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Research Article

Hamiltonicity of 3_tEC Graphs with $\alpha = \kappa + 1$

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A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set of G is the total domination number $\gamma_t(G)$ of G. The graph G is total domination edge-critical, or γ_tEC, if for every edge e in the complement of G, $\gamma_t(G+e) < \gamma_t(G)$. If G is γ_tEC and $\gamma_t(G) = k$, we say that G is k_tEC. In this paper, we show that every 3_tEC graph with $\delta(G) \geq 2$ and $\alpha(G) = \kappa(G) + 1$ has a Hamilton cycle.

1. Introduction

All graphs we considered here are finite, undirected, and simple. We refer to [1] for unexplained terminology and notations. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, the open neighborhood of v is the set $N(v) = \{u \in V\colon uv \in E(G)\}$, and closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, its open neighborhood is the set $N(S) = \cup_{v\in S} N(v)$, and its closed neighborhood is the set $N[S] = N(S) \cup S$. The complement $\overline{G}$ of G has vertex set $V(G)$ and edge set $\{xy | \{x, y\} \subseteq V(G), xy \notin E(G)\}$. The subgraph of G induced by the nonempty vertex subset $S \subseteq V(G)$ will be denoted by $G[S]$, and set $G - S = G[V(G) - S]$. We use $\omega(G - S)$ to denote the number of components of $G - S$. An independent set of the graph G is a set of vertices, no two of which are adjacent. The independence number of G, denoted by $\alpha(G)$, is the order of a maximum independent set in G. The connectivity of G, denoted by $\kappa(G)$, is defined as the order of the minimum vertex subset whose deletion resulting in G is disconnected. We often simply write α for $\alpha(G)$ and κ for $\kappa(G)$, respectively. A Hamilton path of G is a path passing exactly once through every vertex of G. A Hamilton cycle is a closed Hamilton path. We refer to [2–4] for some recent results on the Hamiltonian properties of graphs.

The domination theory of graphs is an important part of graph theory because of its wide range of applications and theoretical significance [5, 6]. A subset $S \subseteq V(G)$ is called a dominating set of a graph G if for every $v \in V(G)$, either $v \in S$ or v is adjacent to a vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. The graph G is said to be domination edge-critical, or γEC, if for every edge $e \in E(\overline{G})$, $\gamma(G+e) < \gamma(G)$. If G is γEC and $\gamma(G) = k$, we say that G is kEC. A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set of G is the total domination number $\gamma_t(G)$ of G. We refer to [7–12] for some recent results on the domination number and total domination number of G. As introduced in [13], a graph G is total domination edge-critical, or γ_tEC, if for every edge $e \in E(\overline{G})$, $\gamma_t(G+e) < \gamma_t(G)$. If G is γ_tEC and $\gamma_t(G) = k$, we say that G is k_tEC. We refer to [14–18] for some results on γ_tEC graphs.

In [11], Pan et al. proved that the total edge domination problem is NP-complete for bipartite graphs with maximum degree 3. In [19], Furuya and Matsumoto showed that the order of a connected 3-edge-critical planar graph is at most 23.

For the sake of convenience, we write $u \Rightarrow v$ (or $v \Rightarrow u$) if $uv \in E(G)$. In general, for subsets S and T of $V(G)$, if every vertex in T has a neighbor in S, we say that S dominates T and write $S \Rightarrow T$.

The study of total domination edge-critical graphs was initiated in [13]. It was shown in [13] that the addition of an edge to a graph can change the total domination number by at most two. If G is γ_tEC graph, $\gamma_t(G) - 2 \leq \gamma_t(G+e) \leq \gamma_t(G)$ for any edge $e \in E(\overline{G})$. Since $\gamma_t(G) \geq 2$ for any graph G, if G is 3_tEC, then $\gamma_t(G+e) = 2$ for any edge $e \in E(\overline{G})$. Also, observe that any graph G with $\gamma_t(G) = 3$ is connected. In [14], Asplund et al. constructed some families of γ_tEC graphs with diameter 2 and determined which γ_tEC graphs have complements that are 3_tEC graphs.

We also use the following results on 3_tEC graphs.

