Research Article

Sun Toughness Conditions for P_2 and P_3 Factor Uniform and Factor Critical Avoidable Graphs

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The security of a network is closely related to the structure of the network graph. The denser the network graph structure is, the better it can resist attacks. Toughness and isolated toughness are used to characterize the vulnerable programs of the network which have been paid attention from mathematics and computer scholars. On this basis, considering the particularity of the sun component structures, sun toughness was introduced in mathematics and applied to computer networks. From the perspective of modern graph theory, this paper presents the sun toughness conditions of the path factor uniform graph and the path factor critical avoidable graph in $P_{\geq 2}$ -factor and $P_{\geq 3}$ -factor settings. Furthermore, examples show that the given boundaries are sharp.

1. Introduction

Network security is one of most important focus points in computer networks. When the network is represented by a graph model, graph parameters are introduced to describe the stability and vulnerability of the network under a specific structure. The classic parameters to measure the vulnerability of a network include toughness, isolated toughness, and binding number. In this paper, we only discuss the theoretical problems in computer network such that all network structures are denoted by graphs, and only simple graphs are considered. That is to say, if there are multiple channels connected between two sites, the model will be idealized, thinking that only one edge connects two corresponding vertices.

For graph $G = (V(G), E(G))$, we use $d_G(x)$ to express the degree of vertex x in G . For $n \geq 2$ being an integer number, P_n and $P_{\geq n}$ denote the path with exactly n and at least n vertices, respectively. A path factor is a special spanning subgraph whose components are paths, and a $P_{\geq n}$ -factor is a special path factor such that each path has at least n vertices. We no longer explain each concept of graph theory one by one since these basic notations and

terminologies can be easily found in other classic graph theory books.

A graph R is factor critical if any induced subgraph with order $|V(R)| - 1$ admits a perfect matching. If $H \cong K_1$, $H \cong K_2$, or H is the corona of a factor critical graph R with $|V(R)| \geq 3$, then H is called a sun. Specially, a sun with order at least six in the last case is called a big sun. We use $\text{sun}(G)$ to denote the number of sun components of G . Let $\omega(G)$ and $i(G)$ be the number of connected components and isolated vertices of a graph G , respectively. Hence, $\omega(G) \geq \text{sun}(G) \geq i(G)$ for any graph G .

Las Vergnas [1] and Kaneko [2] characterized the criterion for the existence of $P_{\geq 2}$ -factors and $P_{\geq 3}$ -factors, respectively.

Lemma 1 (see Las Vergnas [1]). *A graph G contains a $P_{\geq 2}$ -factor $\Leftrightarrow i(G - S) \leq 2|S|$ for any vertex subset S of G .*

Lemma 2 (see Kaneko [2]). *A graph G contains a $P_{\geq 3}$ -factor $\Leftrightarrow \text{sun}(G - S) \leq 2|S|$ for any vertex subset S of G .*

The sun toughness $s(G)$ was introduced by Zhou et al. [3] as

$$s(G) = \min \left\{ \frac{|S|}{\text{sun}(G-S)} : \text{sun}(G-S) \geq 2 \right\}, \quad (1)$$

if G is not a complete graph; otherwise, $s(G) = +\infty$. Considering the previous trick of defining new toughness and isolated toughness parameters by subtracting one from the denominator, Zhu et al. [4] introduced the new parameter of sun toughness denoted by $s'(G)$ which can be formulated (if G is not complete) as follows:

$$s'(G) = \min \left\{ \frac{|S|}{\text{sun}(G-S) - 1} : \text{sun}(G-S) \geq 2 \right\}. \quad (2)$$

For the complete graph, $s'(G) = +\infty$.

If for any edge $e \in E(G)$, a graph G contains a $P_{\geq n}$ -factor covering e , then G is called a $P_{\geq n}$ -factor covered graph. The following two conclusions for $P_{\geq n}$ -factor covered graph for $n = 2$ and 3 were determined by Zhang and Zhou [5].

Lemma 3 (see Zhang and Zhou [5]). *A connected graph G is $P_{\geq 2}$ -factor covered $\Leftrightarrow$*

$$i(G-S) \leq 2|S| - \varepsilon_1(S), \quad (3)$$

for any vertex subset S of G , where

$$\varepsilon_1(S) = \begin{cases} 2, & \text{if } S \text{ is not an independent set,} \\ 1, & \text{if } S \text{ is independent and there exists a nontrivial component of } G-S, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Lemma 4 (see Zhang and Zhou [5]). *A connected graph G is $P_{\geq 3}$ -factor covered $\Leftrightarrow$*

$$\text{sun}(G-S) \leq 2|S| - \varepsilon_2(S), \quad (5)$$

for any $S \subseteq V(G)$, where

$$\varepsilon_2(S) = \begin{cases} 2, & \text{if } S \text{ is not an independent set,} \\ 1, & \text{if } S \text{ is independent and there exists a non-sun component of } G-S, \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Some recent advances in graph factors can be referred to Zhou et al. [6] and Gao et al. [7-9].

The main contribution in this work is to study the existence of $P_{\geq 2}$ -factor or $P_{\geq 3}$ -factor in special settings by means of sun toughness parameter. The results can be divided into two parts which are manifested in the next section: first, the sun toughness conditions for a graph to be $P_{\geq 2}$ -factor uniform and $P_{\geq 3}$ -factor uniform are determined; then, the sun toughness bounds for $P_{\geq 2}$ -factor avoidable graph and $P_{\geq 3}$ -factor avoidable graph are given. The counterexamples presented in sharpness section imply that all these obtained sun toughness bounds are sharp.

2. Main Results and Proofs

The aim of this section is to present our main conclusions and detailed proof.

