^2 + 2H(0) c_3 - 2c_3] e_n^3 - [7c_2 c_3 - 4c_2^3 - 3c_4 + H(0)(13c_2^3 - 14c_2 c_3 + 3c_4) + c_2 H'(0)(2c_3 - 4c_2^2)] e_n^4 - c_2 \left[\frac{c_2^2}{2} [H''(0) - 6H'(0)] + 2c_3 H'(0) \right] e_n^4 + \dots, \quad (46)$$

which, due to equation (32), can be arranged to that in equation (33). $\square$

Theorem 3 (see [12]). *ie following two-step iterative scheme:*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n) + (2 + \alpha)f(x_n)f(x_n) + \theta f^2(x_n)}{f^2(x_n) + \alpha f(x_n)f(x_n) + \beta f^2(x_n)} \frac{f(y_n)}{f'(x_n)}, \quad (47)$$

for solving $f(x) = 0$ has fourth-order convergence, where y_n is computed by equation (11). The error equation reads as

$$e_{n+1} = [(5 + 2\alpha - \theta + \beta)c_2^3 - c_2 c_3] e_n^4 + \mathcal{O}(e_n^5), \quad (48)$$

which is not supplied in [12].

Proof. It is easy to check that the weight function in iterative scheme (47):

$$H(\eta) = \frac{1 + (2 + \alpha)\eta + \theta\eta^2}{1 + \alpha\eta + \beta\eta^2}, \quad (49)$$

satisfies equation (32); hence, iterative scheme (47) is a special case of iterative scheme (30).

We can derive

$$H''(\eta) = \frac{1}{A^4(\eta)} \left\{ A^2(\eta) [A(\eta)B''(\eta) - B(\eta)A'(\eta)] - 2[A(\eta)B'(\eta) - B(\eta)A'(\eta)]A(\eta)A'(\eta) \right\}, \quad (50)$$

where

$$\begin{aligned} A &:= 1 + \alpha\eta + \beta\eta^2, \\ B &:= 1 + (2 + \alpha)\eta + \theta\eta^2. \end{aligned} \quad (51)$$

Inserting $A(0) = 1, A'(0) = \alpha, A''(0) = 2\beta, B(0) = 1, B'(0) = 2 + \alpha, B''(0) = 2\theta$ into equation (50) by taking $\eta = 0$, we have

$$H''(0) = -2(2\alpha - \theta + \beta). \quad (52)$$

Inserting equation (52) into equation (33), we can derive

$$e_{n+1} = [(5 + 2\alpha - \theta + \beta)c_2^3 - c_2 c_3] e_n^4 + \mathcal{O}(e_n^5). \quad (53)$$

This ends the proof of this theorem.

Theorem 2 includes those in [9, 17] as special cases. The family developed by Chicharro et al. [9]:

$$x_{n+1} = x_n - G(\eta_n) \frac{f(x_n)}{f'(x_n)}, \quad (54)$$

with $G(0) = G'(0) = 1$ and $G''(0) = 4$ is a special case because we can derive

$$H(\eta_n)\eta_n = G(\eta_n) - 1. \quad (55)$$

Accordingly,

$$\begin{aligned} H(\eta_n) + \eta_n H'(\eta_n) &= G'(\eta_n), \\ 2H'(\eta_n) + \eta_n H''(\eta_n) &= G''(\eta_n), \end{aligned} \quad (56)$$

and $H(0) = 1$ and $H'(0) = 2$ imply $G(0) = G'(0) = 1$ and $G''(0) = 4$. For H , we have only two constraints, but for G , there are three constraints. Hence, iterative scheme (30) is more general than the iterative scheme (54). Moreover, a further differential of the last term in equation (56),

$$3H''(\eta_n) + \eta_n H'''(\eta_n) = G'''(\eta_n), \quad (57)$$

leads to

$$3H''(0) = G'''(0), \quad (58)$$

and hence the error equation of iterative scheme (54) is

$$e_{n+1} = \left[\left(5 - \frac{G'''(0)}{6} \right) c_2^3 - c_2 c_3 \right] e_n^4 + \mathcal{O}(e_n^5). \quad (59)$$

In [9], Chicharro et al. derived the error equation as $e_{n+1} = (5c_2^3 - c_2 c_3)e_n^4 + \mathcal{O}(e_n^5)$ (equation (2) in [9]), which is incorrect to miss the term $-(G'''(0)c_2^3 e_n^4/6)$ in the error equation.

The general function of $H(\eta)$ is given by

$$H(\eta) = 1 + 2\eta + \int_0^\eta \int_0^\xi F(z) dz d\xi, \quad (60)$$

where $F(z)$ is any integrable function. There are two interesting iterative schemes generated from $F(z) = \cos z$ (COSM) and $F(z) = \sin z$ (SINM):

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - \left(2 + 2 \frac{f(y_n)}{f(x_n)} - \cos \frac{f(y_n)}{f(x_n)} \right) \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - \left(1 + 3 \frac{f(y_n)}{f(x_n)} - \sin \frac{f(y_n)}{f(x_n)} \right) \frac{f(y_n)}{f'(x_n)}. \end{aligned} \quad (61)$$

4. Combinations of Iterative Schemes

In this section, we give some methods to combine the iterative schemes as listed in Table 1, which are special cases of the iterative schemes (47) and (30).