Theorem 1 (see [13]). *For any 3_tEC graph G and nonadjacent vertices u and v, at least one of the following holds:*

(1) $\{u, v\}$ dominates G

(2) There exists $w \in N(u)$ such that $\{u, w\}$ dominates $G - v$, but not v, and we write $uw \rightarrow v$

(3) There exists $w \in N(v)$ such that $\{v, w\}$ dominates $G - u$, but not u, and we write $vw \rightarrow u$

Theorem 2 (see [13]). *A graph G with a cut vertex v is a 3_tEC graph if and only if v is adjacent to a pendent vertex x, and for $W = N(v)\backslash\{x\}$ and $Y = V\backslash N[v]$,*

(1) $G[W]$ is complete and $|W| \geq 2$

(2) $G[Y]$ is complete and $|Y| \geq 2$

(3) Every vertex in W is adjacent to $|Y| - 1$ vertices in Y, and every vertex in Y is adjacent to at least one vertex in W

The following result is immediate from the above theorem.

Corollary 1. *If G is a 3_tEC graph, then $\kappa(G) = 1$ if and only if $\delta(G) = 1$; $\kappa(G) \geq 2$ if and only if $\delta(G) \geq 2$.*

In [20], we prove the following results.

Theorem 3 (see [20]). *If G is a 3_tEC graph, then $\alpha(G) \leq \kappa(G) + 2$, with equality only if $\kappa(G) = \delta(G)$.*

Theorem 4 (see [20]). *Let G be a 3_tEC graph with $\delta(G) \geq 2$. If $\alpha(G) \neq \kappa(G) + 1$, then G is Hamiltonian.*

In this paper, our main result is as follows.

Theorem 5. *Let G be a 3_tEC graph with $\delta(G) \geq 2$. If $\alpha(G) = \kappa(G) + 1$, then G is Hamiltonian.*

Combining Theorems 3–5, we have the following.

Theorem 6. *Every 3_tEC graph with $\delta(G) \geq 2$ is Hamiltonian.*

The remainder of this paper is organized as follows. In Section 2, we list some useful notations and lemmas. The proof of our main result is presented in Section 3.

2. Useful Lemmas

In this section, we list some useful notations and lemmas.

For a 3_tEC graph G, if $\kappa(G) = 1$, then G contains a cut vertex. By Theorem 2, there exists a pendant vertex adjacent to such cut vertex, which means that $\delta(G) = 1$. If $\delta(G) = 1$, then it is clear that $\kappa(G) = 1$. Thus, we have that for a 3_tEC graph G, $\kappa(G) = 1$ if $\delta(G) = 1$. It follows that for a 3_tEC graph G, if $\kappa(G) \neq 1$ if $\delta(G) \neq 1$. Since 3_tEC graph G that we consider is connected, $\kappa(G) \neq 0$, and it does not contain isolated vertices, that is, $\delta(G) \neq 1$. It implies that for a 3_{EC} graph G, $\kappa(G) \geq 2$ if $\delta(G) \geq 2$. Corollary 1. Hence, we may assume $\kappa(G) \geq 2$. Let C be a cycle. We denote by $\overrightarrow{C}$ the cycle C with a given orientation and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in the reverse order, are given by $v\overleftarrow{C}u$. We use u^+ to denote the successor of u and u^- to denote its predecessor.

Let G be a graph and C a longest cycle of G. Suppose G has no Hamilton cycle and H is any component of $G - C$. We set

$$\begin{aligned} N_C(H) &= X = \{x_1, x_2, \ldots, x_k\}, \\ A &= \{a_1, a_2, \ldots, a_k\}, \quad \text{where } a_i = x_i^+, \\ B &= \{b_1, b_2, \ldots, b_k\}, \quad \text{where } b_i = x_{i+1}^-, \\ C_i &= a_i \overrightarrow{C} b_i. \end{aligned} \tag{1}$$

Let $x_i, x_j \in X$ with $i \neq j$. $x_i H x_j$ denotes a longest (x_i, x_j)-path with the internal vertices in $V(H)$. The following Lemmas 1–4 can be found in [21], which play an important role in our proof.