2.1. Factor Uniform Graph. A graph G is a $P_{\geq n}$ -factor uniform graph if for any two edges e_1 and e_2 , G has a $P_{\geq n}$ -factor covering e_1 excluding e_2 (see Zhou and Sun [10]). In this part, we study the sun toughness condition for path factor uniform graphs.

Theorem 1. *Let G be a 2-edge-connected graph. If*

$$s(G) > \begin{cases} \frac{|V(G)| - 4}{2}, & \text{if } |V(G)| \geq 6, \\ 1, & \text{if } |V(G)| \leq 5, \end{cases} \quad (7)$$

or

$$s'(G) > \begin{cases} |V(G)| - 4, & \text{if } |V(G)| \geq 6, \\ 2, & \text{if } |V(G)| \leq 5, \end{cases} \quad (8)$$

then G is a $P_{\geq 2}$ -factor uniform graph.

Proof. The 2-edge-connected condition implies $|V(G)| \geq 3$. If $|V(G)| = 3$, then $G \cong K_3$ and we can verify directly that K_3 is a $P_{\geq 2}$ -factor uniform graph with $s(K_3) = s'(K_3) = +\infty$. If $|V(G)| = 4$, then G is isomorphic to K_4 or C_4 or the graph with edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_3\}$. It is easy to confirm that no matter which G is isomorphic, it is a $P_{\geq 2}$ -factor uniform graph. Therefore, for $|V(G)| \in \{3, 4\}$, 2-edge-connected graph G is a $P_{\geq 2}$ -factor uniform graph. Next, we consider $|V(G)| \geq 5$.

For arbitrary $e \in E(G)$, $G' = G - e$ is connected since $\lambda(G) \geq 2$. To prove Theorem 1, we aim to check that G' is $P_{\geq 2}$ -factor covered. Suppose G' is not a $P_{\geq 2}$ -factor covered graph. By means of Lemma 3, we can find a vertex subset S of G' satisfying

$$i(G' - S) \geq 2|S| - \varepsilon_1(S) + 1. \quad (9)$$

Clearly, the value of $i(G' - S)$ belongs to $\{i(G - S), i(G - S) + 1, i(G - S) + 2\}$. $\square$

Claim 1. S contains at least two vertices.

Proof. If S is an empty set, then using (9), we have $i(G') \geq 1$ which contradicts to G is 2-edge-connected.

Assume that $|S| = 1$. Then, $\varepsilon_1(S) \leq 1$, and hence $i(G' - S) \geq 2$ in light of (9). We get $i(G - S) = 0$ since G is 2-edge-connected. It implies that K_2 is a component in $G - S$ and e is exactly an edge in K_2 . Suppose there is another component W in $G - S$ with $|V(W)| \geq 2$. If $|V(W)| = 2$, then $|V(G)| \geq 5$ and

$$\frac{|V(G)| - 4}{2} < s(G) \leq \frac{|S \cup X|}{\text{sun}(G - S \cup X)} \leq \frac{1 + |V(G)| - 5}{2} = \frac{|V(G)| - 4}{2}, \quad (12)$$

or

$$|V(G)| - 4 < s'(G) \leq \frac{|S \cup X|}{\text{sun}(G - S \cup X) - 1} \leq \frac{1 + |V(G)| - 5}{2 - 1} = |V(G)| - 4, \quad (13)$$

a contradiction.

The last situation is that $G - S$ is isomorphic to K_2 and thus $G \cong K_3$ which is a $P_{\geq 2}$ -factor uniform graph with $s(K_3) = s'(K_3) = +\infty$.

Since $|S| \geq 2$ by Claim 1, we obtain $\varepsilon_1(S) \leq 2$, $i(G' - S) \geq 2|S| - \varepsilon_1(S) + 1 \geq 3$, $i(G - S) \geq i(G' - S) - 2 \geq 1$,

$$s(G) \leq \frac{|S|}{\text{sun}(G - S)} \leq \frac{|S|}{i(G - S)} = \frac{|S|}{i(G' - S)} \leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1} \leq \frac{|S|}{2|S| - 1}. \quad (14)$$

If $|V(G)| \geq 6$, then we obtain $1 \leq ((|V(G)| - 4)/2) < s(G) \leq (|S|/(2|S| - 1))$ which contradicts to $|S| \geq 2$. If $|V(G)| = 5$, then $1 < s(G) \leq (|S|/(2|S| - 1))$, also contradicts to $|S| \geq 2$.

For $s'(G)$, we have

$$\begin{aligned} 1 \leq |V(G)| - 4 < s'(G) &\leq \frac{|S|}{\text{sun}(G - S) - 1} \leq \frac{|S|}{i(G' - S) - 1} \\ &\leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 1} \leq \frac{|S|}{2|S| - 2}. \end{aligned} \quad (15)$$

$$\begin{aligned} 1 \leq \frac{|V(G)| \geq 6 - 4}{2} < s(G) &\leq \frac{|S|}{\text{sun}(G - S)} \leq \frac{|S|}{i(G - S)} = \frac{|S|}{i(G' - S) - 1} \\ &\leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 1} \leq \frac{|S|}{2|S| - 2}, \end{aligned} \quad (16)$$

or

$$1 \leq |V(G)| - 4 < s'(G) \leq \frac{|S|}{\text{sun}(G - S) - 1} \leq \frac{1}{2 - 1} = 1, \quad (11)$$

a contradiction. Otherwise, $|V(W)| \geq 3$, and we select $|V(W)| - 2$ vertices from W and denote it by X such that $W - X$ is isomorphic to K_2 . Thus, we have

and $|V(G)| \geq 5$. The following three cases are discussed according to the relationship between $i(G - S)$ and $i(G' - S)$:

Case 1. $i(G - S) = i(G' - S) \geq 3$.

