Research Article

Hermite-Hadamard, Jensen, and Fractional Integral Inequalities for Generalized P -Convex Stochastic Processes

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The stochastic process is one of the important branches of probability theory which deals with probabilistic models that evolve over time. It starts with probability postulates and includes a captivating arrangement of conclusions from those postulates. In probability theory, a convex function applied on the expected value of a random variable is always bounded above by the expected value of the convex function of that random variable. The purpose of this note is to introduce the class of generalized p -convex stochastic processes. Some well-known results of generalized p -convex functions such as Hermite-Hadamard, Jensen, and fractional integral inequalities are extended for generalized p -stochastic convexity.

1. Introduction

A stochastic process is a mathematical tool commonly defined as a set of random variables in various fields of probability. Verifiably, random variables were related to or listed by a lot of numbers, normally as focuses in time, giving the translation of a stochastic process, speaking to numerical estimations, some systems randomly changing over time, such as the growth of bacterial populations, fluctuations in electrical flow due to thermal noise, or the production of gas molecules. Stochastic systems are commonly used as scientific models of systems that tend to alter in an arbitrary manner. They have applications in various fields, especially in sciences, for instance, chemistry, physics, biology, neuroscience, and ecology, in addition to technology and engineering fields, for example, picture preparing, cryptography, signal processing, telecommunications, PC science, and data theory. Furthermore, apparently, arbitrary changes in money-related markets have inspired the broad utilization of stochastic processes in fund.

Convex stochastic processes and their applications have a fundamental significance in mathematics and in probability. Nikodem [1] in 1980 proposed the idea of convex

stochastic processes in his article. In 1992, λ -convex and Jensen-convex stochastic processes were initiated by Skowronski [2]. More recently, Kotrys presented in [3] the results on convex stochastic processes.

For more details, refer to [4]. Many studies in the literature have been performed on some extensions of convex stochastic processes and on Hermite-Hadamard type inequalities for these extensions [5].

In the present note, we purpose to investigate the idea of generalized p -convex stochastic processes. The notion of inequality as convexity has a significant place in literature [6], as it yields a broader setting in order to investigate the mathematical programming and optimization problems. Therefore, Schur type, Hermite-Hadamard, Jensen, and fractional integral inequalities and some important results for the above said processes will be obtained in this study.

We start by definition of the stochastic process [7].

Definition 1 (see [3]). Assume a probability space $(\Omega, \mathcal{A}, P)$. A random variable is the function $\xi: \Omega \rightarrow \mathbb{R}$ if ξ is $\mathcal{A}$ -measurable; whereas, a stochastic process is the function $\xi: I \times \Omega \rightarrow \mathbb{R}$ if $\xi(t, \cdot)$ is a random variable for every $t \in I$.

Let us review some basic notions about stochastic processes.

Definition 2 (see [3]). The stochastic process $\xi: I \times \Omega \longrightarrow \mathbb{R}$ is as follows:

- (1) Continuous on I , if for all $u_0 \in I$,

$$P - \lim_{u \rightarrow u_0} \xi(u, \cdot) = \xi(u_0, \cdot), \quad (1)$$

where $P - \lim$ represents the limit in the probability;

- (2) Mean-square continuous on I , if for every $u_0 \in I$,

$$\lim_{u \rightarrow u_0} \mathbb{E}(\xi(u, \cdot) - \xi(u_0, \cdot))^2 = 0, \quad (2)$$

Where $\mathbb{E}[\xi(u, \cdot)]$ represents an expectation of the random variable ξ .

It is obvious in probability that if a stochastic process is mean-square continuous, then it is also continuous, but the converse is not true;

- (3) Mean square is differentiable at $t \in I$, if there is a random variable $\xi'(v, \cdot): I \times \Omega \longrightarrow \mathbb{R}$, such that for every $u_0 \in I$,

$$\xi'(u_0, \cdot) = P - \lim_{u \rightarrow u_0} \frac{\xi(u, \cdot) - \xi(u_0, \cdot)}{u - u_0}. \quad (3)$$

Definition 3 (see [3]). Consider $\xi: I \times \Omega \longrightarrow \mathbb{R}$, a stochastic process with $\mathbb{E}[\xi(t, \cdot)^2] < \infty$. We say a random variable $v: \Omega \longrightarrow \mathbb{R}$ to be the mean-square integral of the process ξ on $[a_1, a_2]$ if for each normal sequence of partitions of $[a_1, a_2]$, $a_1 = u_0 < u_1 < \dots < u_n = a_2$, and for all $\Theta_k \in [u_{k-1}, u_k]$, we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\sum_{k=1}^n \xi(\Theta_k, \cdot) (u_k - u_{k-1}) - v(\cdot) \right)^2 \right] = 0. \quad (4)$$

Then, $v(\cdot)$ is written as

$$v(\cdot) = \int_{a_1}^{a_2} \xi(s, \cdot) ds, \text{ (a.e.)} \quad (5)$$

For more on mean-square integrable stochastic processes, refer [8].