Lemma 1 (see [21]). *Let $a_i, a_j \in A$ ($b_i, b_j \in B$, resp.) with $i \neq j$. Then, there is no path in $G - ((V(C) - \{a_i, a_j\}) \cup V(H))$, ($G - ((V(C) - \{b_i, b_j\}) \cup V(H))$, resp.) connecting a_i and a_j (b_i and b_j, resp.).*

Lemma 2 (see [21]). *For any $h \in V(H)$, both $A \cup \{h\}$ and $B \cup \{h\}$ are independent sets.*

Lemma 3 (see [21]). *Let $a_i, a_j \in A$ with $i \neq j$. For any vertex $v \in a_i^+ \overrightarrow{C} a_j^-$, if $va_i \in E(G)$, then $v^- a_j \notin E(G)$.*

Lemma 4 (see [21]). *Let $a_i \in A$ and $b_j \in B$ with $i \neq j + 1$. For any vertex $v \in a_{j+1} \overrightarrow{C} x_i$, if $vb_j \in E(G)$, then $v^- a_i \notin E(G)$. Similarly, for any vertex $v \in x_{j+1} \overrightarrow{C} b_{i-1}$, if $vb_j \in E(G)$, then $v^+ a_i \notin E(G)$.*

In [22], Simmons obtained the following result.

Lemma 5 (see [22]). *Let G be a 3_tEC graph with $\delta(G) \geq 2$ and T a cut vertex of G. Then, $\omega(G - T) \leq |T| + 1$.*

We give the following useful lemma.

Lemma 6. *If W is an independent set of $k \geq 3$ vertices of a $3_t EC$ graph G, then there exist an ordering $w_1, w_2, \ldots, w_k$ of the vertices of W and a path $x_1 x_2, \ldots, x_{k-1}$ in $V(G)\backslash W$ such that $w_i x_i \rightarrow w_{i+1}$ for $i = 1, 2, \ldots, k-1$.*

Proof. First, we show the first part of the statement. Since W is an independent set of G, it is a clique of $\overline{G}[W]$. For any two vertices $u, v \in W$, since $k \geq 3$, $\{u, v\}$ is not a total dominating set of $G + uv$. By Theorem 1, if there exists a vertex $x \in V(G)$ with $ux \rightarrow v$, then we orient uv from u to v, and if there exists a vertex $x \in V(G)$ with $vx \rightarrow u$, we orient it from v to u in $\overline{G}[W]$. Now, $\overline{G}[W]$ becomes a tournament under this orientation. By Rédei theorem [1], let $w_1 w_2, \ldots, w_k$ be a directed Hamilton path of $\overline{G}[W]$. For an integer $i \in \{1, \ldots, k-1\}$, we consider w_i and w_{i+1}. By our convention, there exists a neighbor x_i of w_i such that $w_i x_i \rightarrow w_{i+1}$. Thus, $x_i \in V(G)\backslash W$.

Now, we show that $x_i \neq x_j$, for any $i, j \in \{1, 2, \ldots, k-1\}$ with for each $i \neq j$. Since $w_j x_j \rightarrow w_{j+1}$, $x_j w_{j+1} \notin E(G)$. On the contrary, since $w_i x_i \rightarrow w_{i+1}$ and $w_{i+1} \neq w_{j+1}$, $x_i w_{j+1} \in E(G)$. This implies that $x_i \neq x_j$.

Since $w_i x_i \rightarrow w_{i+1}$ and $x_{i+1} \neq w_{i+1}$, $x_i x_{i+1} \in E(G)$. This proves $x_1 x_2, \ldots, x_{k-1}$ is a path in $G - W$. □

3. Proof of Theorem 5

Let G be a graph and C a longest cycle of G. Suppose G has no Hamilton cycle and H is any component of $G - C$. We consider the following two cases separately to complete the proof.

Case 1. $\kappa(G) = 2$.

Suppose G has no Hamilton cycle, and let C be a longest cycle of G. Since $\alpha(G) = \kappa(G) + 1$, $\alpha(G) = 3$. By Lemma 2, $|N_C(H)| \leq 2$. If $|N_C(H)| = 1$, then $\kappa(G) = 1$. This contradicts $\kappa(G) = 2$. Thus,

$$\left|N_C(H)\right| = 2 \text{ for any component } H \text{ of } G - C. \tag{2}$$

Claim 1. If $|C_i| \geq 2$, then $a_i b_i \in E(G)$ for each $1 \leq i \leq 2$.