From Table 1, we can observe that there exists a cubic term c_2^3 in the error equation for most iterative schemes. Indeed, this term is a dominant factor to enlarge the error, and thus we can combine two iterative schemes by eliminating this term.

Theorem 4. For the following two-step iterative scheme:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - w_1 \frac{f^2(x_n) + (2 + \alpha_1)f(x_n)f(y_n) + \theta_1 f^2(y_n)}{f^2(x_n) + \alpha_1 f(x_n)f(y_n) + \beta_1 f^2(y_n)} \frac{f(y_n)}{f'(x_n)} \\ &\quad - w_2 \frac{f^2(x_n) + (2 + \alpha_2)f(x_n)f(y_n) + \theta_2 f^2(y_n)}{f^2(x_n) + \alpha_2 f(x_n)f(y_n) + \beta_2 f^2(y_n)} \frac{f(y_n)}{f'(x_n)}, \end{aligned} \quad (62)$$

if

$$a_1 := 5 + 2\alpha_1 - \theta_1 + \beta_1 \neq a_2 := 5 + 2\alpha_2 - \theta_2 + \beta_2, \quad (63)$$

$$\begin{aligned} w_1 &= -\frac{a_2}{a_1 - a_2}, \\ w_2 &= \frac{a_1}{a_1 - a_2}, \end{aligned} \quad (64)$$

then the error equation reads as

$$e_{n+1} = -c_2 c_3 e_n^4 + \mathcal{O}(e_n^5). \quad (65)$$

Proof. The weighting factors are subjected to

$$w_1 + w_2 = 1. \quad (66)$$

We seek the combination of iterative scheme (47) with two sets of the parameters $(\alpha_1, \beta_1, \theta_1)$ and $(\alpha_2, \beta_2, \theta_2)$ and demand the coefficient preceding $c_2^3 e_n^4$ being zero,

$$w_1 a_1 + w_2 a_2 = w_1 (5 + 2\alpha_1 - \theta_1 + \beta_1) + w_2 (5 + 2\alpha_2 - \theta_2 + \beta_2) = 0. \quad (67)$$

Solving equations (66) and (67), we can derive equation (64), and the error equation (48) reduces to that in equation (65).

We cannot exhaust all the combinations of the iterative schemes; however, we list the following two: one is the combination of equations (16) and (19), namely, the KOM:

$$x_{n+1} = x_n + \frac{f(x_n)}{2\gamma f'(x_n)} + \frac{1}{2\gamma} \frac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)} - \frac{(1 + 2\gamma)[f(x_n)f(x_n) - f(y_n)]}{2\gamma f'(x_n)[f(x_n) - 2f(y_n)]}. \quad (68)$$

TABLE 1: The comparison of different iterative schemes on the error equations.

Algorithm	α	β	θ	Error equation (e_{n+1})
(12)	-2	1	0	$(2c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(13)	-2	2	0	$(3c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(14)	0	0	1	$(4c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(15)	0	0	0	$(5c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(16)	$\gamma - 2$	0	0	$((1 + 2\gamma)c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(17)	-1/2	-1/4	3/4	$(3c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(18)	-1	0	0	$(3c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(19)	-2	0	0	$(c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(20)	-1	0	-1	$(4c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(21)	2	1	4	$(6c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$
(22)	-2	1	-1	$(3c_2^3 - c_2c_3)e_n^4 + \mathcal{O}(e_n^5)$

The other one is the combination of equations (12) and (19), namely, the COM:

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)} + \left(\frac{f(x_n)}{f(x_n) - f(y_n)} \right)^2 \frac{f(y_n)}{f'(x_n)} - \frac{2f(x_n)[f(x_n) - f(y_n)]}{f'(x_n)[f(x_n) - 2f(y_n)]}. \quad (69)$$

5. Second Family of Optimal Fourth-Order Iterative Schemes

In Theorem 2, we have derived a new family of optimal fourth-order iterative schemes with the assumption that the H -function satisfies $H(0) = 1$ and $H'(0) = 2$. We can relax the conditions to $H(0) = 1$ and derive the following result.

Theorem 5. Suppose that there are two different functions $H_1(\eta)$ and $H_2(\eta)$ satisfying

$$H_1(0) = 1, \quad (70)$$

$$H_2(0) = 1,$$

$$H_1'(0) \neq H_2'(0). \quad (71)$$

The following two-step iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - [w_1 H_1(\eta_n) + w_2 H_2(\eta_n)] \frac{f(y_n)}{f'(x_n)}, \quad (72)$$

for solving $f(x) = 0$ has fourth-order convergence, where y_n is computed by equation (11), and η is defined by equation (31). The corresponding error equation is

$$e_{n+1} = \left[\left(5 - \frac{w_1 H_1''(0) + w_2 H_2''(0)}{2} \right) c_2^3 - c_2 c_3 \right] e_n^4 + \mathcal{O}(e_n^5), \quad (73)$$