We deduce

It implies $|S| < 2$, a contradiction.

Case 2. $i(G - S) = i(G' - S) - 1 \geq 2$.

In this case, we acquire $|V(G)| \geq 6$ and

or

$$\begin{aligned} 2 \leq |V(G)| - 4 < s'(G) &\leq \frac{|S|}{\text{sun}(G-S) - 1} \leq \frac{|S|}{i(G-S) - 1} = \frac{|S|}{i(G'-S) - 1 - 1} \\ &\leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 2} \leq \frac{|S|}{2|S| - 3}. \end{aligned} \quad (17)$$

It implies $|S| < 2$, a contradiction.

Case 3. $i(G-S) = i(G'-S) - 2$.

In this case, $i(G-S) \geq 1$ since $i(G'-S) \geq 3$. We confirm that K_1 and K_2 are components in $G-S$, $e \in E(K_2)$, and $|V(G)| \geq 5$. If $|V(G)| \geq 8$, then

$$\begin{aligned} 2 \leq \frac{|V(G)| - 4}{2} < s(G) &\leq \frac{|S|}{\text{sun}(G-S)} \leq \frac{|S|}{i(G-S)} = \frac{|S|}{i(G'-S) - 2} \\ &\leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 2} \leq \frac{|S|}{2|S| - 3}, \end{aligned} \quad (18)$$

which reveals $|S| < 2$, a contradiction.

If $|V(G)| = 5$, then $G = K_2 \vee (K_1 \cup K_2)$ and thus $s(G) = 1$ which contradicts to $s(G) > 1$ if $|V(G)| = 5$.

If $|V(G)| = 6$, then $G = K_2 \vee (2K_1 \cup K_2)$ and thus $s(G) = (2/3)$ contradicts to $s(G) > ((|V(G)| - 4)/2) = 1$.

If $|V(G)| = 7$, then G is isomorphic to $K_2 \vee (3K_1 \cup K_2)$ or $K_2 \vee (K_1 \cup 2K_2)$ and thus $s(G) = (1/2)$ or $(2/3)$ contradicts to $s(G) > ((|V(G)| - 4)/2) = (3/2)$.

Note that $\text{sun}(G-S) \geq i(G-S) + 1$ because of the presence of at least one K_2 component in $G-S$. For $s'(G)$, if $|V(G)| \geq 6$, we infer

$$\begin{aligned} 2 \leq |V(G)| - 4 < s'(G) &\leq \frac{|S|}{\text{sun}(G-S) - 1} \leq \frac{|S|}{i(G-S)} = \frac{|S|}{i(G'-S) - 2} \\ &\leq \frac{|S|}{2|S| - \varepsilon_1(S) + 1 - 2} \leq \frac{|S|}{2|S| - 3}, \end{aligned} \quad (19)$$

which reveals $|S| < 2$, a contradiction. If $|V(G)| = 5$, then $G = K_2 \vee (K_1 \cup K_2)$ and thus $s'(G) = 2$ which contradicts to $s'(G) > 2$ if $|V(G)| = 5$.

Therefore, the desired result is proved. $\square$

Theorem 2. Let G be a 2-edge-connected graph. If

$$s(G) > \begin{cases} \frac{|V(G)| - 4}{2}, & \text{if } |V(G)| \geq 6, \\ 1, & \text{if } |V(G)| \leq 5, \end{cases} \quad (20)$$

or

$$s'(G) > \begin{cases} |V(G)| - 4, & \text{if } |V(G)| \geq 6, \\ 2, & \text{if } |V(G)| \leq 5, \end{cases} \quad (21)$$

then G is a $P_{\geq 3}$ -factor uniform graph.

Proof. For $|V(G)| \leq 4$, we get the conclusion in light of the corresponding discussion presented in Theorem 1. Hence, we consider $|V(G)| \geq 5$ in what follows.

For any $e \in E(G)$, $G' = G - e$ is a connected graph in terms of $\lambda(G) \geq 2$, and we prove the theorem by proving that G' is $P_{\geq 3}$ -factor covered. Suppose that G' is not $P_{\geq 3}$ -factor covered, and there is a subset S of $V(G')$ satisfying

$$\text{sun}(G'-S) \geq 2|S| - \varepsilon_2(S) + 1. \quad (22)$$

$\square$

Claim 2. S contains at least two elements.

Proof. If there is no vertex in S , then $\varepsilon_2(S) = 0$, $\text{sun}(G') = 1$, and G' is a big sun. We confirm that G is a graph which is obtained by inserting an edge in a big sun. Obviously, we can select vertex set X (at least two vertices in X) from $V(G)$, such that it produces $|X|$ sun components in $G - X$. By the definition of sun toughness, we obtain

$$1 < s(G) \leq \frac{|X|}{\text{sun}(G-X)} \leq \frac{|X|}{|X|} = 1, \quad (23)$$

or

$$2 < s'(G) \leq \frac{|X|}{\text{sun}(G-X)-1} \leq \frac{|X|}{|X|-1} = 1 + \frac{1}{|X|-1} \leq 1 + \frac{1}{2-1} = 2, \quad (24)$$

a contradiction.

If $|S| = 1$, then $\varepsilon_2(S) \leq 1$ and $\text{sun}(G' - S) \geq 2$ by (22). If $\text{sun}(G - S) \geq 2$, then

$$1 < s(G) \leq \frac{|S|}{\text{sun}(G-S)} \leq \frac{1}{2}, \quad (25)$$

or

$$2 < s'(G) \leq \frac{|S|}{\text{sun}(G-S)-1} \leq 1, \quad (26)$$

a contradiction.