Theorem 1 (see [3]). Let us consider the Jensen-convex stochastic process $\xi: I \times \Omega \longrightarrow \mathbb{R}$ that is mean-square continuous on I ; then, we have

$$\xi\left(\frac{a_1 + a_2}{2}, \cdot\right) \leq \frac{1}{a_2 - a_1} \int_{a_1}^{a_2} \xi(u, \cdot) du \leq \frac{\xi(a_1, \cdot) + \xi(a_2, \cdot)}{2}, \text{ (a.e.)} \quad (6)$$

for all $a_1, a_2 \in I$, $a_1 < a_2$. The above inequality is Hermite-Hadamard inequality for stochastic convexity.

Let us present some important generalizations of convex stochastic processes.

Definition 4 (see [9]). A stochastic process $\xi: I \times \Omega \longrightarrow \mathbb{R}$ is said to be generalized convex if for $\theta \in [0, 1]$ and $u, v \in I$,

$$\xi(\theta u + (1 - \theta)v, \cdot) \leq \xi(v, \cdot) + \theta \eta(\xi(u, \cdot), \xi(v, \cdot)), \text{ (a.e.)} \quad (7)$$

Definition 5 (see [10]). An interval I is a p -convex set if $[\theta u^p + (1 - \theta)v^p]^{(1/p)} \in I$ for all $u, v \in I$, $\theta \in [0, 1]$, and $p = 2m + 1$ or $p = (r/n)$, $r = 2s + 1$, $n = 2t + 1$, and $m, s, t \in \mathbb{N}$.

Definition 6 (see [10]). A function $f: I \longrightarrow \mathbb{R}$ is p -convex, if for $\theta \in [0, 1]$ and $u, v \in I$, we have

$$f\left([\theta u^p + (1 - \theta)v^p]^{(1/p)}\right) \leq \theta f(u) + (1 - \theta)f(v), \quad (8)$$

where I is a p -convex set.

Remark 1 (see [11]). If $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$, then

$$[tx^p + (1 - t)y^p]^{(1/p)} \in I, \quad \text{for all } x, y \in I \text{ and } t \in [0, 1]. \quad (9)$$

According to Remark 1, we can give a different version of the definition of p -convex function as follows:

Definition 7 (see [11]). If $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. A function $f: I \longrightarrow \mathbb{R}$ is said to be a p -convex function if

$$f\left([tx^p + (1 - t)y^p]^{(1/p)}\right) \leq tf(x) + (1 - t)f(y), \quad (10)$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality (10) is reversed, then f is said to be p -concave.

Definition 8 (see [12]). A process $\xi: I \times \Omega \longrightarrow \mathbb{R}$, where I is a p -convex set, is said to be a p -convex stochastic process if for $\theta \in [0, 1]$ and $u, v \in I$, we have

$$\xi\left([\theta u^p + (1 - \theta)v^p]^{(1/p)}, \cdot\right) \leq \theta \xi(u, \cdot) + (1 - \theta)\xi(v, \cdot), \text{ (a.e.)} \quad (11)$$

In [13], the following functions are defined.

Definition 9

$$\beta(u, v) = \int_0^1 \theta^{u-1} (1 - \theta)^{v-1} d\theta, \quad u, v > 0. \quad (12)$$

Definition 10. For $w > v > 0$, $|z| < 1$,

$${}_2F_1(u, v; w; z) = \frac{1}{\beta(v, w - v)} \int_0^1 \theta^{v-1} (1 - \theta)^{w-v-1} (1 - z\theta)^{-u} d\theta. \quad (13)$$

Now, we are in position to define the main notion of this article.

Definition 11. A stochastic process $\xi: I \times \Omega \longrightarrow \mathbb{R}$ is said to be generalized p -convex, if for $\theta \in [0, 1]$ and , we have

$$\xi\left([\theta u^p + (1-\theta)v^p]^{(1/p)}, \cdot\right) \leq \xi(v, \cdot) + \theta\eta(\xi(u, \cdot), \xi(v, \cdot)), \text{ (a.e.)} \quad (14)$$

If the inequality in (14) is reversed, then ξ is the generalized p -concave.

Remark 2. It is obvious that the inequality (14) reduces to the convex stochastic process for $p = 1$ and $\eta(x, y) = x - y$.

Example 1. Consider a stochastic process $\xi: (0, \infty) \times \Omega \rightarrow \mathbb{R}$ defined by $\xi(u, \cdot) = u^p$, $p \neq 0$, and $\eta(x, y) = x - y$; then, ξ is the generalized p -convex.

We organize our study as follows. First we derive some basic properties for this generalization. In next section, Schur type inequality is obtained. The third, fourth, and fifth sections are devoted to Hermite-Hadamard, Jensen, and fractional integral inequalities for generalized p -convex stochastic processes.

Proposition 1. Let $\xi_1, \xi_2: I \times \Omega \rightarrow \mathbb{R}$ be two generalized p -convex stochastic processes:

- (1) If η is additive, then $\xi_1 + \xi_2: I \times \Omega \rightarrow \mathbb{R}$ is also a generalized p -convex stochastic process
- (2) If η is nonnegatively homogeneous, then $\lambda\xi_1: I \times \Omega \rightarrow \mathbb{R}$, for any $\lambda \geq 0$, is the generalized p -convex stochastic process

The proof Proposition 1 is straightforward.

