Research Article

The Hermite–Hadamard–Jensen–Mercer Type Inequalities for Riemann–Liouville Fractional Integral

Hua Wang ¹, Jamroz Khan ², Muhammad Adil Khan ³, Sadia Khalid ⁴,
and Rewayat Khan ⁵

¹School of Mathematics and Statistics Changsha University of Science and Technology, Changsha 410114, China

²Government College of Management Sciences, Higher Education Department KPK, Hangu, Pakistan

³Department of Mathematics, University of Peshawar, Peshawar, Pakistan

⁴Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore, Pakistan

⁵Department of Mathematics, Abbotabad University of Science and Technology, Abbotabad, Pakistan

Correspondence should be addressed to Muhammad Adil Khan; adilswati@gmail.com

Received 1 February 2021; Revised 22 March 2021; Accepted 10 April 2021; Published 15 May 2021

Academic Editor: Ahmet Ocak Akdemir

Copyright © 2021 Hua Wang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we give Hermite–Hadamard type inequalities of the Jensen–Mercer type for Riemann–Liouville fractional integrals. We prove integral identities, and with the help of these identities and some other eminent inequalities, such as Jensen, Hölder, and power mean inequalities, we obtain bounds for the difference of the newly obtained inequalities.

1. Introduction

The concept of convex functions plays a vital role in both pure and applied mathematics. Convex functions also have many applications in other branches of science such as finance, economics, and engineering.

Definition 1 (see [1]). A function $\psi: [m, M] \rightarrow \mathbb{R}$ is convex if

$$\psi(sx + (1-s)y) \leq s\psi(x) + (1-s)\psi(y), \quad (1)$$

for all $x, y \in [m, M]$ and $s \in [0, 1]$.

If the inequality in (1) is strict for $x \neq y$, then ψ is said to be a strictly convex function, and if $-\psi$ is convex, then ψ is said to be a concave function [2, 3].

Many important inequalities such as Jensen, Jensen–Mercer, Hermite–Hadamard, and support line inequalities hold for convex functions. The classical Jensen's inequality is among the most prominent inequalities stated as follows [4, 5].

If $\psi: [m, M] \rightarrow \mathbb{R}$ is convex, then

$$\psi\left(\sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n w_i \psi(x_i), \quad (2)$$

for all $x_i \in [m, M]$ and $w_i \in [0, 1]$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n w_i = 1$.

In [6], Mercer presented a type of Jensen's inequality called Jensen–Mercer inequality.

Theorem 1. If $\psi: [m, M] \rightarrow \mathbb{R}$ is convex, then

$$\psi\left(m + M - \sum_{i=1}^n w_i x_i\right) \leq \psi(m) + \psi(M) - \sum_{i=1}^n w_i \psi(x_i), \quad (3)$$

for each $x_i \in [m, M]$ and $w_i \in [0, 1]$ ($i = 1, 2, \dots, n$) with $\sum_{i=1}^n w_i = 1$.

For a convex function, there exist at least one line lies on or below the graph of the function.

Definition 2 (see [7]). A function $\psi: I \rightarrow \mathbb{R}$ has a support at $x_0 \in I$ if

$$\psi(x_0) + c(u - x_0) \leq \psi(u), \quad (4)$$

for all $x_0 \in I$ and for each $u \in [m, M] \subset I$. Inequality (4) is said to be the support line inequality.

The following theorem connects the support line inequality with convex functions.

Theorem 2 (see [7]). *ie function $\psi: [m, M] \longrightarrow \mathbb{R}$ is convex if and only if ψ has at least one line of support at each $x_0 \in [m, M]$.*

The Hermite–Hadamard inequality is one of the most investigated inequality in the theory of convex functions due to its geometrical significance and applications. Because of the importance of Hermite–Hadamard inequality, there is an ample amount of research work dedicated to the extensions, generalizations, refinements, and applications of the Hermite–Hadamard inequality. The Hermite–Hadamard inequality is given below [8].

Let $\psi: I \longrightarrow \mathbb{R}$ be a convex function, where I is an interval and $m, M \in I$ such that $m < M$. Then,

$$\psi\left(\frac{m+M}{2}\right) \leq \frac{1}{M-m} \int_m^M \psi(x) dx \leq \frac{\psi(m) + \psi(M)}{2}. \quad (5)$$

If ψ is concave, then (5) holds in the reversed direction. For more results associated with Hermite–Hadamard inequality, see [9–18].

The Hermite–Hadamard inequality has been extended by means of fractional integral operators. Most popular of them is the Riemann–Liouville fractional operator given in the following definition [19–22].

Definition 3 (see [23, 24]). Let ψ be an integrable function defined on $[m, M]$. Then, the integrals $J_{m^+}^\alpha \psi(x)$ and $J_{M^-}^\alpha \psi(x)$ defined by

$$J_{m^+}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_m^x (x-s)^{\alpha-1} \psi(s) ds, \quad x > m, \quad (6)$$

$$J_{M^-}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_x^M (s-x)^{\alpha-1} \psi(s) ds, \quad x < M, \quad (7)$$

are called the left and right Riemann–Liouville fractional integrals of order $\alpha > 0$ respectively. Here, Γ represents gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-s} s^{\alpha-1} ds$.

In [25, 26], authors used the following lemmas to obtain trapezoidal and midpoint type inequalities.

Lemma 1 (see [25]). *Let $\psi: I^\circ \longrightarrow \mathbb{R}$ (where I° is the interior of I) be a differentiable function and $m, M \in I^\circ$ such that $m < M$. If $\psi' \in L[m, M]$, then*

$$\frac{\psi(m) + \psi(M)}{2} - \frac{1}{M-m} \int_m^M \psi(u) du = \frac{M-m}{2} \int_0^1 (1-2s) \psi'(sm + (1-s)M) ds. \quad (8)$$

Lemma 2 (see [26]). *Let all the assumptions of Lemma 1 hold. Then,*

$$\begin{aligned} & \frac{1}{M-m} \int_m^M \psi(u) du - \psi\left(\frac{m+M}{2}\right) \\ &= (M-m) \left(\int_0^{(1/2)} s \psi'(sm + (1-s)M) ds + \int_{(1/2)}^1 (s-1) \psi'(sm + (1-s)M) ds \right). \end{aligned} \quad (9)$$

In this article, we establish fractional Hermite–Hadamard–Jensen–Mercer type inequalities. We give identities involving fractional integrals, and from these identities, we derive trapezoidal and midpoint type inequalities.

Throughout this article, α represents a positive real number.