If $\text{sun}(G - S) = 1$, then if there is a non-sun component W in $G - S$, it produces a new sun component after deleting e from W . Thus, there is a vertex x in W such that $d_{G-S}(x) = 1$ and assume $xy \in E(G - S)$. We deduce $\text{sun}(G - S \cup \{y\}) \geq 2$ and

$$1 < s(G) \leq \frac{|S \cup \{y\}|}{\text{sun}(G - S \cup \{y\})} \leq 1, \quad (27)$$

or

$$2 < s'(G) \leq \frac{|S \cup \{y\}|}{\text{sun}(G - S \cup \{y\}) - 1} \leq 2, \quad (28)$$

a contradiction. The other circumstances are described as follows: there is a K_2 component in $G - S$, the deleting edge e is exactly belongs to $E(K_2)$, and $G - S - V(K_2)$ contains a non-sun component which leads $\varepsilon_2(S) = 1$. In this case, we have $|V(G)| \geq 6$. The worst situation is that $G - S - V(K_2)$ is a complete graph K_t with order at least 3. In order to get at least two sun components, $t - 2$ vertices need to be deleted from K_t to obtain the second K_2 component. This extreme situation can be discussed using a trick similar to Claim 1 in the proofing of Theorem 1.

If $\text{sun}(G - S) = 0$, then there is a non-sun component M in $G - S$ such that it produces two new sun components after deleting e from M . Thus, there is a vertex set $Z \subseteq V(M)$ such that deleting Z (with $|Z| \geq 1$) from M will produce $|Z| + 1$ sun components. We yield $\text{sun}(G - S \cup Z) \geq 2$ and

$$\begin{aligned} 1 < s(G) &\leq \frac{|S \cup Z|}{\text{sun}(G - S \cup Z)} = \frac{|S| + |Z|}{\text{sun}(G - S) + |Z| + 1} \\ &\leq \frac{1 + |Z|}{\text{sun}(G' - S) - 2 + |Z| + 1} \leq \frac{1 + |Z|}{2|S| - \varepsilon_2(S) + 1 + |Z| - 1} \leq \frac{1 + |Z|}{|Z| + 1} = 1, \end{aligned} \quad (29)$$

or

$$\begin{aligned} 2 < s'(G) &\leq \frac{|S \cup Z|}{\text{sun}(G - S \cup Z) - 1} = \frac{|S| + |Z|}{\text{sun}(G - S) + |Z| + 1 - 1} \\ &\leq \frac{1 + |Z|}{\text{sun}(G' - S) - 2 + |Z|} \leq \frac{1 + |Z|}{2|S| - \varepsilon_2(S) + 1 + |Z| - 2} \\ &\leq \frac{1 + |Z|}{|Z|} = 1 + \frac{1}{|Z|} \leq 2, \end{aligned} \quad (30)$$

a contradiction.

From $|S| \geq 2$, we acquire $\varepsilon_2(S) \leq 2$, $\text{sun}(G' - S) \geq 3$, and $\text{sun}(G - S) \geq \text{sun}(G' - S) - 2 \geq 1$. If $\text{sun}(G - S) = \text{sun}(G' - S)$ or $\text{sun}(G - S) = \text{sun}(G' - S) - 1$, we deduce

$$\begin{aligned} 1 < s(G) &\leq \frac{|S|}{\text{sun}(G - S)} \leq \frac{|S|}{\text{sun}(G' - S) - 1} \leq \frac{|S|}{2|S| - \varepsilon_2(S) + 1 - 1} \\ &\leq \frac{|S|}{2|S| - 2} = \frac{1}{2} + \frac{1}{2|S| - 2} \leq \frac{1}{2} + \frac{1}{2 \times 2 - 2} = 1, \end{aligned} \quad (31)$$

or

$$\begin{aligned} 2 < s'(G) &\leq \frac{|S|}{\text{sun}(G - S) - 1} \leq \frac{|S|}{\text{sun}(G' - S) - 1 - 1} \leq \frac{|S|}{2|S| - \varepsilon_2(S) + 1 - 2} \\ &\leq \frac{|S|}{2|S| - 3} = \frac{1}{2} + \frac{3}{2(2|S| - 3)} \leq \frac{1}{2} + \frac{3}{2(2 \times 2 - 3)} = 2, \end{aligned} \quad (32)$$

a contradiction.

If $\text{sun}(G - S) = \text{sun}(G' - S) - 2$, then edge e belongs to a non-sun component W in $G - S$, while removing e will

produce two sun components. There is a vertex set $X \subseteq V(W)$ such that deleting X (clearly $|X| \geq 1$) from W will produce $|X| + 1$ sun components. Hence, we get

$$\begin{aligned}
1 < s(G) &\leq \frac{|S \cup X|}{\text{sun}(G - S \cup X)} \leq \frac{|S| + |X|}{\text{sun}(G - S) + |X| + 1} \\
&\leq \frac{|S| + |X|}{\text{sun}(G' - S) - 2 + |X| + 1} \leq \frac{|S| + |X|}{2|S| - \varepsilon_2(S) + 1 + |X| - 1} \leq \frac{|S| + |X|}{2|S| + |X| - 2} \\
&= 1 - \frac{|S| - 2}{2|S| + |X| - 2} \leq 1,
\end{aligned} \tag{33}$$

a contradiction.