Theorem 2. Assume a nonempty collection $\{\xi_j: I \times \Omega \rightarrow \mathbb{R}, j \in J\}$ of generalized p -convex stochastic processes, such that

- (1) There exist $\alpha \in [0, \infty]$ and $\beta \in [-1, \infty]$, such that $\eta(u, v) = \alpha u + \beta v$ for all $u, v \in \mathbb{R}$
- (2) For each $u \in I$, $\max_{j \in J} \xi_j(u, \cdot)$ exists in $\mathbb{R}$; then, the stochastic process defined by $\xi(u, \cdot) = \max_{j \in J} \xi_j(u, \cdot)$ for all $u \in I$ is the generalized p -convex.

Proof. For any $u, v \in I$ and $\theta \in [0, 1]$, we have

$$\begin{aligned} \xi\left([\theta u^p + (1-\theta)v^p]^{(1/p)}, \cdot\right) &= \max_{j \in J} \xi_j\left([\theta u^p + (1-\theta)v^p]^{(1/p)}, \cdot\right) \\ &\leq \max_{j \in J} \{\xi_j(v, \cdot) + \theta\eta(\xi_j(u, \cdot), \xi_j(v, \cdot))\} \\ &= \max_{j \in J} \{\xi_j(v, \cdot) + \theta(\alpha\xi_j(u, \cdot) + \beta\xi_j(v, \cdot))\} \\ &= \max_{j \in J} \{(1+\beta\theta)\xi_j(v, \cdot) + \alpha\theta\xi_j(u, \cdot)\} \\ &\leq (1+\beta\theta) \max_{j \in J} \xi_j(v, \cdot) + \alpha\theta \max_{j \in J} \xi_j(u, \cdot) \\ &= (1+\beta\theta)\xi(v, \cdot) + \alpha\theta\xi(u, \cdot) \\ &= \xi(v, \cdot) + \theta(\alpha\xi(u, \cdot) + \beta\xi(v, \cdot)) \\ &= \xi(v, \cdot) + \theta\eta(\xi(u, \cdot), \xi(v, \cdot)), \end{aligned} \quad (15)$$

which is as required. $\square$

2. Schur Type Inequality

Theorem 3. For $I \subset (0, \infty)$ and $p > 0$, let $\xi: I \times \Omega \rightarrow \mathbb{R}$ is the generalized p -convex stochastic process. Then, $\forall u_1, u_2, u_3 \in I$, such that $u_1 < u_2 < u_3$ and $u_3^p - u_1^p, u_3^p - u_2^p, u_2^p - u_1^p \in (0, 1)$, and we have

$$\begin{aligned} &\xi(u_3, \cdot)(u_3^p - u_1^p) - \xi(u_2, \cdot)(u_3^p - u_1^p) \\ &\quad + (u_3^p - u_2^p)\eta(\xi(u_1, \cdot), \xi(u_3, \cdot)) \geq 0, \text{ (a.e.)} \end{aligned} \quad (16)$$

Proof. Let $u_1, u_2, u_3 \in I$ be given. Then, we can easily see that

$$\begin{aligned} &\frac{u_3^p - u_2^p}{u_3^p - u_1^p} \frac{u_2^p - u_1^p}{u_3^p - u_1^p} \in (0, 1), \\ &\frac{u_3^p - u_2^p}{u_3^p - u_1^p} + \frac{u_2^p - u_1^p}{u_3^p - u_1^p} = 1. \end{aligned} \quad (17)$$

Setting $\theta = (u_3^p - u_2^p / (u_3^p - u_1^p))$, $u_2^p = \theta u_1^p + (1 - \theta)u_3^p$. As ξ is generalized p -convex, so

$$\xi(u_2, \cdot) \leq \xi(u_3, \cdot) + \frac{u_3^p - u_2^p}{u_3^p - u_1^p} \eta(\xi(u_1, \cdot), \xi(u_3, \cdot)) \quad (18)$$

By assuming $u_3^p - u_1^p > 0$ and multiplying the above inequality by $u_3^p - u_1^p$, we get inequality (16). $\square$

3. Hermite-Hadamard Type Inequality

Theorem 4. For $I \subset (0, \infty)$ and $p > 0$, let a mean-square generalized p -convex stochastic process $\xi: [u_1, u_2] \times \Omega \rightarrow \mathbb{R}$, which is integrable. Then, for any $u_1, u_2 \in I$, ($u_1 < u_2$), the following inequality holds almost everywhere:

$$\begin{aligned} &\xi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{(1/p)}, \cdot\right) - \frac{p}{2(u_2^p - u_1^p)} \int_{u_1}^{u_2} x^{p-1} \eta \\ &\quad \left(\left[\xi\left[u_1^p + u_2^p - x^p\right]^{(1/p)}, \cdot\right), \xi(x, \cdot)\right) dx \\ &\leq \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} x^{p-1} \xi(x, \cdot) dx \\ &\leq \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} + \frac{1}{4} [\eta(\xi(u_1, \cdot), \xi(u_2, \cdot)) \\ &\quad + \eta(\xi(u_2, \cdot), \xi(u_1, \cdot))]. \end{aligned} \quad (19)$$