2. Main Results

We begin this section with our first main result.

Theorem 3. *Suppose $\psi: [m, M] \longrightarrow \mathbb{R}$ is a convex function and $x, y \in [m, M]$ such that $x < y$. Then,*

$$\begin{aligned}\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) &\leq \psi(m)+\psi(M)-\frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J_{x^{+}}^{\alpha}\psi(y) \\ &\leq \psi(m)+\psi(M)-\psi\left(\frac{\alpha x+y}{\alpha+1}\right),\end{aligned}\quad (10)$$

$$\begin{aligned}\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) &\leq \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}}J_{(m+M-x)^{-}}^{\alpha}\psi(m+M-y) \\ &\leq \frac{\alpha\psi(m+M-x)+\psi(m+M-y)}{\alpha+1} \\ &\leq \psi(m)+\psi(M)-\frac{\alpha\psi(x)+\psi(y)}{\alpha+1}.\end{aligned}\quad (11)$$

Proof. Since ψ is convex, it has support line at each point $x_0 \in [m, M]$, that is,

$$\psi(x_0) + c(u - x_0) \leq \psi(u), \quad (12)$$

for each $u \in [m, M]$. Substituting $x_0 = m + M - (\alpha x + y)/(\alpha + 1)$ and $u = m + M - sx - (1 - s)y$, where $s \in [0, 1]$, in inequality (12), we obtain

$$\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) + c\left[-sx - (1-s)y + \frac{\alpha x+y}{\alpha+1}\right] \quad (13)$$

$$\leq \psi(m+M-sx-(1-s)y).$$

Multiplying (13) with $\alpha s^{\alpha-1}$ and integrate with respect to s , we obtain

$$\begin{aligned}&\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) + c\left[-\frac{\alpha x+y}{\alpha+1} + \frac{\alpha x+y}{\alpha+1}\right] \\ &\leq \alpha \int_0^1 s^{\alpha-1} \psi(m+M-sx-(1-s)y) ds \\ &\implies \psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) \leq \alpha \int_0^1 s^{\alpha-1} \psi(m+M-sx-(1-s)y) ds.\end{aligned}\quad (14)$$

Using Mercer's inequality, we obtain

$$\begin{aligned}\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) &\leq \alpha \int_0^1 s^{\alpha-1} (\psi(m)+\psi(M)-(s\psi(x)+(1-s)\psi(y))) ds \\ &\implies \psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) \leq \psi(m)+\psi(M)-\alpha \int_0^1 s^{\alpha-1} (s\psi(x)+(1-s)\psi(y)) ds.\end{aligned}\quad (15)$$

Since ψ is convex, we have $-(s\psi(x)+(1-s)\psi(y)) \leq -\psi(sx+y(1-s))$ and (15) becomes

$$\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) \leq \psi(m)+\psi(M)-\alpha \int_0^1 s^{\alpha-1} \psi(sx+(1-s)y) ds. \quad (16)$$

Substituting $sx + (1-s)y = w$ in (16), we obtain

$$\begin{aligned} \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) &\leq \psi(m) + \psi(M) - \frac{\alpha}{(y-x)^\alpha} \int_x^y (y-w)^{\alpha-1} \psi(w) dw \\ &\implies \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) \leq \psi(m) + \psi(M) - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{x^+}^\alpha \psi(y). \end{aligned} \quad (17)$$

Now, we prove the second inequality of (10). Put $x_0 = (\alpha x + y)/(\alpha + 1)$ and $u = sx + (1-s)y$ in (12), we obtain

$$\psi\left(\frac{\alpha x + y}{\alpha + 1}\right) + c \left[sx + (1-s)y - \frac{\alpha x + y}{\alpha + 1} \right] \leq \psi(sx + (1-s)y). \quad (18)$$

Multiplying the above inequality with $\alpha s^{\alpha-1}$ and integrating and using $\int_0^1 s^{\alpha-1} ds = 1/\alpha$ and $\int_0^1 s^\alpha ds = 1/(\alpha + 1)$, we obtain

$$\psi\left(\frac{\alpha x + y}{\alpha + 1}\right) \leq \alpha \int_0^1 s^{\alpha-1} \psi(sx + (1-s)y) ds. \quad (19)$$

Put $sx + (1-s)y = w$, and we obtain

$$\begin{aligned} \psi\left(\frac{\alpha x + y}{\alpha + 1}\right) &\leq \frac{\alpha}{(y-x)^\alpha} \int_x^y (y-w)^{\alpha-1} \psi(w) dw \\ &\implies \psi\left(\frac{\alpha x + y}{\alpha + 1}\right) \leq \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{x^+}^\alpha \psi(y) \\ &\implies -\frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{x^+}^\alpha \psi(y) \leq -\psi\left(\frac{\alpha x + y}{\alpha + 1}\right). \end{aligned} \quad (20)$$

Adding $\psi(m) + \psi(M)$ on both sides of (20), we obtain

$$\psi(m) + \psi(M) - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{x^+}^\alpha \psi(y) \leq \psi(m) + \psi(M) - \psi\left(\frac{\alpha x + y}{\alpha + 1}\right), \quad (21)$$

and on combining (17) and (21), we obtain (10).

Now, we prove the inequalities in (11). Let $u = m + M - sx - (1-s)y \implies s = (u - m - M + y)/(y - x)$ and (14) become

$$\begin{aligned} \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) &\leq \alpha \int_{m+M-y}^{m+M-x} \left(\frac{u - m - M + y}{y - x} \right)^{\alpha-1} \psi(u) \frac{du}{y-x} \\ &= \frac{\alpha \Gamma(\alpha)}{(y-x)^\alpha} \frac{1}{\Gamma(\alpha)} \int_{m+M-y}^{m+M-x} (u - (m + M - y))^{\alpha-1} \psi(u) du \\ &\implies \psi\left(m + M - \frac{\alpha x + y}{\alpha + 1}\right) \leq \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m + M - y). \end{aligned} \quad (22)$$

Now, we prove the other two inequalities of (11). As ψ is a convex function, we have

$$\begin{aligned} \psi(m + M - sx - (1-s)y) &= \psi(s(m + M - x) + (1-s)(m + M - y)) \\ &\implies \psi(m + M - sx - (1-s)y) \leq s\psi(m + M - x) + (1-s)\psi(m + M - y) \\ &\leq \psi(m) + \psi(M) - s\psi(x) - (1-s)\psi(y). \end{aligned} \quad (23)$$