For $s'(G)$, if $|S| \geq 3$, we obtain

$$\begin{aligned}
2 < s'(G) &\leq \frac{|S \cup X|}{\text{sun}(G - S \cup X) - 1} \leq \frac{|S| + |X|}{\text{sun}(G - S) + |X| + 1 - 1} \\
&\leq \frac{|S| + |X|}{\text{sun}(G' - S) - 2 + |X|} \leq \frac{|S| + |X|}{2|S| - \varepsilon_2(S) + 1 + |X| - 2} \leq \frac{|S| + |X|}{2|S| + |X| - 3} \\
&= 1 - \frac{|S| - 3}{2|S| + |X| - 3} \leq 1,
\end{aligned} \tag{34}$$

a contradiction. If $|S| = 2$, then

$$\begin{aligned}
2 < s'(G) &\leq \frac{|S| + |X|}{2|S| + |X| - 3} = \frac{|X| + 2}{|X| + 1} \\
&= 1 + \frac{1}{|X| + 1} \leq 1 + \frac{1}{1 + 1} = \frac{3}{2},
\end{aligned} \tag{35}$$

a contradiction. $\square$

2.2. Factor Critical Deleted Graph. A graph G is a $P_{\geq n}$ -factor deleted graph (or called $P_{\geq n}$ -factor avoidable graph) if for arbitrary edge $e \in E(G)$, G admits a $P_{\geq n}$ -factor excluding e . A graph G is called a $(P_{\geq n}, k)$ -factor critical deleted graph (or called $(P_{\geq n}, k)$ -factor critical avoidable graph) if for any $U \subseteq V(G)$ with $|U| = k$, $G - U$ is a $P_{\geq n}$ -factor deleted graph (or $P_{\geq n}$ -factor avoidable graph).

In this subsection, we present sun toughness conditions for $(P_{\geq 2}, k)$ -factor critical deleted graph and $(P_{\geq 3}, k)$ -factor critical deleted graph.

Theorem 3. Let $k \in \mathbb{N} \cup \{0\}$ and G be a $(k + 2)$ -connected graph. If $s(G) > ((k + 2)/4)$ (resp. $s'(G) > ((k + 2)/3)$), then G is $(P_{\geq 2}, k)$ -factor critical deleted.

Proof. Following from connectivity condition, Theorem 3 is obviously established if G is a complete graph. We only consider that G is not complete. Let U be a vertex subset with k vertices, $G' = G - U$, $e \in E(G')$, and $G'' = G' - e$. Hence, G'' is connected by means of G being $(k + 2)$ -connected. We prove Theorem 3 in view of proving that G'' has $P_{\geq 2}$ -factor. On the contrary, we assume that G'' has no $P_{\geq 2}$ -factor. Using Lemma 1, there is a vertex subset S of G'' satisfying

$$i(G'' - S) \geq 2|S| + 1. \tag{36}$$

Clearly, $S \neq \emptyset$ since G'' is connected. If $|S| = 1$, then according to $i(G'' - S) = i(G' - e - S) \leq i(G' - S) + 2$ and (36), we have $i(G' - S) \geq i(G'' - S) - 2 \geq 2|S| + 1 - 2 = 1$. It implies that $G' - S$ admits an isolated vertex x , and thus $d_G(x) \leq d_{G'}(x) + |U| \leq d_{G' - S}(x) + |S| + k = k + 1$ which contradicts to G is $k + 2$ -connected. Therefore, $|S| \geq 2$.

Notice that

$$\begin{aligned}
\text{sun}(G - U \cup S) &\geq \text{sun}(G - (U \cup S) - e) - 2 = \text{sun}(G'' - S) - 2 \\
&\geq i(G'' - S) - 2 \geq 2|S| + 1 - 2 \geq 3.
\end{aligned} \tag{37}$$

In fact, the “=”s in the derivation above cannot hold simultaneously. If $\text{sun}(G - U \cup S) = \text{sun}(G - (U \cup S) - e) - 2$, then there is a non-sun component W in $G' - S$, and deleting e from W will add two new sun components in $G'' - S$. On the other hand, if $\text{sun}(G'' - S) - 2 = i(G'' - S) - 2$, then all sun components in $G'' - S$ are

isolated vertices, and thus K_2 is a sun component in $G' - S$ and $e \in E(K_2)$. Thus, $\text{sun}(G - U \cup S) = \text{sun}(G - (U \cup S) - e) - 2$ and $\text{sun}(G'' - S) - 2 = i(G'' - S) - 2$ cannot be hold simultaneously, and $\text{sun}(G - U \cup S) \geq i(G'' - S) - 1 \geq 2|S| + 1 - 1 = 2|S|$.

In light of the definition of sun toughness, we acquire

$$\begin{aligned}
\frac{k+2}{4} < s(G) &\leq \frac{|U \cup S|}{\text{sun}(G - U \cup S)} \leq \frac{|U| + |S|}{i(G'' - S) - 1} \\
&\leq \frac{k + |S|}{2|S| + 1 - 1} = \frac{k + |S|}{2|S|} = \frac{1}{2} + \frac{k}{2|S|} \\
&\leq \frac{1}{2} + \frac{k}{2 \times 2} = \frac{1}{2} + \frac{k}{4} = \frac{k+2}{4},
\end{aligned} \tag{38}$$

or

$$\begin{aligned}
\frac{k+2}{3} < s'(G) &\leq \frac{|U \cup S|}{\text{sun}(G - U \cup S) - 1} \leq \frac{|U| + |S|}{i(G'' - S) - 1 - 1} \\
&\leq \frac{k + |S|}{2|S| + 1 - 2} = \frac{k + |S|}{2|S| - 1} = \frac{1}{2} + \frac{k + (1/2)}{2|S| - 1} \\
&\leq \frac{1}{2} + \frac{k + (1/2)}{2 \times 2 - 1} = \frac{1}{2} + \frac{k + (1/2)}{3} = \frac{k+2}{3},
\end{aligned} \tag{39}$$

a contradiction. $\square$

Theorem 4. Let $k \in \mathbb{N} \cup \{0\}$ and G be a $(k+2)$ -connected graph. If

$$\begin{aligned}
s(G) &\begin{cases} \geq 1, & \text{if } k \in \{0, 1\}, \\ > \frac{k+2}{4}, & \text{if } k \geq 2, \end{cases} \\
\left(\text{resp. } s'(G) \right) &\begin{cases} \geq 1 & \text{if } k = 0 \\ > \frac{k+2}{3} & \text{if } k \geq 1 \end{cases}
\end{aligned} \tag{40}$$

then G is $(P_{\geq 3}, k)$ -factor critical deleted.