Proof. Take $x^p = \theta u_1^p + (1 - \theta)u_2^p$ and $y^p = (1 - \theta)u_1^p + \theta u_2^p$; so,

$$\xi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{(1/p)}, \cdot\right) = \xi\left(\left[\frac{x^p + y^p}{2}\right]^{(1/p)}, \cdot\right). \quad (20)$$

Since ξ is the generalized p -convex, so we have

$$\begin{aligned} \xi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{(1/p)}, \cdot\right) &= \xi\left(\frac{1}{2}\left[\theta u_1^p + (1-\theta)u_2^p\right]^{(1/p)}, \cdot\right)^p \\ &\quad + \frac{1}{2}\left(\left[(1-\theta)u_1^p + \theta u_2^p\right]^{(1/p)}\right)^p \\ &\leq \xi\left(\left[(1-\theta)u_1^p + \theta u_2^p\right]^{(1/p)}, \cdot\right) \\ &\quad + \frac{1}{2}\eta\left(\xi\left(\left[\theta u_1^p + (1-\theta)u_2^p\right]^{(1/p)}, \cdot\right), \right. \\ &\quad \left. \xi\left(\left[(1-\theta)u_1^p + \theta u_2^p\right]^{(1/p)}, \cdot\right)\right). \end{aligned} \quad (21)$$

Integrating w.r.t “ θ ,” the above inequality on $[0, 1]$,

$$\begin{aligned} \xi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{(1/p)}, \cdot\right) &\leq \int_0^1 \xi\left(\left[(1-\theta)u_1^p + \theta u_2^p\right]^{(1/p)}, \cdot\right) d\theta \\ &\quad + \frac{1}{2} \int_0^1 \eta\left(\xi\left(\left[\theta u_1^p + (1-\theta)u_2^p\right]^{(1/p)}, \cdot\right), \right. \\ &\quad \left. \xi\left(\left[(1-\theta)u_1^p + \theta u_2^p\right]^{(1/p)}, \cdot\right)\right) d\theta, \end{aligned} \quad (22)$$

which implies

$$\begin{aligned} &\xi\left(\left[\frac{u_1^p + u_2^p}{2}\right]^{(1/p)}, \cdot\right) - \frac{p}{2(u_2^p - u_1^p)} \\ &\quad \times \int_{u_1}^{u_2} x^{p-1} \eta\left(\xi\left(\left[u_1^p + u_2^p - x^p\right]^{(1/p)}, \cdot\right), \xi(x, \cdot)\right) dx \\ &\leq \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} x^{p-1} \xi(x, \cdot) dx. \end{aligned} \quad (23)$$

Now,

$$\begin{aligned} &\int_{u_1}^{u_2} x^{p-1} \xi(x, \cdot) dx \\ &= \frac{u_2^p - u_1^p}{p} \int_0^1 \xi\left(\left[\theta u_1^p + (1-\theta)u_2^p\right]^{(1/p)}, \cdot\right) d\theta \\ &\leq \frac{u_2^p - u_1^p}{p} \left(\xi(u_2, \cdot) + \int_0^1 \theta \eta(\xi(u_1, \cdot), \xi(u_2, \cdot)) d\theta \right), \end{aligned} \quad (24)$$

which implies

$$\begin{aligned} &\frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} x^{p-1} \xi(x, \cdot) dx \\ &\leq \xi(u_2, \cdot) + \int_0^1 \theta \eta(\xi(u_1, \cdot), \xi(u_2, \cdot)) d\theta. \end{aligned} \quad (25)$$

Similarly,

$$\begin{aligned} &\frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} x^{p-1} \xi(x, \cdot) dx \\ &\leq \xi(u_1, \cdot) + \int_0^1 \theta \eta(\xi(u_2, \cdot), \xi(u_1, \cdot)) d\theta. \end{aligned} \quad (26)$$

Adding (25) and (26),

$$\begin{aligned} &\frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} x^{p-1} \xi(x, \cdot) dx \\ &\leq \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} \\ &\quad + \frac{1}{4} [\eta(\xi(u_1, \cdot), \xi(u_2, \cdot)) + \eta(\xi(u_2, \cdot), \xi(u_1, \cdot))]. \end{aligned} \quad (27)$$

Combining (23) and (27), we obtain the inequality (22). $\square$

Remark 3. For $p = 1$ and $\eta(u, v) = u - v$ in (22), we get Hermite-Hadamard inequality (6) for the convex stochastic process.

4. Jensen Type Inequality

The following result will be helpful in the derivation of Jensen's type inequality for the generalized p -convex stochastic process.