Multiplying with $\alpha s^{\alpha-1}$ and integrating, we obtain

$$\begin{aligned}
 & \alpha \int_0^1 s^{\alpha-1} \psi(m+M-sx-(1-s)y) ds \\
 & \leq \alpha \psi(m+M-x) \int_0^1 s^{\alpha} ds + \alpha \psi(m+M-y) \int_0^1 (s^{\alpha-1} - s^{\alpha}) ds \\
 & \leq \psi(m) + \psi(M) - \alpha \psi(x) \int_0^1 s^{\alpha} ds - \alpha \psi(y) \int_0^1 (s^{\alpha-1} - s^{\alpha}) ds \\
 & \implies \alpha \int_0^1 s^{\alpha-1} \psi(m+M-sx-(1-s)y) ds \\
 & \leq \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} \\
 & \leq \psi(m) + \psi(M) - \frac{\alpha \psi(x) + \psi(y)}{\alpha+1}.
 \end{aligned} \tag{24}$$

By changing of variable, (24) becomes

$$\begin{aligned}
 & \alpha \int_{m+M-y}^{m+M-x} \left(\frac{u-m-M+y}{y-x} \right)^{\alpha-1} \psi(u) \frac{du}{y-x} \\
 & \leq \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} \leq \psi(m) + \psi(M) - \frac{\alpha \psi(x) + \psi(y)}{\alpha+1} \\
 & \implies \frac{\alpha \Gamma(\alpha)}{(y-x)^{\alpha}} \frac{1}{\Gamma(\alpha)} \int_{m+M-y}^{m+M-x} (u-(m+M-y))^{\alpha-1} \psi(u) du \\
 & \leq \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} \leq \psi(m) + \psi(M) - \frac{\alpha \psi(x) + \psi(y)}{\alpha+1} \\
 & \implies \frac{\Gamma(\alpha+1)}{(y-x)^{\alpha}} J_{(m+M-x)^-}^{\alpha} \psi(m+M-y) \\
 & \leq \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} \leq \psi(m) + \psi(M) - \frac{\alpha \psi(x) + \psi(y)}{\alpha+1},
 \end{aligned} \tag{25}$$

and on combining (22) and (25), we obtain (11). $\square$

Remark 1. If we put $\alpha = 1$ in Theorem 3 and in the obtained expressions substitute $u = sx + (1-s)y$ and $u = m + M - v$, respectively, we obtain

$$\begin{aligned} \psi\left(m + M - \frac{x+y}{2}\right) &\leq \psi(m) + \psi(M) - \int_0^1 \psi(sx + (1-s)y)ds \\ &\leq \psi(m) + \psi(M) - \psi\left(\frac{x+y}{2}\right), \end{aligned} \quad (26)$$

$$\begin{aligned} \psi\left(m + M - \frac{x+y}{2}\right) &\leq \frac{1}{y-x} \int_x^y \psi(m + M - v)dv \\ &\leq \frac{\psi(m + M - x) + \psi(m + M - y)}{2} \\ &\leq \psi(m) + \psi(M) - \frac{\psi(x) + \psi(y)}{2}, \end{aligned} \quad (27)$$

respectively. Inequalities (26) and (27) have been proved in [27].

Remark 2. Substituting $\alpha = 1$, $x = m$, and $y = M$ in (11), one can obtain Hermite–Hadamard inequality.

of Hermite–Hadamard–Jensen–Mercer type inequalities, first, we present the following lemmas.

Lemma 3. Let $x, y \in [m, M]$ such that $x < y$ and let $\psi' \in L[m, M]$. Then,

3. Bounds for the Difference of Hermite–Hadamard–Jensen–Mercer Type Inequalities

Throughout this section, we consider $\psi: [m, M] \rightarrow \mathbb{R}$ is a differentiable function. To give the bounds for the difference

$$\begin{aligned} &\frac{\alpha\psi(m + M - x) + \psi(m + M - y)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(y - x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m + M - y) \\ &= \frac{y - x}{\alpha + 1} \int_0^1 ((\alpha + 1)s^\alpha - 1) \psi'(m + M - sx - (1-s)y) ds. \end{aligned} \quad (28)$$

Proof. Using the techniques of integration, we have

$$\begin{aligned}
 & \frac{y-x}{\alpha+1} \int_0^1 [(\alpha+1)s^\alpha - 1] \psi'(m+M-sx-(1-s)y) ds \\
 &= (y-x) \int_0^1 s^\alpha \psi'(m+M-sx-(1-s)y) ds \\
 & \quad - \frac{y-x}{\alpha+1} \int_0^1 \psi'(m+M-sx-(1-s)y) ds \\
 &= s^\alpha \psi(m+M-sx-(1-s)y) \Big|_0^1 - \alpha \int_0^1 s^{\alpha-1} \psi(m+M-sx-(1-s)y) ds \\
 & \quad - \frac{1}{\alpha+1} \psi(m+M-sx-(1-s)y) \Big|_0^1, \\
 &= \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \alpha \int_0^1 s^{\alpha-1} \psi(m+M-sx-(1-s)y) ds \\
 &= \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} \\
 & \quad - \frac{\alpha}{(y-x)^\alpha} \int_{m+M-y}^{m+M-x} (u-(m+M-y))^{\alpha-1} \psi(u) du \\
 &= \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y).
 \end{aligned} \tag{29}$$

(30)

Remark 3. Substituting $\alpha = 1$, $x = m$, and $y = M$ in (28), we obtain (8).

Lemma 4. Let all the assumptions of Lemma 3 hold. Then,

$$\begin{aligned}
 & \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \\
 &= (y-x) \left(\int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) \psi'(m+M-sx-(1-s)y) ds - \int_0^{\alpha/(\alpha+1)} s^\alpha \psi'(m+M-sx-(1-s)y) ds \right).
 \end{aligned} \tag{31}$$

Proof. Using techniques of integration, we have

$$\begin{aligned}
 & (y-x) \int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) \psi'(m+M-sx-(1-s)y) ds \\
 & - (y-x) \int_0^{\alpha/(\alpha+1)} s^\alpha \psi'(m+M-sx-(1-s)y) ds \\
 & = (y-x) \left[\int_{\alpha/(\alpha+1)}^1 \psi'(m+M-sx-(1-s)y) ds \right] \\
 & - (y-x) \left[\int_0^1 s^\alpha \psi'(m+M-sx-(1-s)y) ds \right] \\
 & = \psi(m+M-sx-(1-s)y) \Big|_{\alpha/(\alpha+1)}^1 - s^\alpha \psi(m+M-sx-(1-s)y) \Big|_0^1 \\
 & + \alpha \int_0^1 s^{\alpha-1} \psi(m+M-sx-(1-s)y) ds \\
 & = -\psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right) + \alpha \int_0^1 s^{\alpha-1} \psi(m+M-sx-(1-s)y) ds \\
 & = \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M-\frac{\alpha x+y}{\alpha+1}\right).
 \end{aligned} \tag{32}$$

Remark 4. If we put $\alpha = 1$, $x = m$, and $y = M$ in (31), we obtain (9).