where

$$w_1 = \frac{H_2'(0) - 2}{H_2'(0) - H_1'(0)}, \quad (74)$$

$$w_2 = \frac{2 - H_1'(0)}{H_2'(0) - H_1'(0)}.$$

Proof. From equations (46) and (70), it follows that the error equations corresponding to H_1 and H_2 are, respectively,

$$\begin{aligned} e_{n+1} &= [2 - H_1'(0)]c_2^2 e_n^3 - A_1 e_n^4 + \dots, \\ e_{n+1} &= [2 - H_2'(0)]c_2^2 e_n^3 - A_2 e_n^4 + \dots, \end{aligned} \quad (75)$$

where

$$\begin{aligned} A_1 &:= 9c_2^3 - 7c_2c_3 + c_2H_1'(0)(2c_3 - 4c_2^2) + c_2 \left[\frac{c_2^2}{2} [H_1''(0) - 6H_1'(0)] + 2c_3H_1'(0) \right], \\ A_2 &:= 9c_2^3 - 7c_2c_3 + c_2H_2'(0)(2c_3 - 4c_2^2) + c_2 \left[\frac{c_2^2}{2} [H_2''(0) - 6H_2'(0)] + 2c_3H_2'(0) \right]. \end{aligned} \quad (76)$$

TABLE 2: The comparison of different methods for the number of iterations.

Functions	$f_1, x_0 = -0.3$	$f_2, x_0 = 0$	$f_3, x_0 = 3$	$f_4, x_0 = 3.5$	$f_5, x_0 = 1$
NM	55	5	7	11	7
KM	49	3	4	7	8
GM	38	3	6	7	11
CM1	12	3	4	5	4
CM2	24	3	4	6	5
OM	56	3	4	5	4
AM	24	3	4	7	5
COSM	45	2	3	6	5
SINM	45	2	3	6	5
KOM	5	3	4	5	6
COM	5	3	4	4	4

We seek a combination of the two iterative schemes corresponding to H_1 and H_2 and ask the coefficient preceding e_n^3 to be zero, such that we have to solve w_1 and w_2 from

$$\begin{aligned} w_1 + w_2 &= 1, \\ w_1[2 - H_1'(0)] + w_2[2 - H_2'(0)] &= 0, \end{aligned} \quad (77)$$

whose solutions are given by equation (74). At the same time, the combined error equation is given by

$$\begin{aligned} e_{n+1} &= -w_1 A_1 e_n^4 - w_2 A_2 e_n^4 + \dots, \\ &= -\left[9c_2^3 - 7c_2c_3 + 2c_2(2c_3 - 4c_2^2) + \frac{c_2^3}{2}[w_1 H_1''(0) + w_2 H_2''(0)] + 4c_2c_3 - 6c_2^3\right]e_n^4 + \dots, \end{aligned} \quad (78)$$

which can be arranged to that in equation (73).

The family in equation (72) includes some optimal fourth-order iterative schemes with two parameters w_1 and w_2 , whose error equation again belongs to the type in equation (23). It can be seen that the functions with $H(0) = 1$ are very general, and for this class of iterative schemes, the conjecture of Kung and Traub is also true. $\square$

6. Numerical Experiments

In this section, we give numerical tests of the proposed combined iterative schemes. The test examples are given by

$$\begin{aligned} f_1(x) &= x^3 + 4x^2 - 10, \\ f_2(x) &= x^2 - e^x - 3x + 2, \\ f_3(x) &= (x-1)^3 - 2, \\ f_4(x) &= (x+2)e^x - 1, \\ f_5(x) &= \sin^2 x - x^2 + 1. \end{aligned} \quad (79)$$

The corresponding solutions are, respectively, $r_1 = 1.3652300134$, $r_2 = 0.2575302854$, $r_3 = 2.2599210499$, $r_4 = -0.442854401002$, and $r_5 = 1.4044916482$.

In Table 2, for different functions, we list the number of iterations (NI) obtained by the presently developed algorithms, which are compared to the NM, the CM1 in equation (12), the CM2 in equation (15), the KM in equation (16) with $\gamma = 3$, the OM in equation (19), the AM in equation (20), the

GM in equation (21), the KOM in equation (68) with $\gamma = 3$, and the COM in equation (69).

7. Conclusions

Employing a new weight function, the nonlinear equations were solved by using a new family of the fourth-order iterative scheme, which is optimal according to the conjecture of Kung and Traub, and it was proven to be of fourth-order convergence with E.I. = 1.5874. Theorem 1 indicated that if one can develop a fourth-order iterative scheme based on the evaluations of $[f(x_n), f'(x_n), f(y_n)]$ whose coefficient preceding $c_2c_3e_n^4$ is not -1 , then the Kung–Traub conjecture would be disproved. We also proposed a combination of two fourth-order iterative schemes of which the dominant term $c_2^3e_n^4$ in the error equation is eliminated. Upon comparing some examples to other methods, we found that the combined iterative scheme converges faster. The present iterative scheme was competitive to other optimal fourth-order iterative schemes.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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