Proof. First, we show that if $|C_1| \geq 2$, then $a_1 b_1 \in E(G)$; otherwise, we suppose that $a_1 b_1 \notin E(G)$. Since $\alpha(G) = 3$, by Lemma 2, we have $|C_2| \geq 2$ and $a_2 b_1, a_1 b_2 \in E(G)$. Since $\{a_1, b_1, h\}$ is an independent set, where $h \in H$, we have $\{a_1, b_1\} \Rightarrow C_2$. This implies that there is some vertex $v \in C_2$ such that $v b_1, v^+ a_1 \in E(G)$, which contradicts Lemma 4. Hence, we have $a_1 b_1 \in E(G)$ if $|C_1| \geq 2$. Similarly, if $|C_2| \geq 2$, then $a_2 b_2 \in E(G)$. □

Claim 2. For any $u \in C_1$ with $a_1 u^+ \in E(G)$ and $v \in C_2$ with $a_2 v^+ \in E(G)$, $uv \notin E(G)$; for any $u \in C_1$ with $b_1 u^- \in E(G)$ and $v \in C_2$ with $b_2 v^- \in E(G)$, $uv \notin E(G)$.

Proof. Otherwise, $u\overleftarrow{C}a_1 u^+ \overrightarrow{C} x_2 H x_1 \overleftarrow{C} v^+ a_2 \overrightarrow{C} vu$ is a cycle longer than C, a contradiction. By symmetry, for any $u \in C_1$ with $b_1 u^- \in E(G)$ and $v \in C_2$ with $b_2 v^- \in E(G)$, $uv \notin E(G)$. □

Claim 3. For each vertex $v \in C_i \backslash \{a_i\}, a_i v \in E(G)$, and for each vertex $v \in C_i \backslash \{b_i\}, b_i v \in E(G)$, where $i = 1, 2$.

Proof. Without loss of generality, we show that, for each vertex $v \in C_1\{a_1\}, a_1 v \in E(G)$. By contradiction, assume that y is the last vertex in $\overrightarrow{C}[a_1^+, b_1]$ which is not adjacent to a_1; thus, $a_1 y \notin E(G)$. If $a_2 y \in E(G)$, then $x_2 \overleftarrow{C} y^+ a_1 \overrightarrow{C} y a_2 \overrightarrow{C} x_1 H x_2$ is a cycle longer than C, a contradiction. We have $a_2 y \notin E(G)$. Thus, $\{a_1, y, a_2, h\}$ is an independent set of four vertices which contradicts $\alpha(G) = 3$, where $h \in H$. □

Claim 4. $\omega(G - C) = 1$.

Proof. Suppose to the contrary that H' is another component of $G - C$. Take $h \in H$ and $h' \in H'$. Since $\alpha(G) = 3$, we have $N_C(H') \cap A \neq \varnothing$; otherwise, $A \cup \{h, h'\}$ is an independent set of four vertices. Similarly, $N_C(H') \cap B \neq \varnothing$. By Lemma 1, we may assume $N_C(H') \cap A = \{a_1\}$. If $b_2 \in N_C(H')$, then $C' = a_1 \overrightarrow{C} b_2 H' a_1$ is a cycle not less than C, and $H_0 = H \cup \{x_1\}$ is a component of $G - C'$, but $\{a_1, b_2, x_2\} \subseteq N_{C'}(H_0)$, $|N_{C'}(H_0)| \geq 3$, which contradicts (2). Hence, we have $b_1 \in N_C(H')$. By the maximality of C, we have $|C_1| \geq 3$. By Claims 1, 2, and 3, we have $a_1^+ a_2 \notin E(G)$; thus, $\{a_1^+, a_2, h, h'\}$ is an independent set of four vertices which contradicts $\alpha(G) = 3$. Thus, we have $\omega(G - C) = 1$. □

Claim 5. $\omega(G - T - H) = 1$, where $T = \{x_1, x_2\}$.

Proof. By Lemma 5 and Claim 4, we have $\omega(G - T - H) \leq 2$. If $\omega(G - T - H) = 2$, then $E(C_1, C_2) = \varnothing$. Consider a_1 and h, $a_1 h \notin E(G)$, where $h \in H$. $\{a_1, h\}$ is not a domination set of G since $\{a_1, h\}$ cannot dominate C_2. By Theorem 1, there exists a vertex x such that $a_1 x \rightarrow h$ or $xh \rightarrow a_1$. If $a_1 x \rightarrow h$, then $x \in N(a_1)$ and $x \in C_1 \backslash \{a_1\}$, but this is impossible since C_2 is not dominated. Thus, $xh \rightarrow a_1$, we have $x = x_2$, and hence,