We use Lemmas 3 and 4 and obtain bounds for the difference of the inequalities in (11).

Theorem 4. Let $|\psi'|$ be a convex function defined on $[m, M]$ and let $x, y \in [m, M]$ such that $x < y$. Then,

$$\begin{aligned}
 & \left| \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\
 & \leq \frac{y-x}{\alpha+1} \left[P_1(\alpha) |\psi'(m)| + P_1(\alpha) |\psi'(M)| - P_2(\alpha) |\psi'(x)| - P_3(\alpha) |\psi'(y)| \right],
 \end{aligned} \tag{33}$$

where

$$\begin{aligned}
 P_1(\alpha) &= \frac{2\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \\
 P_2(\alpha) &= \frac{\alpha[2 + (\alpha+1)^{2/\alpha}]}{2(\alpha+2)(\alpha+1)^{2/\alpha}}, \\
 P_3(\alpha) &= \frac{\alpha[4(\alpha+2)(\alpha+1)^{(1/\alpha)-1} - (\alpha+1)^{2/\alpha} - 2]}{2(\alpha+2)(\alpha+1)^{2/\alpha}}.
 \end{aligned} \tag{34}$$

Proof. From Lemma 3, we have

$$\begin{aligned} & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\ & \leq \frac{y-x}{\alpha+1} \int_0^1 |(\alpha+1)s^\alpha - 1| |\psi'(m+M-sx - (1-s)y)| ds. \end{aligned} \quad (35)$$

Since $|\psi'|$ is convex, using Mercer's inequality, we obtain

$$\begin{aligned} & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\ & \leq \frac{y-x}{\alpha+1} \int_0^1 |(\alpha+1)s^\alpha - 1| (|\psi'(m)| + |\psi'(M)| - s|\psi'(x)| - (1-s)|\psi'(y)|) ds \\ & = \frac{y-x}{\alpha+1} \left(\int_0^{1/(\alpha+1)^{1/\alpha}} (1 - (\alpha+1)s^\alpha) (|\psi'(m)| + |\psi'(M)| - s|\psi'(x)| - (1-s)|\psi'(y)|) ds \right. \\ & \quad \left. + \int_{1/(\alpha+1)^{1/\alpha}}^1 ((\alpha+1)s^\alpha - 1) (|\psi'(m)| + |\psi'(M)| - s|\psi'(x)| - (1-s)|\psi'(y)|) ds \right), \end{aligned} \quad (36)$$

equivalent to

$$\begin{aligned} & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\ & \leq \frac{y-x}{\alpha+1} \left([L_1(\alpha) + N_1(\alpha)] |\psi'(m)| + [L_1(\alpha) + N_1(\alpha)] |\psi'(M)| \right. \\ & \quad \left. - [L_2(\alpha) + N_2(\alpha)] |\psi'(x)| - [L_3(\alpha) + N_3(\alpha)] |\psi'(y)| \right), \end{aligned} \quad (37)$$

where

$$\begin{aligned}
L_1(\alpha) &= \int_0^{1/(\alpha+1)^{1/\alpha}} [1 - (\alpha+1)s^\alpha] ds = \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \\
L_2(\alpha) &= \int_0^{1/(\alpha+1)^{1/\alpha}} [1 - (\alpha+1)s^\alpha] s ds = \frac{\alpha}{2(\alpha+2)(\alpha+1)^{2/\alpha}}, \\
L_3(\alpha) &= \int_0^{1/(\alpha+1)^{1/\alpha}} (1 - (\alpha+1)s^\alpha)(1-s) ds = \frac{\alpha(2(\alpha+2)(\alpha+1)^{(1/\alpha)-1} - 1)}{2(\alpha+2)(\alpha+1)^{2/\alpha}}, \\
N_1(\alpha) &= \int_{1/(\alpha+1)^{1/\alpha}}^1 [(\alpha+1)s^\alpha - 1] ds = \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}, \\
N_2(\alpha) &= \int_{1/(\alpha+1)^{1/\alpha}}^1 [(\alpha+1)s^\alpha - 1] s ds = \frac{\alpha((\alpha+1)^{2/\alpha} + 1)}{2(\alpha+2)(\alpha+1)^{(2/\alpha)}}, \\
N_3(\alpha) &= \int_{1/(\alpha+1)^{1/\alpha}}^1 [(\alpha+1)s^\alpha - 1](1-s) ds \\
&= \frac{\alpha(2(\alpha+2)(\alpha+1)^{(1/\alpha)-1} - (\alpha+1)^{2/\alpha} - 1)}{2(\alpha+2)(\alpha+1)^{2/\alpha}}.
\end{aligned} \tag{38}$$

Substituting these values in (37), we get (33). $\square$

Theorem 5. Let $|\psi'|^q$ be a convex function for $q \geq 1$ and let $x, y \in [m, M]$ such that $x < y$. Then,

Remark 5. If we put $\alpha = 1$, $x = m$, and $y = M$ in (33), we get the inequality given in Theorem 2.2 of [25].

$$\begin{aligned}
&\left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\
&\leq \frac{y-x}{\alpha+1} (P_1(\alpha))^{1-1/q} \\
&\quad \times (P_1(\alpha)|\psi'(m)|^q + P_1(\alpha)|\psi'(M)|^q - P_2(\alpha)|\psi'(x)|^q - P_3(\alpha)|\psi'(y)|^q)^{1/q},
\end{aligned} \tag{39}$$

where $P_1(\alpha)$, $P_2(\alpha)$, and $P_3(\alpha)$ are the same as defined in Theorem 4.