Lemma 1. Let $w_1, \dots, w_n$ be the positive real numbers ($n \geq 2$). Assume $\xi: I \times \Omega \rightarrow \mathbb{R}$ be a generalized p -convex stochastic process and $u_1, u_2, \dots, u_n \in I$; then, we have almost everywhere

$$\begin{aligned} &\xi\left(\left[\frac{1}{W_n} \sum_{i=1}^n w_i u_i^p\right]^{(1/p)}, \cdot\right) \\ &= \xi\left(\left[\frac{W_{n-1}}{W_n} \sum_{i=1}^{n-1} \frac{w_i}{W_{n-1}} u_i^p + \frac{w_n}{W_n} u_n^p\right]^{(1/p)}, \cdot\right) \\ &\leq \xi(u_n, \cdot) + \frac{W_{n-1}}{W_n} \eta\left(\xi\left(\left[\sum_{i=1}^{n-1} \frac{w_i}{W_{n-1}} u_i^p\right]^{(1/p)}, \cdot\right), \xi(u_n, \cdot)\right), \end{aligned} \quad (28)$$

where $W_n = \sum_{i=1}^n w_i$.

Theorem 5 (Jensen type inequality). Let $\xi: I \times \Omega \rightarrow \mathbb{R}$ be a generalized p -convex stochastic process and $\eta: A \times B \rightarrow \mathbb{R}$ be nondecreasing, nonnegatively sublinear in the first variable; then, we have almost everywhere

$$\xi\left(\left[\frac{1}{W_n} \sum_{i=1}^n w_i u_i^p\right]^{(1/p)}, \cdot\right) \leq \xi(u_n, \cdot) + \sum_{i=1}^n \left(\frac{w_i}{W_n}\right) \eta_{\xi}(u_i, u_{i+1}, \dots, u_n, \cdot), \quad (29)$$

where $W_n = \sum_{i=1}^n w_i$ and $\eta_\xi(u_i, u_{i+1}, \dots, u_n, \cdot) = \eta(\eta_\xi(u_i, u_{i+1}, \dots, u_{n-1}, \cdot), \xi(u_n, \cdot))$ and $\eta_\xi(u, \cdot) = \xi(u, \cdot) \forall u \in I$.

Proof. Since η is nondecreasing, nonnegatively sublinear in the first variable, so using Lemma 1, we get

$$\begin{aligned}
 \xi\left(\left[\frac{1}{W_n} \sum_{i=1}^n w_i x_i^p\right]^{(1/p)}, \cdot\right) &= \xi\left(\left[\frac{w_n}{W_n} u_n^p + \sum_{i=1}^{n-1} \frac{w_i}{W_n} u_i^p\right]^{(1/p)}, \cdot\right) \\
 &= \xi\left(\left[\frac{W_{n-1}}{W_n} \sum_{i=1}^{n-1} \frac{w_i}{W_{n-1}} u_i^p + \frac{w_n}{W_n} u_n^p\right]^{(1/p)}, \cdot\right) \\
 &\leq \xi(u_n, \cdot) + \frac{W_{n-1}}{W_n} \eta\left(\xi\left(\left[\sum_{i=1}^{n-1} \frac{w_i}{W_{n-1}} u_i^p\right]^{(1/p)}, \cdot\right), \xi(u_n, \cdot)\right) \\
 &= \xi(u_n, \cdot) + \frac{W_{n-1}}{W_n} \eta\left(\xi\left(\left[\frac{W_{n-2}}{W_{n-1}} \sum_{i=1}^{n-2} \frac{w_i}{W_{n-2}} u_i^p + \frac{w_{n-1}}{W_{n-1}} u_{n-1}^p\right]^{(1/p)}, \cdot\right), \xi(u_n, \cdot)\right) \\
 &\leq \xi(u_n, \cdot) + \frac{W_{n-1}}{W_n} \eta\left(\xi(u_{n-1}, \cdot) + \frac{W_{n-2}}{W_{n-1}} \eta\left(\xi\left(\left[\sum_{i=1}^{n-2} \frac{w_i}{W_{n-2}} u_i^p\right]^{(1/p)}, \cdot\right), \xi(u_{n-1}, \cdot)\right), \xi(u_n, \cdot)\right) \\
 &\leq \xi(u_n, \cdot) + \frac{W_{n-1}}{W_n} \eta(\xi(u_{n-1}, \cdot), \xi(u_n, \cdot)) + \frac{W_{n-2}}{W_n} \eta \\
 &\quad \times \left(\eta\left(\xi\left(\left[\sum_{i=1}^{n-2} \frac{w_i}{W_{n-2}} u_i^p\right]^{(1/p)}, \cdot\right), \xi(u_{n-1}, \cdot)\right), \xi(u_n, \cdot)\right) \\
 &\leq \dots \leq \xi(u_n, \cdot) + \frac{W_{n-1}}{W_n} \eta(\xi(u_{n-1}, \cdot), \xi(u_n, \cdot)) + \frac{W_{n-2}}{W_n} \\
 &\quad \times \eta(\eta(\xi(u_{n-2}, \cdot), \xi(u_n, \cdot)), \xi(u_n, \cdot)) \\
 &\quad + \dots + \frac{W_1}{W_n} \eta(\eta(\dots \eta(\xi(u_1, \cdot), \xi(u_2, \cdot)), \xi(u_3, \cdot)) \dots), \xi(u_{n-1}, \cdot), \xi(u_n, \cdot)) \\
 &= \xi(u_n, \cdot) + \frac{W_{n-1}}{W_n} \eta_\xi(u_{n-1}, u_n, \cdot) + \frac{W_{n-2}}{W_n} \eta_\xi(u_{n-2}, u_{n-1}, u_n, \cdot) \\
 &\quad + \dots + \frac{W_1}{W_n} \eta_\xi(u_1, u_2, \dots, u_{n-1}, u_n, \cdot) \\
 &= \xi(u_n, \cdot) + \sum_{i=1}^{n-1} \left(\frac{W_i}{W_n}\right) \eta_\xi(u_i, u_{i+1}, \dots, u_n, \cdot).
 \end{aligned}$$