$$x_2 \Rightarrow V(G) \backslash \{a_1, x_1\}. \tag{3}$$

Consider a_2 and h, $a_2 h \notin E(G)$. $\{a_2, h\}$ cannot dominate G since C_1 is not dominated; by Theorem 1, there exists a vertex x such that $a_2 x \rightarrow h$ or $hx \rightarrow a_2$. If $a_2 x \rightarrow h$, then $x \in C_2 \backslash \{a_2\}$, but this is impossible since C_1 is not dominated. Thus, $hx \rightarrow a_2$, we have $x = x_1$, and hence,

$$x_1 \Rightarrow V(G) \backslash \{a_2, x_2\}. \tag{4}$$

Consider b_1 and b_2, $b_1 b_2 \notin E(G)$, and $\{b_1, b_2\}$ cannot dominate G since $V(H)$ is not dominated; by Theorem 1, there exists a vertex z such that $b_1 z \rightarrow b_2$ or $b_2 z \rightarrow b_1$. If $b_1 z \rightarrow b_2$, in order to dominate $V(H)$, then $z = x_2$, but this is impossible because of (3). If $b_2 z \rightarrow b_1$, then $z = x_1$, but this is impossible because of (4). Thus, we get $\omega(G - T - H) = 1$. □

Claim 6. $E(C_1, C_2) = \{a_1b_2, a_2b_1\}$.

Proof. By Claim 5, we have $E(C_1, C_2) \neq \varnothing$. By Claims 2 and 3, we have $E(C_1, C_2) \subseteq \{a_1b_2, a_2b_1\}$. Assume $a_1b_2 \in E(G)$; by Lemma 1, $|C_i| \geq 2$ for $i = 1, 2$. We now show that $a_2b_1 \in E(G)$. If $a_2b_1 \notin E(G)$, then $\{a_2, b_1\}$ cannot dominate G since $V(H)$ is not dominated, and there exists a vertex z such that $a_2z \rightarrow b_1$ or $b_1z \rightarrow a_2$. If $a_2z \rightarrow b_1$, then z dominates $V(H)$, so $z = x_1$. We have $a_2x_1 \rightarrow b_1$ and $a_2x_1 \in E(G)$, but $a_2x_1 \notin E(G)$ by Lemma 3, a contradiction. If $b_1z \rightarrow a_2$, then $z = x_1$, and $a_1b_2\overleftarrow{C}x_2Hx_1b_1\overleftarrow{C}a_1$ is a cycle longer than C, a contradiction. Thus, $a_2b_1 \in E(G)$. Hence, $E(C_1, C_2) = \{a_1b_2, a_2b_1\}$.

By Claim 6, we have $|C_i| \geq 3$ for $i = 1, 2$. Otherwise, we suppose that $|C_1| = 2$; then, $a_1x_1Hx_2b_1a_2\overrightarrow{C}b_2a_1$ is a cycle longer than C, a contradiction. By symmetry, we have $|C_2| \geq 3$. Consider a_1 and a_2, $\{a_1, a_2\}$ cannot dominate G, and without loss of generality, there exists a vertex y such that $a_1y \rightarrow a_2$; in order to dominate $V(H)$, then $y = x_1$ and $a_1x_1 \rightarrow a_2$. By Claims 2 and 3, $x_1 \Rightarrow C_2\backslash\{a_2, b_2\}$, and hence, $x_1Hx_2a_2b_1\overleftarrow{C}a_1b_2\overleftarrow{C}a_2^+x_1$ is a cycle longer than C, a contradiction. Thus, G has a Hamilton cycle. □

Case 2. $\kappa(G) \geq 3$.

Let G be a graph and C a longest cycle of G. Suppose G has no Hamilton cycle and H is a component of $G - C$. Set $N_G(H) \cap V(C) = X$, and define A, B, and C_i as before; then, $\kappa(G) \leq k$. By Lemma 2, $\alpha(G) \geq k + 1$. We have $\kappa(G) \geq k$ since $\alpha(G) = \kappa(G) + 1$. Thus, $\kappa(G) = k$.