Proof. From Lemma 3, we have (35). Applying power mean inequality, we obtain

$$\begin{aligned}
&\left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\
&\leq \frac{y-x}{\alpha+1} \left(\int_0^1 |(\alpha+1)s^\alpha - 1| ds \right)^{1-1/q} \\
&\quad \times \left(\int_0^1 |((\alpha+1)s^\alpha - 1)| |\psi'(m+M-sx - (1-s)y)|^q ds \right)^{1/q}.
\end{aligned} \tag{40}$$

Since $|\psi'|^q$ is convex, using Mercer's inequality, we have

$$\begin{aligned}
 & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\
 & \leq \frac{y-x}{\alpha+1} \left(\int_0^{1/(\alpha+1)^{1/\alpha}} [1 - (\alpha+1)s^\alpha] ds + \int_{1/(\alpha+1)^{1/\alpha}}^1 [(\alpha+1)s^\alpha - 1] ds \right)^{1-1/q} \\
 & \quad \times \left(\int_0^1 [(\alpha+1)s^\alpha - 1] (|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q) ds \right)^{1/q} \\
 & = \frac{y-x}{\alpha+1} \left(\frac{2\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}} \right)^{1-(1/q)} \\
 & \quad \times \left(\int_0^{1/(\alpha+1)^{1/\alpha}} (1 - (\alpha+1)s^\alpha) (|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q) ds \right. \\
 & \quad \left. + \int_{1/(\alpha+1)^{1/\alpha}}^1 ((\alpha+1)s^\alpha - 1) (|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q) ds \right)^{1/q}.
 \end{aligned} \tag{41}$$

This implies that

$$\begin{aligned}
 & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\
 & \leq \frac{y-x}{\alpha+1} \left(\frac{2\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}} \right)^{1-(1/q)} \left((L_1(\alpha) + N_1(\alpha)) |\psi'(m)|^q + (L_1(\alpha) + N_1(\alpha)) |\psi'(M)|^q \right. \\
 & \quad \left. - (L_2(\alpha) + N_2(\alpha)) |\psi'(x)|^q - (L_3(\alpha) + N_3(\alpha)) |\psi'(y)|^q \right)^{1/q}
 \end{aligned} \tag{42}$$

Substituting the values of L_1, L_2, L_3, N_1, N_2 , and N_3 as given in the proof of Theorem 4 in (42), we get (39). $\square$

Remark 6. If we put $\alpha = 1$, $x = m$, and $y = M$ in (39), we obtain the inequality proved in Theorem 1 of [28].

In the following theorem, we derive trapezoidal type inequality.

Theorem 6. Let $p, q > 1$ and $|\psi'|^q$ be a convex function, and let $x, y \in [m, M]$ such that $x < y$. Then,

$$\begin{aligned}
 & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\
 & \leq \frac{y-x}{\alpha+1} (L_4(\alpha, p))^{1/p} \left(|\psi'(m)|^q + |\psi'(M)|^q - \frac{|\psi'(x)|^q + |\psi'(y)|^q}{2} \right)^{1/q},
 \end{aligned} \tag{43}$$

where $L_4(\alpha, p) = \int_0^1 |(\alpha+1)s^\alpha - 1|^p ds$ such that $(1/p) + (1/q) = 1$.

Proof. Using Lemma 3, we have (35). Applying Hölder's inequality, we obtain

$$\begin{aligned}
& \left| \frac{\alpha \psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\
& \leq \frac{y-x}{\alpha+1} \left(\int_0^1 |(\alpha+1)s^\alpha - 1|^p ds \right)^{1/p} \left(\int_0^1 |\psi'(m+M-sx - (1-s)y)|^q ds \right)^{1/q} \\
& \leq \frac{y-x}{\alpha+1} (L_4(\alpha, p))^{1/p} \left(\int_0^1 (|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q) ds \right)^{1/q} \\
& = \frac{y-x}{\alpha+1} (L_4(\alpha, p))^{1/p} \left(|\psi'(m)|^q + |\psi'(M)|^q - \frac{|\psi'(x)|^q + |\psi'(y)|^q}{2} \right)^{1/q}.
\end{aligned} \tag{44}$$

Remark 7. Substituting $\alpha = 1$, $x = m$, and $y = M$ in Theorem 6, we obtain Theorem 2.3 of [25].

Theorem 7. Let $|\psi'|$ be a convex function and let $x, y \in [m, M]$ such that $x < y$. Then,

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
& \leq (y-x) (P_5(\alpha) |\psi'(x)| + P_6(\alpha) |\psi'(y)|),
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
P_5(\alpha) &= \frac{-\alpha}{2(\alpha+2)(\alpha+1)^2}, \\
P_6(\alpha) &= \frac{\alpha}{2(\alpha+2)(\alpha+1)^2}.
\end{aligned} \tag{46}$$

Proof. Using Lemma 4, we have

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
& \leq (y-x) \int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) |\psi'(m+M-sx - (1-s)y)| ds \\
& \quad - (y-x) \int_0^{\alpha/(\alpha+1)} s^\alpha |\psi'(m+M-sx - (1-s)y)| ds.
\end{aligned} \tag{47}$$

Since $|\psi'|$ is convex, using Mercer's inequality, we obtain

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
& \leq (y-x) \int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) (|\psi'(m)| + |\psi'(M)| - s|\psi'(x)| - (1-s)|\psi'(y)|) ds \\
& \quad - (y-x) \int_0^{\alpha/(\alpha+1)} s^\alpha (|\psi'(m)| + |\psi'(M)| - s|\psi'(x)| - (1-s)|\psi'(y)|) ds,
\end{aligned} \tag{48}$$

which implies that

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
& \leq (y-x) ((N_5(\alpha) - L_5(\alpha)) |\psi'(x)| + (N_6(\alpha) - L_6(\alpha)) |\psi'(y)|),
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
 L_5(\alpha) &= \int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) s ds = \frac{\alpha(\alpha+1)^\alpha + 2\alpha^{\alpha+2}}{2(\alpha+2)(\alpha+1)^{\alpha+2}}, \\
 L_6(\alpha) &= \int_{\alpha/(\alpha+1)}^1 (1-s^\alpha)(1-s) ds = \frac{4\alpha^{\alpha+1} - \alpha(\alpha+1)^\alpha}{2(\alpha+2)(\alpha+1)^{\alpha+2}}, \\
 N_5(\alpha) &= \int_0^{\alpha/(\alpha+1)} s^{\alpha+1} ds = \frac{\alpha^{\alpha+2}}{(\alpha+2)(\alpha+1)^{\alpha+2}}, \\
 N_6(\alpha) &= \int_0^{\alpha/(\alpha+1)} s^\alpha (1-s) ds = \frac{2\alpha^{\alpha+1}}{(\alpha+2)(\alpha+1)^{\alpha+2}}.
 \end{aligned} \tag{50}$$

Substituting these values in (49), we get (45).