Proof. Theorem 4 is clearly held if G is a complete graph using connectivity condition. Only noncomplete graphs are considered in what follows. Let U be a vertex subset with k vertices, $G' = G - U$, $e \in E(G')$, and $G'' = G' - e$. Hence, G'' is connected by means of G being $(k+2)$ -connected. We prove Theorem 4 in view of proving that G'' has $P_{\geq 3}$ -factor. If not, we assume that G'' has no $P_{\geq 3}$ -factor. In light of Lemma 2, there is a vertex subset S of G'' satisfying

$$\text{sun}(G'' - S) \geq 2|S| + 1. \tag{41}$$

$\square$

Claim 3. $|S| \geq 2$.

Proof. If S is an empty set, then using (41) and connectivity condition of G , we confirm that G'' is a big sun. Thus, $|V(G'')| \geq 6$ and let R be the factor critical subgraph of G'' . We verify that R contains at least 3 vertices and G' is constructed by inserting an edge into a big sun G'' . By observation, we can select vertex set X (at least two vertices in X) from $V(G')$, such that it produces $|X|$ sun components in $G' - X$. Then, the contradiction can be deduced using the trick as presented in Claim 2.

If $|S| = 1$, then $\text{sun}(G'' - S) \geq 3$ by (41) and $\text{sun}(G' - S) \geq \text{sun}(G'' - S) - 2 \geq 1$. If $\text{sun}(G' - S) \geq 2$, then it contradicts to G is $(k+2)$ -connected.

If $\text{sun}(G' - S) = 1$, then $\text{sun}(G'' - S) = 3$ and there is a non-sun component W in $G' - S$ such that it produces two new sun components after deleting e from W . It reveals that there is a sun component and a non-sun component in $G' - S$ which contradicts to $G' - S$ is connected.

By Claim 3 and (41), we yield $\text{sun}(G'' - S) \geq 2|S| + 1 \geq 5$ and $\text{sun}(G' - S) \geq \text{sun}(G'' - S) - 2 \geq 3$. Now, we consider the following cases:

Case 1. $\text{sun}(G' - S) = \text{sun}(G'' - S) \geq 5$.

In this case, we have

$$s(G) \leq \frac{|U \cup S|}{\text{sun}(G - U \cup S)} = \frac{|U| + |S|}{\text{sun}(G' - S)} = \frac{k + |S|}{\text{sun}(G'' - S)} \leq \frac{k + |S|}{2|S| + 1}. \tag{42}$$

If $k = 0$, then $1 \leq s(G) \leq ((k + |S|)/(2|S| + 1)) < (1/2)$, a contradiction. If $k \geq 1$, we have

$$s(G) \leq \frac{k + |S|}{2|S| + 1} = \frac{1}{2} + \frac{k - (1/2)}{2|S| + 1} \leq \frac{1}{2} + \frac{k - (1/2)}{2 \times 2 + 1} = \frac{k+2}{5}. \tag{43}$$

If $k = 1$, then $1 \leq s(G) \leq ((k+2)/5)$, a contradiction. If $k \geq 2$, then $((k+2)/4) < s(G) \leq ((k+2)/5)$, a contradiction.

For $s'(G)$, by its definition, we get

$$\begin{aligned}
s'(G) &\leq \frac{|U \cup S|}{\text{sun}(G - U \cup S) - 1} = \frac{|U| + |S|}{\text{sun}(G' - S) - 1} \\
&= \frac{k + |S|}{\text{sun}(G'' - S) - 1} \leq \frac{k + |S|}{2|S| + 1 - 1} = \frac{1}{2} + \frac{k}{2|S|} \\
&\leq \frac{1}{2} + \frac{k}{2 \times 2} = \frac{k+2}{4}.
\end{aligned} \tag{44}$$

If $k = 0$, then $1 \leq s'(G) \leq (2/4)$, a contradiction. If $k \geq 1$, then $((k+2)/3) < s'(G) \leq ((k+2)/4)$, a contradiction.

Case 2. $\text{sun}(G' - S) = \text{sun}(G'' - S) - 1$.

In this case, we consider two situations. If there is a non-sun component W in $G' - S$ and deleting e from W , then we produce a new sun component. Then, there

is a vertex $y \in V(W)$ with degree 1 in $G' - S$, and we set $xy \in E(G' - S)$. Hence, using the definition of sun component, we have

$$\begin{aligned} s(G) &\leq \frac{|U \cup S \cup \{x\}|}{\text{sun}(G - U \cup S \cup \{x\})} = \frac{|U| + |S \cup \{x\}|}{\text{sun}(G' - S \cup \{x\})} \\ &= \frac{k + |S| + 1}{\text{sun}(G' - S) + 1} = \frac{k + |S| + 1}{\text{sun}(G'' - S) - 1 + 1} \leq \frac{k + |S| + 1}{2|S| + 1} \\ &= \frac{1}{2} + \frac{k + (1/2)}{2|S| + 1} \leq \frac{1}{2} + \frac{k + (1/2)}{2 \times 2 + 1} = \frac{k + 3}{5}. \end{aligned} \quad (45)$$

If $k \in \{0, 1\}$, then $1 \leq s(G) \leq ((k + 3)/5)$, a contradiction. If $k \geq 2$, then $((k + 2)/4) < s(G) \leq ((k + 3)/5)$ which implies $k < 2$, a contradiction.