Considering A and h, where $h \in H$, by Lemma 2, $A \cup \{h\}$ is an independent set. Let $T = y_1y_2, \ldots, y_{k-1}$ be the path of A defined in Lemma 6. Since $|A| \geq 3$, by Lemma 6, we have $a_iy_i \rightarrow a_{i+1}$ and $ha_i \notin E(G)$ for $i = 1, 2, \ldots, k - 1$; then, $y_ia_i \in E(G)$ and $y_ih \in E(G)$. Thus, $\{y_1, \ldots, y_{k-1}\} \subseteq N(h)$ and $y_i \in X$. We may assume $a_{i_j}x_{i_j} \rightarrow a_{i_{j+1}}$, $1 \leq j \leq k - 1$. Thus, we can obtain that $V(H) \subseteq N_G(x_{i_j})$ for $1 \leq j \leq k - 1$. Take $u \in N_G(x_{i_k}) \cap V(H)$. Since $a_{i_j}x_{i_j} \rightarrow a_{i_{j+1}}$ and $ua_{i_j} \notin E(G)$, we have $ux_{i_j} \in E(G)$; then, $u \in (\cap_{i=1}^{k} N_G(x_i)) \cap V(H)$, that is,

$$ux_i \in E(G), \quad \text{for } 1 \leq i \leq k. \tag{5}$$

Claim 7. $|A \cup B| \geq k + 1$.

Proof. If $|A \cup B| = k$, then $|C_i| = 1$ for $1 \leq i \leq k$. Since $a_{i_1}x_{i_1} \rightarrow a_{i_2}$, $a_{i_2}x_{i_1} \notin E(G)$. By Lemma 2, $N_G(a_{i_2}) \cap V(C) \subseteq X$; thus, $|N_G(a_{i_2}) \cap V(C)| \leq k - 1$. Since $\kappa(G) = k$, there exists a component H' of $G - C$ such that $N_G(a_{i_2}) \cap V(H') \neq \varnothing$ and $|N_C(H') \cap V(C)| \geq k$. By Lemma 1, there is no path in $G - ((V(C) - \{a_i, a_j\}) \cup V(H))$ connecting a_i and a_j, so $N_G(H') \cap V(C) \subseteq X \cup \{a_{i_2}\}\backslash\{x_{i_2}, x_{i_2+1}\}$, which implies that $\kappa(G) \leq k - 1$, a contradiction. Thus, we get $|A \cup B| \geq k + 1$. □

Claim 8. There exist a vertex $v \in A \cup B$ and a vertex z such that $vz \rightarrow u$, where u is the vertex specified in (5).

Proof. For any $v \in A \cup B$, $uv \notin E(G)$, and $\{u, v\}$ cannot dominate G; thus, there exists a vertex z such that $uz \rightarrow v$ or $vz \rightarrow u$. For $v_i, v_j \in A \cup B$ with $v_i \neq v_j$, $i \neq j$, if there exist vertices z_i, z_j such that $uz_i \rightarrow v_i$ and $uz_j \rightarrow v_j$, then we claim that $z_i \neq z_j$. Otherwise, we set $z_i = z_j = z$. Since $uz \rightarrow v_i$ and $uv_j \notin E(G)$, $zv_j \in E(G)$, but $uz \rightarrow v_j$, and we have $zv_j \notin E(G)$; this is impossible; thus, $z_i \neq z_j$. For every $v_i \in A \cup B$, if there exists a vertex z_i such that $uz_i \rightarrow v_i$, then $z_i \in N(u)$ and $z_i \in X$, but this is impossible since $|A \cup B| \geq k + 1$ and $|X| = k$. Thus, there exist a vertex $v \in A \cup B$ and a vertex z such that $vz \rightarrow u$.

By symmetry, we may assume that $a_1z \rightarrow u$. Obviously, $z \notin X$. □

Claim 9. If $a_1z \rightarrow u$, then $z = b_k$.

Proof. Since $a_1z \rightarrow u$, $a_iz \in E(G)$ for $1 \leq i \leq k$, and by Lemma 1, we can assume that $z \in C_i$. Since $\kappa(G) \geq 3$ and A is an independent set, we have $z \notin A$.

If $i \neq k$, then we show that $z \notin C_i\backslash\{a_i, b_i\}$. Otherwise, we suppose $z \in C_i\backslash\{a_i, b_i\}$; then, $z^+ \notin X$ and $z^+u \notin E(G)$. Since $a_1z \rightarrow u$ and $a_1a_{i+1} \notin E(G)$, $za_{i+1} \in E(G)$. By Lemma 3, we have $z^+a_j \notin E(G)$ for $j \neq i + 1$. If $z^+a_{i+1} \in E(G)$, we have $zb_{i+1} \notin E(G)$ by Lemma 4; hence, $a_1b_{i+1} \in E(G)$. Note that $a_1z \rightarrow u$ and $za_{i+1}, a_1b_{i+1} \in E(G)$; thus, there exists a vertex $v \in C_{i+1}$ such that $vz \in E(G)$ and $v^+a_1 \in E(G)$, but $ux_{i+1}\overleftarrow{C}z^+a_{i+1}\overrightarrow{C}vz\overleftarrow{C}a_1v^+\overrightarrow{C}x_1u$ is a cycle longer than C, a contradiction. Hence, $z^+a_{i+1} \notin E(G)$. $A \cup \{z^+, u\}$ is an independent set of G with $k + 2$ vertices; hence, $\alpha \geq k + 2$, again a contradiction.