(30)

□

5. Fractional Integral Inequalities

Lemma 2 (see [12]). Assume a stochastic process $\xi: I \subset (0, \infty) \times \Omega \rightarrow \mathbb{R}$ which is mean-square differentiable on I^o and $u_1, u_2 \in I^o$ with $u_1 < u_2$. If $\xi' \in L[u_1, u_2]$, then we have almost everywhere

$$\begin{aligned} & \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} \frac{\xi(x, \cdot)}{x^{1-p}} dx \\ &= \frac{u_2^p - u_1^p}{2p} \int_0^1 \frac{1 - 2\theta}{[\theta u_1^p + (1 - \theta)u_2^p]^{1-(1/p)}} \\ & \quad \times \xi'([\theta u_1^p + (1 - \theta)u_2^p]^{(1/p)}, \cdot) d\theta. \end{aligned} \quad (31)$$

Theorem 6. For $I \subset (0, \infty)$ and $p > 0$ and under the assumptions of Lemma 2 with $|\xi'|^q$, a generalized p -convex stochastic process on $[u_1, u_2]$ for $q \geq 1$, then we have almost everywhere

$$\begin{aligned} & \left| \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} \frac{\xi(x, \cdot)}{x^{1-p}} dx \right| \\ & \leq \frac{u_2^p - u_1^p}{2p} L_1^{1-(1/q)} [L_1 |\xi'(u_2, \cdot)|^q]^{(1/q)} \\ & \quad + L_2 (\eta(|\xi'(u_1, \cdot)|^q, |\xi'(u_2, \cdot)|^q))^{(1/q)}, \end{aligned} \quad (32)$$

where

$$\begin{aligned} L_1(u_1, u_2; p) &= \frac{1}{4} \left(\frac{u_1^p - u_2^p}{2} \right)^{(1/p)-1} \left[{}_2F_1 \left(1 - \frac{1}{p}, 2, 3; \frac{u_1^p - u_2^p}{u_1^p + u_2^p} \right) \right. \\ & \quad \left. + {}_2F_1 \left(1 - \frac{1}{p}, 2, 3; \frac{u_2^p - u_1^p}{c^p + u_2^p} \right) \right], \\ L_2(u_1, u_2; p) &= \frac{1}{24} \left(\frac{u_1^p + u_2^p}{2} \right)^{(1/p)-1} \left[{}_2F_1 \left(1 - \frac{1}{p}, 2, 4; \frac{u_1^p - u_2^p}{u_1^p + u_2^p} \right) \right. \\ & \quad \left. + {}_2F_1 \left(1 - \frac{1}{p}, 2, 4; \frac{u_2^p - u_1^p}{u_1^p + v^p} \right) \right]. \end{aligned} \quad (33)$$

Proof. By making use of Lemma 2 and power mean-integral inequality, we have

$$\begin{aligned} & \left| \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{1}{u_2^p - u_1^p} \int_{u_1}^{u_2} \frac{\xi(x, \cdot)}{x^{1-p}} dx \right| \\ & \leq \frac{u_2^p - u_1^p}{2p} \int_0^1 \left| \frac{1 - 2\theta}{[\theta u_1^p + (1 - \theta)u_2^p]^{1-(1/p)}} \right| \\ & \quad \times \left| \xi'([\theta u_1^p + (1 - \theta)u_2^p]^{(1/p)}, \cdot) \right| d\theta \\ & \leq \frac{u_2^p - u_1^p}{2p} \left(\int_0^1 \frac{|1 - 2\theta|}{[\theta u_1^p + (1 - \theta)u_2^p]^{1-(1/p)}} dt \right)^{1-(1/q)} \\ & \quad \times \left(\int_0^1 \frac{|1 - 2\theta|}{[\theta u_1^p + (1 - \theta)u_2^p]^{1-(1/p)}} \left| \xi'([\theta u_1^p + (1 - \theta)u_2^p]^{(1/p)}, \cdot) \right|^q d\theta \right)^{(1/q)}. \end{aligned} \quad (34)$$