For $s'(G)$, we acquire

$$\begin{aligned} s'(G) &\leq \frac{|U \cup S \cup \{x\}|}{\text{sun}(G - U \cup S \cup \{x\}) - 1} = \frac{|U| + |S \cup \{x\}|}{\text{sun}(G' - S \cup \{x\}) - 1} \\ &= \frac{k + |S| + 1}{\text{sun}(G' - S) + 1 - 1} = \frac{k + |S| + 1}{\text{sun}(G'' - S) - 1} \leq \frac{k + |S| + 1}{2|S| + 1 - 1} \\ &= \frac{1}{2} + \frac{k + 1}{2|S|} \leq \frac{1}{2} + \frac{k + 1}{2 \times 2} = \frac{k + 3}{4}. \end{aligned} \quad (46)$$

If $k = 0$, then $1 \leq s'(G) \leq (3/4)$, a contradiction. If $k \geq 1$, then $((k + 2)/3) < s'(G) \leq ((k + 3)/4)$ which implies $k < 1$, a contradiction.

Another situation is stated as follows: K_2 is a sun component in $G' - S$, $e \in E(K_2)$, and thus the number of sun component adds 1 after removing e from $G' - S$. Hence, we yield

$$\begin{aligned} s(G) &\leq \frac{|U \cup S|}{\text{sun}(G - U \cup S)} = \frac{|U| + |S|}{\text{sun}(G' - S)} \\ &= \frac{k + |S|}{\text{sun}(G'' - S) - 1} \leq \frac{k + |S|}{2|S| + 1 - 1} = \frac{1}{2} + \frac{k}{2|S|} \quad (47) \\ &\leq \frac{1}{2} + \frac{k}{2 \times 2} = \frac{k + 2}{4}. \end{aligned}$$

If $k \in \{0, 1\}$, then $1 \leq s(G) \leq ((k + 2)/4)$, a contradiction. If $k \geq 2$, then $((k + 2)/4) < s(G) \leq ((k + 2)/4)$, a contradiction.

For $s'(G)$, we get

$$\begin{aligned} s'(G) &\leq \frac{|U \cup S|}{\text{sun}(G - U \cup S) - 1} = \frac{|U| + |S|}{\text{sun}(G' - S) - 1} \\ &= \frac{k + |S|}{\text{sun}(G'' - S) - 1 - 1} \leq \frac{k + |S|}{2|S| + 1 - 2} = \frac{1}{2} + \frac{k + (1/2)}{2|S| - 1} \\ &\leq \frac{1}{2} + \frac{k + (1/2)}{2 \times 2 - 1} = \frac{k + 2}{3}. \end{aligned} \quad (48)$$

If $k = 0$, then $1 \leq s'(G) \leq (2/3)$, a contradiction. If $k \geq 1$, then $((k + 2)/3) < s'(G) \leq ((k + 2)/3)$, a contradiction.

Case 3. $\text{sun}(G' - S) = \text{sun}(G'' - S) - 2$.

In this case, edge e belongs to a non-sun component W in $G' - S$, and removing e will produce two sun components. We can find a vertex set $X \subset V(W)$ such that deleting X

(clearly $|X| \geq 1$) from W will produce $|X| + 1$ sun components. Hence, in view of the definition of sun toughness, we infer

$$\begin{aligned}
 s(G) &\leq \frac{|U \cup S \cup X|}{\text{sun}(G - U \cup S \cup X)} = \frac{|U| + |S| + |X|}{\text{sun}(G' - S) + |X| + 1} \\
 &= \frac{k + |S| + |X|}{\text{sun}(G'' - S) - 2 + |X| + 1} \leq \frac{k + |S| + |X|}{2|S| + 1 + |X| - 1} \\
 &= \frac{k + |S| + |X|}{2|S| + |X|} \\
 &= \frac{1}{2} + \frac{k + (1/2)|X|}{2|S| + |X|} \leq \frac{1}{2} + \frac{k + (1/2)|X|}{2 \times 2 + |X|} = \frac{k + |X| + 2}{4 + |X|} \\
 &= 1 + \frac{k - 2}{4 + |X|}.
 \end{aligned} \tag{49}$$

$$\begin{aligned}
 s'(G) &\leq \frac{|U \cup S \cup X|}{\text{sun}(G - U \cup S \cup X) - 1} = \frac{|U| + |S| + |X|}{\text{sun}(G' - S) + |X| + 1 - 1} \\
 &= \frac{k + |S| + |X|}{\text{sun}(G'' - S) - 2 + |X|} \leq \frac{k + |S| + |X|}{2|S| + 1 + |X| - 2} = \frac{k + |S| + |X|}{2|S| + |X| - 1} \\
 &= \frac{1}{2} + \frac{k + (1/2)|X| + (1/2)}{2|S| + |X| - 1} \leq \frac{1}{2} + \frac{k + (1/2)|X| + (1/2)}{2 \times 2 + |X| - 1} = \frac{k + |X| + 2}{3 + |X|} \\
 &= 1 + \frac{k - 2}{3 + |X|}.
 \end{aligned} \tag{50}$$

If $k = 0$, then $1 \leq s'(G) \leq 1 - (1/(3 + |X|)) < 1$, a contradiction. If $k \geq 1$, then $((k + 2)/3) < s'(G) \leq 1 + ((k - 1)/3 + |X|) \leq ((k + 3)/4)$ which reveals $k < 1$, a contradiction.

In all, this finishes the proof of Theorem 4. $\square$

If $k = 0, 1$, then $1 \leq s(G) \leq 1 - ((2 - k)/(4 + |X|)) < 1$, a contradiction. If $k \geq 2$, then $((k + 2)/4) < s(G) \leq 1 + ((k - 2)/(4 + |X|)) \leq 1 + ((k - 2)/(4 + 1)) = ((k + 3)/5)$, which implies $k < 2$, a contradiction.