If $i \neq k$, then $z \neq b_i$. Otherwise, we have $a_1b_i \rightarrow u$ and $a_1b_{i+1}, b_ia_{i+1} \in E(G)$. This implies that there exists a vertex $v \in C_{i+1}$ such that $vb_i \in E(G)$ and $v^+a_1 \in E(G)$, which contradicts Lemma 4. Thus, $z \neq b_i$ for $i \neq k$, and $z \notin X\backslash\{x_1\}$.

If $i = k$ and $z \neq b_k$, then we have $z^+ \notin X$, and hence, $z^+u \notin E(G)$. Since $a_1z \rightarrow u$, $za_1 \in E(G)$. By Lemma 4, we get $z^+b_j, z^-b_j \notin E(G)$ for $j \neq k$. If $z^+z^- \in E(G)$, then $ux_2\overleftarrow{C}a_1za_2\overrightarrow{C}z^-z^+\overrightarrow{C}x_1u$ is a cycle longer than C, a contradiction. Hence, $z^+z^- \notin E(G)$. $B \cup \{u, z^+, z^-\}\backslash\{b_k\}$ is an independent set of $k + 2$ vertices; hence, $\alpha \geq k + 2$, again a contradiction. Thus, we have $z = b_k$ and $a_1b_k \rightarrow u$.

By symmetry, if $a_iz \rightarrow u$, then $z = b_{i-1}$ for any $2 \leq i \leq k$.

By Claim 9, we have $a_1b_k \rightarrow u$, $a_1b_k \in E(G)$. Thus, by Lemma 2, we have $|C_1| \geq 2$ and $|C_k| \geq 2$ which imply $|A \cup B| \geq k + 2$. Since $a_1b_k \in E(G)$, by Lemma 3, we have $x_1a_j \notin E(G)$ for $j \neq 1$, and $x_1b_i \notin E(G)$ for $i \neq k$. Thus, we have $N(x_1) \cap (A \cup B\backslash\{a_1, b_k\}) = \varnothing$. Since $k \geq 3$, by Lemma 2, we can see that, for any vertex $v \in A \cup B$, $ux_1 \rightarrow v$ is impossible. For any $v \in A \cup B\backslash\{a_1, b_k\}$, considering u and v, $uv \notin E(G)$, and $\{u, v\}$ cannot dominate G. So, there exists a vertex z such that $uz \rightarrow v$ or $vz \rightarrow u$. If for any $v_i, v_j \in A \cup B\backslash\{a_1, b_k\}$, $v_i \neq v_j$ for $i \neq j$, there exist vertices z_i and z_j such that $uz_i \rightarrow v_i$ and $uz_j \rightarrow v_j$, then by the same analogous argument to the proof of Claim 8, we have $z_i \neq z_j$. For every $v \in A \cup B\backslash\{a_1, b_k\}$, if there exists a vertex z such that $uz \rightarrow v$, then we have $z \in X\backslash\{x_1\}$, but this is impossible since $|A \cup B\backslash\{a_1, b_k\}| \geq k$ and $|X\backslash\{x_1\}| = k - 1$. Thus, there exist a vertex $v \in A \cup B \backslash \{a_1, b_k\}$ and a vertex z such that $vz \rightarrow u$. Assume, without loss of generality, that $v \in A$, say $v = a_l$. Since $N(x_1) \cap (A \cup B \backslash \{a_1, b_k\}) = \emptyset$ and $a_l z \rightarrow u$, $z \neq b_{l-1}$, which contradicts Claim 9.

The proof of Theorem 5 is completed. We show that every 3_tEC graph with $\delta(G) \geq 2$ and $\alpha(G) = \kappa(G) + 1$ has a Hamilton cycle. Combining Theorems 3–5, we show that every 3_tEC graph with $\delta(G) \geq 2$ is Hamiltonian. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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