In next theorem, we use power mean inequality and derive midpoint type inequality. $\square$

Theorem 8. Let $|\psi'|^q$ be a convex function for $q \geq 1$ and let $x, y \in (m, M)$ such that $x < y$. Then,

$$\begin{aligned}
 & \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
 & \leq (y-x) (L_8(\alpha))^{1-(1/q)} \\
 & \quad \times \left((L_8(\alpha) |\psi'(m)|^q + L_8(\alpha) |\psi'(M)|^q - L_5(\alpha) |\psi'(x)|^q - L_6(\alpha) |\psi'(y)|^q)^{1/q} \right. \\
 & \quad \left. - (L_8(\alpha) |\psi'(m)|^q + L_8(\alpha) |\psi'(M)|^q - N_5(\alpha) |\psi'(x)|^q - N_6(\alpha) |\psi'(y)|^q)^{1/q} \right),
 \end{aligned} \tag{51}$$

where

$$L_8(\alpha) = \int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) ds = \int_0^{\alpha/(\alpha+1)} s^\alpha ds = \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}, \tag{52}$$

and $L_5(\alpha)$, $L_6(\alpha)$, $N_5(\alpha)$, and $N_6(\alpha)$ are given in the proof of Theorem 7.

Proof. Using power mean inequality in (47), we obtain

$$\begin{aligned}
 & \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
 & \leq (y-x) \left(\int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) ds \right)^{1-(1/q)} \left(\int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) |\psi'(m+M-sx-(1-s)y)|^q ds \right)^{1/q} \\
 & \quad - (y-x) \left(\int_0^{\alpha/(\alpha+1)} s^\alpha ds \right)^{1-(1/q)} \left(\int_0^{\alpha/(\alpha+1)} s^\alpha |\psi'(m+M-sx-(1-s)y)|^q ds \right)^{1/q}.
 \end{aligned} \tag{53}$$

Since $|\psi'|^q$ is convex, using Mercer's inequality, we obtain

$$\begin{aligned}
 & \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
 & \leq (y-x) \left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}} \right)^{1-(1/q)} \times \left(\left(\int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) (|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q) ds \right)^{1/q} \right. \\
 & \quad \left. - \left(\int_0^{\alpha/(\alpha+1)} s^\alpha (|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q) ds \right)^{1/q} \right) \\
 & = (y-x) (L_8(\alpha))^{1-(1/q)} \times (L_8(\alpha) |\psi'(m)|^q + L_8(\alpha) |\psi'(M)|^q - L_5(\alpha) |\psi'(x)|^q - L_6(\alpha) |\psi'(y)|^q)^{1/q} \\
 & \quad - (L_8(\alpha) |\psi'(m)|^q + L_8(\alpha) |\psi'(M)|^q - N_5(\alpha) |\psi'(x)|^q - N_6(\alpha) |\psi'(y)|^q)^{1/q}.
 \end{aligned} \tag{54}$$

Remark 8. Substituting $\alpha = 1$, $x = m$, and $y = M$ in Theorem 8, we obtain the following midpoint type inequality:

$$\begin{aligned}
 & \left| \frac{1}{M-m} \int_m^M f(u) du - \psi\left(\frac{m+M}{2}\right) \right| \\
 & \leq \frac{M-m}{8} \left(\left(\frac{|\psi'(m)|^q + 2|\psi'(M)|^q}{3} \right)^{1/q} - \left(\frac{2|\psi'(m)|^q + |\psi'(M)|^q}{3} \right)^{1/q} \right).
 \end{aligned} \tag{55}$$

Another midpoint type inequality is presented in the following theorem.

Theorem 9. Let $p, q > 1$ and $|\psi'|^q$ be convex, and let $x, y \in [m, M]$ such that $x < y$. Then,

$$\begin{aligned}
 & \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
 & \leq (y-x) (L_9(\alpha, p))^{1/p} \\
 & \quad \times \left(\frac{1}{\alpha+1} |\psi'(m)|^q + \frac{1}{\alpha+1} |\psi'(M)|^q - \frac{2\alpha+1}{2(\alpha+1)^2} |\psi'(x)|^q - \frac{1}{2(\alpha+1)^2} |\psi'(y)|^q \right)^{1/q} \\
 & \quad - (y-x) (L_{10}(\alpha, p))^{1/p} \\
 & \quad \times \left(\frac{\alpha}{\alpha+1} |\psi'(m)|^q + \frac{\alpha}{\alpha+1} |\psi'(M)|^q - \frac{\alpha^2}{2(\alpha+1)^2} |\psi'(x)|^q - \frac{\alpha(\alpha+2)}{2(\alpha+1)^2} |\psi'(y)|^q \right)^{1/q},
 \end{aligned} \tag{56}$$

where $L_9(\alpha, p) = \int_{\alpha/(\alpha+1)}^1 (1-s^\alpha)^p ds$ and $L_{10}(\alpha, p) = \int_0^{\alpha/(\alpha+1)} s^{\alpha p} ds$ such that $(1/p) + (1/q) = 1$. *Proof.* Applying Hölder's inequality in (47), we have

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\ & \leq (y-x) \left(\int_{\alpha/(\alpha+1)}^1 (1-s^\alpha)^p ds \right)^{1/p} \left(\int_{\alpha/(\alpha+1)}^1 |\psi'(m+M-sx-(1-s)y)|^q ds \right)^{1/q} \\ & \quad - (y-x) \left(\int_0^{\alpha/(\alpha+1)} s^{\alpha p} dt \right)^{1/p} \left(\int_0^{\alpha/(\alpha+1)} |\psi'(m+M-sx-(1-s)y)|^q ds \right)^{1/q}. \end{aligned} \quad (57)$$

As $|\psi'|^q$ is convex, applying Mercer's inequality, we obtain

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\ & \leq (y-x) (L_9(\alpha, p))^{1/p} \times \left(\int_{\alpha/(\alpha+1)}^1 (|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q) ds \right)^{1/q} \\ & \quad - (y-x) (L_{10}(\alpha, p))^{1/p} \\ & \quad \times \left(\int_0^{\alpha/(\alpha+1)} (|\psi'(m)|^q + |\psi'(M)|^q - s|\psi'(x)|^q - (1-s)|\psi'(y)|^q) ds \right)^{1/q} \\ & = (y-x) (L_9(\alpha, p))^{1/p} \times \left(\frac{1}{\alpha+1} |\psi'(m)|^q + \frac{1}{\alpha+1} |\psi'(M)|^q - \frac{2\alpha+1}{2(\alpha+1)^2} |\psi'(x)|^q - \frac{1}{2(\alpha+1)^2} |\psi'(y)|^q \right)^{1/q} \\ & \quad - (y-x) (L_{10}(\alpha, p))^{1/p} \\ & \quad \times \left(\frac{\alpha}{\alpha+1} |\psi'(m)|^q + \frac{\alpha}{\alpha+1} |\psi'(M)|^q - \frac{\alpha^2}{2(\alpha+1)^2} |\psi'(x)|^q - \frac{\alpha(\alpha+2)}{2(\alpha+1)^2} |\psi'(y)|^q \right)^{1/q}. \end{aligned} \quad (58)$$