Hence, by generalized p -convexity of $|\xi'|^q$ on $[u_1, u_2]$, we have

$$\begin{aligned} & \left| \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{1}{u_2^p - u_1^p} \int_a^b \frac{\xi(x, \cdot)}{x^{1-p}} dx \right| \\ & \leq \frac{u_2^p - u_1^p}{2p} \left(\int_0^1 \frac{|1 - 2\theta|}{[\theta u_1^p + (1 - \theta)u_2^p]^{1-(1/p)}} dt \right)^{1-(1/q)} \\ & \quad \left(\int_0^1 \frac{|1 - 2\theta| [|\xi'(u_2, \cdot)|^q + \theta \eta(|\xi'(u_1, \cdot)|^q, |\xi'(u_2, \cdot)|^q)]}{[\theta u_1^p + (1 - \theta)u_2^p]^{1-(1/p)}} dt \right)^{(1/q)} \\ & \leq \frac{u_2^p - u_1^p}{2p} L_1^{1-(1/q)} [L_1 |\xi'(u_2, \cdot)|^q]^{(1/q)} \\ & \quad + L_2 (\eta(|\xi'(u_1, \cdot)|^q, |\xi'(u_2, \cdot)|^q))^{(1/q)}. \end{aligned} \quad (35)$$

It is easy to check that

$$\begin{aligned} & \int_0^1 \frac{|1 - 2\theta|}{[\theta u_1^p + (1 - \theta)u_2^p]^{1-(1/p)}} d\theta = L_1(u_1, u_2; p), \\ & \int_0^1 \frac{|1 - 2\theta|\theta}{[\theta u_1^p + (1 - \theta)u_2^p]^{1-(1/p)}} d\theta = L_2(u_1, u_2; p). \end{aligned} \quad (36)$$

□

Remark 4. By setting $\eta(u_1, u_2) = u_1 - u_2$ in (32), we get Theorem 4 of [12].

We will get the following Corollary by taking $q = 1$ in (32).

Corollary 1. If $|\xi'|$ is generalized p -convex on $[u_1, u_2]$, then we have almost everywhere

$$\left| \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} \xi(x, \cdot) x^{1-p} dx \right| \leq \frac{u_2^p - u_1^p}{2p} [L_1 |\xi'(u_2, \cdot)| + L_2 (\eta(|\xi'(u_1, \cdot)|, |\xi'(u_2, \cdot)|))], \quad (37)$$

where L_1 and L_2 are defined in Theorem 6.

Remark 5. If we take $\eta(u_1, u_2) = u_1 - u_2$ in (37), then we have Corollary 4 of [12].

Theorem 7. For $I \subset (0, \infty)$ and $p > 0$ and under the assumptions of Lemma 2 with $|\xi'|^q$, a generalized p -convex stochastic process on $[u_1, u_2]$ for $1 < q$, $(1/r) + (1/q) = 1$, then we have almost everywhere

$$\left| \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} \frac{\xi(x, \cdot)}{x^{1-p}} dx \right| \leq \frac{u_2^p - u_1^p}{2p} \left(\frac{1}{r+1} \right)^{(1/r)} [L_4 |\xi'(u_2, \cdot)|^q + L_5 (\eta(|\xi'(u_1, \cdot)|^q, |\xi'(u_2, \cdot)|^q))]^{(1/q)}, \quad (38)$$

where

$$L_3 = L_3(u_1, u_2; p; q)$$

$$= \begin{cases} \frac{1}{u_1^{q(p-q)}} \cdot {}_2F_1\left(q - \frac{q}{p}, 1; 2; 1 - \left(\frac{u_2}{u_1}\right)^p\right), & \text{if } p < 0, \\ \frac{1}{u_2^{q(p-q)}} \cdot {}_2F_1\left(q - \frac{q}{p}, 1; 2; 1 - \left(\frac{u_1}{u_2}\right)^p\right), & \text{if } p > 0, \end{cases}$$

$$L_4 = L_4(u_1, u_2; p; q)$$

$$= \begin{cases} \frac{1}{2u_1^{q(p-q)}} \cdot {}_2F_1\left(q - \frac{q}{p}, 1; 3; 1 - \left(\frac{u_2}{u_1}\right)^p\right), & \text{if } p < 0, \\ \frac{1}{2u_2^{q(p-q)}} \cdot {}_2F_1\left(q - \frac{q}{p}, 2; 3; 1 - \left(\frac{u_1}{u_2}\right)^p\right), & \text{if } p > 0. \end{cases} \quad (39)$$

Proof. By making use of Lemma 2, Holder's inequality, and generalized p -convexity of $|\xi'|^q$ on $[u_1, u_2]$, we have

$$\begin{aligned} & \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} \frac{\xi(x, \cdot)}{x^{1-p}} dx \\ & \leq \frac{u_2^p - u_1^p}{2p} \left(\int_0^1 |1 - 2\theta|^r dt \right)^{(1/r)} \\ & \quad \times \left(\int_0^1 \frac{1}{[\theta u_1^p + (1-\theta)u_2^p]^{q-(q/p)}} \left| \xi' \left([\theta u_1^p + (1-\theta)u_2^p]^{(1/p)}, \cdot \right) \right|^q dt \right)^{(1/q)} \\ & \leq \frac{u_2^p - u_1^p}{2p} \left(\frac{1}{r+1} \right)^{(1/r)} \\ & \quad \times \left(\int_0^1 \frac{|\xi'(u_2, \cdot)|^q + \theta \eta(|\xi'(u_1, \cdot)|^q, |\xi'(u_2, \cdot)|^q)}{[\theta u_1^p + (1-\theta)u_2^p]^{q-(q/p)}} d\theta \right)^{(1/q)}, \end{aligned} \quad (40)$$