Remark 9. Substituting $\alpha = 1$, $x = m$, and $y = M$ in Theorem 9, we obtain

$$\begin{aligned} & \left| \frac{1}{M-m} \int_m^M f(u) du - \psi\left(\frac{m+M}{2}\right) \right| \\ & \leq \frac{M-m}{4(p+1)^{1/p}} \left(\left(\frac{|\psi'(m)|^q + 3|\psi'(M)|^q}{4} \right)^{1/q} - \left(\frac{3|\psi'(m)|^q + |\psi'(M)|^q}{4} \right)^{1/q} \right). \end{aligned} \quad (59)$$

Theorem 10. Let $x, y \in [m, M]$ such that $x < y$. If $|\psi'|$ is concave on $[m, M]$, then

$$\begin{aligned} & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\ & \leq \frac{y-x}{\alpha+1} \left(L_1(\alpha) |\psi'(m+M-L_{11}(\alpha)x - L_{12}(\alpha)y)| \right. \\ & \quad \left. + N_1(\alpha) |\psi'(m+M-L_{13}(\alpha)x - L_{14}(\alpha)y)| \right), \end{aligned} \quad (60)$$

where L_1 and N_1 are given in the proof of Theorem 4 and

$$\begin{aligned} L_{11}(\alpha) &= \frac{(\alpha+1)^{1-(1/\alpha)}}{2(\alpha+2)}, \\ L_{12}(\alpha) &= \frac{(\alpha+1)^{1-(1/\alpha)} [2(\alpha+2)(\alpha+1)^{(1/\alpha)-1} - 1]}{2(\alpha+2)}, \\ L_{13}(\alpha) &= \frac{(\alpha+1)^{1-(1/\alpha)} [(\alpha+1)^{2/\alpha} + 1]}{2(\alpha+2)}, \\ L_{14}(\alpha) &= \frac{(\alpha+1)^{1-(1/\alpha)} [2(\alpha+2)(\alpha+1)^{(1/\alpha)-1} - (\alpha+1)^{2/\alpha} - 1]}{2(\alpha+2)}. \end{aligned} \quad (61)$$

Proof. Lemma 3 implies that

$$\begin{aligned} & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\ & \leq \frac{y-x}{\alpha+1} \int_0^1 |(\alpha+1)s^\alpha - 1| |\psi'(m+M-sx - (1-s)y)| ds \\ & = \frac{y-x}{\alpha+1} \left(\int_0^{1/(\alpha+1)^{1/\alpha}} (1 - (\alpha+1)s^\alpha) |\psi'(m+M-sx - (1-s)y)| ds \right. \\ & \quad \left. + \int_{1/(\alpha+1)^{1/\alpha}}^1 ((\alpha+1)s^\alpha - 1) |\psi'(m+M-sx - (1-s)y)| ds \right). \end{aligned} \quad (62)$$

Since $|\psi'|$ is concave, using Jensen's inequality, we obtain

$$\begin{aligned}
 & \left| \frac{\alpha\psi(m+M-x) + \psi(m+M-y)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) \right| \\
 & \leq \frac{y-x}{\alpha+1} \left(\int_0^{1/(\alpha+1)^{1/\alpha}} (1-(\alpha+1)s^\alpha) ds \times \left| \psi' \left(\frac{\int_0^{1/(\alpha+1)^{1/\alpha}} (1-(\alpha+1)s^\alpha)(m+M-sx-(1-s)y) ds}{\int_0^{1/(\alpha+1)^{1/\alpha}} (1-(\alpha+1)s^\alpha) ds} \right) \right| \right. \\
 & \quad \left. + \int_{1/(\alpha+1)^{1/\alpha}}^1 ((\alpha+1)s^\alpha - 1) ds \times \left| \psi' \left(\frac{\int_{1/(\alpha+1)^{1/\alpha}}^1 ((\alpha+1)s^\alpha - 1)(m+M-sx-(1-s)y) ds}{\int_{1/(\alpha+1)^{1/\alpha}}^1 ((\alpha+1)s^\alpha - 1) ds} \right) \right| \right) \\
 & = \frac{y-x}{\alpha+1} (L_1(\alpha) |\psi'(m+M-L_{11}(\alpha)x-L_{12}(\alpha)y)| \\
 & \quad + N_1(\alpha) |\psi'(m+M-L_{13}(\alpha)x-L_{14}(\alpha)y)|).
 \end{aligned} \tag{63}$$

Theorem 11. Let $|\psi'|$ be a concave function, and let $x, y \in [m, M]$ such that $x < y$. Then,

$$\begin{aligned}
 & \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
 & \leq (y-x) \left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}} \right) (|\psi'(m+M-L_{15}(\alpha)x-L_{16}(\alpha)y)| \\
 & \quad - |\psi'(m+M-L_{17}(\alpha)x-L_{18}(\alpha)y)|),
 \end{aligned} \tag{64}$$

where

$$L_{15}(\alpha) = \frac{(\alpha+1)^\alpha + 2\alpha^{\alpha+1}}{2\alpha^\alpha(\alpha+2)},$$

$$L_{16}(\alpha) = \frac{4\alpha^\alpha - (\alpha+1)^\alpha}{2\alpha^\alpha(\alpha+2)}, \tag{65}$$

$$L_{17}(\alpha) = \frac{\alpha}{\alpha+2},$$

$$L_{18}(\alpha) = \frac{2}{\alpha+2}.$$

Proof. From Lemma 4, we have (47). As $|\psi'|$ is concave, using Jensen's inequality, we obtain

$$\begin{aligned}
 & \left| \frac{\Gamma(\alpha+1)}{(y-x)^\alpha} J_{(m+M-x)^-}^\alpha \psi(m+M-y) - \psi\left(m+M - \frac{\alpha x + y}{\alpha+1}\right) \right| \\
 & \leq (y-x) \int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) ds \left| \psi' \left(\frac{\int_{\alpha/(\alpha+1)}^1 (1-s^\alpha)(m+M-sx-(1-s)y) ds}{\int_{\alpha/(\alpha+1)}^1 (1-s^\alpha) ds} \right) \right| \\
 & \quad - (y-x) \int_0^{\alpha/(\alpha+1)} s^\alpha ds \left| \psi' \left(\frac{\int_0^{\alpha/(\alpha+1)} s^\alpha (m+M-sx-(1-s)y) ds}{\int_0^{\alpha/(\alpha+1)} s^\alpha ds} \right) \right| \\
 & = (y-x) \left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}} \right) (|\psi'(m+M-L_{15}(\alpha)x-L_{16}(\alpha)y)| \\
 & \quad - |\psi'(m+M-L_{17}(\alpha)x-L_{18}(\alpha)y)|).
 \end{aligned} \tag{66}$$

4. Conclusion

In this paper, we establish the fractional Hermite–Hadamard type inequalities of Mercer type by using support line inequality. We expect that this work will lead to the new fractional integral studies for Hermite–Hadamard inequality. It is an open problem to prove inequalities (10) and (11) by any other method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to this paper. All authors read and approved the final manuscript.

References

- [1] J. Pečarić, F. Prochan, and Y. L. Tong, *Convex Functions, Partial Ordering and Statistical Applications*, Academic Press, New York, NY, USA, 1991.
- [2] A. Ekinici, M. E. Özdemir, and E. Set, “New integral inequalities of Ostrowski type for quasi-convex functions with applications,” *Turkish Journal of Science*, vol. 5, no. 3, pp. 290–304, 2020.
- [3] M. Gürbüz and M. E. Özdemir, “On some inequalities for product of different kinds of convex functions,” *Turkish Journal of Science*, vol. 5, no. 1, pp. 23–27, 2020.
- [4] M. Adil Khan, J. Khan, and J. Pečarić, “Generalization of Jensen’s and Jensen-Steffensen’s inequalities by generalized majorization theorem,” *Journal of Mathematical Inequalities*, vol. 11, no. 4, pp. 1049–1074, 2017.
- [5] J. Khan, M. Adil Khan, and J. Pecaric, “On Jensen’s type inequalities via generalized majorization inequalities,” *FILOMAT*, vol. 32, no. 16, pp. 5719–5733, 2018.
- [6] A. Mercer, “A variant of Jensen’s inequality,” *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 4, p. 73, 2003.
- [7] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York, NY, USA, 1973.
- [8] J. Hadamard, “Étude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann,” *Journal de Mathématiques Pures et Appliquées*, vol. 58, pp. 171–215, 1893.
- [9] M. Adil Khan, A. Iqbal, M. Suleman, and Y. M. Chu, “Hermite-Hadamard type inequalities for fractional integrals via Green’s function,” *Journal of Inequalities and Applications*, vol. 2018, p. 161, 2018.
- [10] M. Adil Khan, T. U. Khan, and Y. M. Chu, “Generalized Hermite-Hadamard type inequalities for quasi-convex functions with applications,” *Journal of Inequalities and Special Functions*, vol. 11, no. 1, pp. 24–42, 2020.
- [11] M. Adil Khan, Y. Khurshid, T. S. Du, and Y. M. Chu, “Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals,” *Journal of Function Spaces*, vol. 2018, Article ID 5357463, , 2018.
- [12] S. I. Butt, M. Umar, S. Rashid, A. O. Akdemir, and Y. M. Chu, “New Hermite-Jensen-Mercer-type inequalities via k-fractional integrals,” *Advances in Difference Equations*, vol. 635, pp. 1–24, 2020.
- [13] S. S. Dragomir and C. E. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, Footscray, Australia, 2000.
- [14] M. Gürbüz, A. O. Akdemir, S. Rashid, and E. Set, “Hermite-Hadamard inequality for fractional integrals of Caputo-Fabrizio type and related inequalities,” *Journal of Inequalities and Applications*, vol. 172, pp. 1–10, 2020.
- [15] A. Iqbal, M. Adil Khan, N. Mohammed, E. R. Nwaeze, and Y. M. Chu, “Revisiting the Hermite-Hadamard fractional integrals via Green’s function,” *AIMS Mathematics*, vol. 5, pp. 6087–6107, 2020.
- [16] M. E. Özdemir, M. Avci, and H. Kavurmaci, “Hermite-Hadamard type inequalities for s-convex and s-concave functions via fractional integrals,” *Turkish Journal of Science*, vol. 1, no. 1, pp. 28–40, 2016.
- [17] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, “Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities,” *Mathematical and Computer Modeling*, vol. 57, no. 9–10, pp. 2403–2407, 2013.
- [18] E. Set, İ. İşcan, M. Zeki Sarikaya, and M. Emin Özdemir, “On new inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals,” *Applied Mathematics and Computation*, vol. 259, pp. 875–881, 2015.
- [19] S. I. Butt, M. Nadeem, and G. Farid, “On Caputo fractional derivatives via exponential s-convex functions,” *Turkish Journal of Science*, vol. 5, no. 2, pp. 140–146, 2020.
- [20] A. Ekinici and M. E. Özdemir, “Some new integral inequalities via Riemann-Liouville integral operators,” *Applied and Computational Mathematics*, vol. 3, no. 18, pp. 288–295, 2019.
- [21] M. K. Sajid, R. S. Ali, and I. Nayab, “Some results of generalized k-fractional integral operator with k-Bessel function,” *Turkish Journal of Science*, vol. 5, no. 3, pp. 157–169, 2020.
- [22] E. Set, A. O. Akdemir, and F. Demirci, “Grüss type inequalities for fractional integral operator involving the extended generalized Mittag-Leffler function,” *Applied and Computational Mathematics*, vol. 19, no. 3, pp. 402–414, 2020.
- [23] M. Kunt, D. Karapinar, S. Turhan, and İ. İşcan, “The left Riemann-Liouville fractional Hermite-Hadamard type inequalities for convex functions,” *Mathematica Slovaca*, vol. 69, no. 4, pp. 773–784, 2019.
- [24] S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Hoboken, NJ, USA, 1993.
- [25] S. S. Dragomir and R. P. Agarwal, “Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,” *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.
- [26] U. S. Kirmaci, “Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula,” *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004.
- [27] M. Kian and M. S. Moslehian, “Refinement of the operator Jensen-Mercer inequality,” *The Electronic Journal of Linear Algebra*, vol. 26, p. 50, 2013.
- [28] C. E. M. Pearce and J. Pečarić, “Inequalities for differentiable mappings with application to special means and quadrature formulæ,” *Applied Mathematics Letters*, vol. 13, no. 2, pp. 51–55, 2000.