where an easy calculation gives

$$\int_0^1 \frac{1}{[\theta u_1^p + (1-\theta)u_2^p]^{q-(q/p)}} d\theta = L_3(u_1, u_2; p; q), \quad (41)$$

$$\int_0^1 \frac{\theta}{[\theta u_1^p + (1-\theta)u_2^p]^{q-(q/p)}} d\theta = L_4(u_1, d; p; q). \quad (42)$$

Substituting equations (41) and (42) into (38), the proof is completed. $\square$

Remark 6. By taking $\eta(u_1, u_2) = u_1 - u_2$ in Theorem 7, then we obtain Theorem 6 of [12].

Theorem 8. For $I \subset (0, \infty)$ and $p > 0$ and under the assumptions of Lemma 2 with $|\xi'|^q$, a generalized p -convex on $[u_1, u_2]$ for $1 < q$, $(1/r) + (1/q) = 1$, then we have almost everywhere

$$\begin{aligned} & \left| \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} \frac{\xi(x, \cdot)}{x^{1-p}} dx \right| \\ & \leq \frac{u_2^p - u_1^p}{2p} L_5^{(1/r)} \left(\frac{1}{q+1} \right)^{(1/q)} \\ & \quad \left(|\xi'(u_2, \cdot)|^q + \frac{1}{2} \eta(|\xi'(u_1, \cdot)|^q, |\xi'(u_2, \cdot)|^q) \right)^{(1/q)}, \end{aligned} \quad (43)$$

where

$$L_5 = L_5(u_1, u_2; p; r)$$

$$= \begin{cases} \frac{1}{u_1^{pr-r}} \cdot {}_2F_1\left(r - \frac{r}{p}, 1; 2; 1 - \left(\frac{u_2}{u_1}\right)^p\right), & \text{if } p < 0, \\ \frac{1}{u_2^{pr-r}} \cdot {}_2F_1\left(r - \frac{r}{p}, 1; 2; 1 - \left(\frac{u_1}{u_2}\right)^p\right), & \text{if } p > 0. \end{cases} \quad (44)$$

Proof. From Lemma 2, Holder's inequality, and generalized p -convexity of $|\xi'|^q$ on $[u_1, u_2]$, we have

$$\left| \frac{\xi(u_1, \cdot) + \xi(u_2, \cdot)}{2} - \frac{p}{u_2^p - u_1^p} \int_{u_1}^{u_2} \frac{\xi(x, \cdot)}{x^{1-p}} dx \right|$$

$$\leq \frac{u_2^p - u_1^p}{2p} \left(\int_0^1 \frac{1}{[\theta u_1^p + (1-\theta)u_2^p]^{q-(q/p)}} d\theta \right)^{(1/r)}$$

$$\times \left(\int_0^1 |1 - 2\theta|^q \left| \xi' \left([\theta u_1^p + (1-\theta)u_2^p]^{(1/p)}, \cdot \right) \right|^q d\theta \right)^{(1/q)}$$

$$\leq \frac{u_2^p - u_1^p}{2p} L_6^{(1/r)} \left(\frac{1}{q+1} \right)^{(1/q)}$$

$$\left(\left| \xi'(u_2, \cdot) \right|^q + \frac{1}{2} \left(\eta \left(\left| \xi'(u_1, \cdot) \right|^q, \left| \xi'(u_2, \cdot) \right|^q \right) \right) \right)^{(1/q)}, \quad (45)$$

where an easy calculation gives

$$\int_0^1 \frac{1}{[\theta u_1^p + (1-\theta)u_2^p]^{q-(r/p)}} d\theta = L_5(c, u_2; p; r), \quad (46)$$

$$\int_0^1 |1 - 2\theta|^q d\theta = \frac{1}{(q+1)}. \quad (47)$$

Substituting (46) and (47) into (43), we obtain the required result. $\square$

6. Conclusion

There are many applications of stochastic processes, for instance, the Kolmogorov–Smirnov test on equality of distributions. The other application includes sequential analysis and quickest detection. In this study, we have presented a new class of convex stochastic processes which are generalized p -convex and established Jensen, Hermite-Hadamard, and fractional integral inequalities for this class. Our conclusions are applicable, since the expected value of a random variable is consistently bounded above by the expected value of the convex function of that random variable. It will be interesting to find parallel results by using the proposed definition in this study in the setting of other fractional integrals [14, 15].

Data Availability

The data used to support the findings of this study are included within this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Fangfang Ma validated the results, prepared the final draft of the manuscript, and arranged the funding for this study. Waqas Nazeer proved the main results. Mamoon Ghafoor wrote the first draft of the manuscript.

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