For $s'(G)$, we deduce

3. Sharpness

3.1. Sharpness of Theorems 1 and 2. The sun toughness bounds in Theorems 1 and 2 are tight. For $|V(G)| \geq 6$, set $G = K_1 \vee (K_2 \cup K_t)$ which is 2-edge-connected graph where $t \geq 3$ and thus K_t is not a sun. Select $t - 2$ vertices in K_t and denote this vertex set by X . We yield

$$\begin{aligned}
 s(G) &= \frac{|V(K_1) \cup X|}{\text{sun}(G - V(K_1) \cup X)} = \frac{1 + |V(G)| - 5}{2} = \frac{|V(G)| - 4}{2}, \\
 s'(G) &= \frac{|V(K_1) \cup X|}{\text{sun}(G - V(K_1) \cup X) - 1} = \frac{1 + |V(G)| - 5}{2 - 1} = |V(G)| - 4.
 \end{aligned} \tag{51}$$

Select $e \in E(K_2)$ and set $G' = G - e = K_1 \vee (2K_1 \cup K_t)$. Let S be the vertex set in first K_1 in G' . We have $\varepsilon_1(S) = \varepsilon_2(S) = 1$,

$$\begin{aligned}
 i(G' - S) &= 2 > 1 = 2|S| - \varepsilon_1(S), \\
 \text{sun}(G' - S) &= 2 > 1 = 2|S| - \varepsilon_2(S).
 \end{aligned} \tag{52}$$

Hence, in view of Lemmas 3 and 4, G' is neither $P_{\geq 2}$ -factor covered nor $P_{\geq 3}$ -factor covered, and G is neither a $P_{\geq 2}$ -factor uniform graph nor a $P_{\geq 3}$ -factor uniform graph.

For $|V(G)| = 5$, we consider $G = K_2 \vee (K_1 \cup K_2)$. Clearly, $s(G) = 1$ and $s'(G) = 2$. Choose $e \in E(K_1 \cup K_2)$ and $G' = G - e = K_2 \vee (3K_1)$. Set $S = V(K_2)$ in G' and then $\varepsilon_1(S) = \varepsilon_2(S) = 2$,

$$\begin{aligned} i(G' - S) &= 3 > 2 = 2|S| - \varepsilon_1(S), \\ \text{sun}(G' - S) &= 3 > 2 = 2|S| - \varepsilon_2(S). \end{aligned} \quad (53)$$

Therefore, in light of Lemmas 3 and 4, G' is neither $P_{\geq 2}$ -factor covered nor $P_{\geq 3}$ -factor covered and G is neither a $P_{\geq 2}$ -factor uniform graph nor a $P_{\geq 3}$ -factor uniform graph.

3.2. Sharpness of Theorem 3. To state the sharpness of Theorem 3, we consider graph $G = K_{k+2} \vee (K_2 \cup 3K_1)$. Clearly, G is $(k+2)$ -connected graph, $s(G) = ((k+2)/4)$, and $s'(G) = ((k+2)/3)$. Let $U \subset V(K_{k+2})$ with $|U| = k$ and $e \in E(K_2)$. Then, $G - U - e = K_2 \vee (5K_1)$. Set $S = V(K_2) \subset V(G - U - e)$, and we have

$$i(G - U - e - S) = 5 > 4 = 2|S|. \quad (54)$$

Thus, $G - U - e$ has no $P_{\geq 2}$ -factor, $G - U$ is not a $P_{\geq 2}$ -factor avoidable graph, and G is not a $(P_{\geq 2}, k)$ -factor critical avoidable graph.

3.3. Sharpness of Theorem 4. We use the following graphs to show that the sun toughness bounds described in Theorem 4

FIGURE 1: The structure of W .

are tight. First, considering $G = K_{k+2} \vee (3K_2 \cup W)$ where the subgraph W is constructed as Figure 1, there are two big suns H_1 and H_2 connected by an edge e , and W is a non-sun component in $G - V(K_{k+2})$.

Let R_1 and R_2 be the factor critical subgraph of W where both $|V(R_1)|$ and $|V(R_2)|$ are big numbers. We confirm that

$$\begin{aligned} s(G) &= \frac{|V(K_{k+2}) \cup V(R_1) \cup (V(R_2) - \{w\})|}{\text{sun}(G - V(K_{k+2}) \cup V(R_1) \cup (V(R_2) - \{w\}))} = \frac{k+2 + |V(R_1)| + |V(R_2)| - 1}{3 + |V(R_1)| + |V(R_2)|} \\ &= 1 + \frac{k-2}{3 + |V(R_1)| + |V(R_2)|}, \end{aligned} \quad (55)$$

which implies that if $k < 2$, then $s(G) < 1$, and $s(G) \rightarrow 1$ as $|V(R_1)| + |V(R_2)| \rightarrow +\infty$.

Also, we verify

$$\begin{aligned} s'(G) &= \frac{|V(K_{k+2}) \cup V(R_1) \cup (V(R_2) - \{w\})|}{\text{sun}(G - V(K_{k+2}) \cup V(R_1) \cup (V(R_2) - \{w\})) - 1} = \frac{k+2 + |V(R_1)| + |V(R_2)| - 1}{3 + |V(R_1)| + |V(R_2)| - 1} \\ &= 1 + \frac{k-1}{2 + |V(R_1)| + |V(R_2)|}, \end{aligned} \quad (56)$$

which implies that if $k < 1$, then $s'(G) < 1$, and $s'(G) \rightarrow 1$ as $|V(R_1)| + |V(R_2)| \rightarrow +\infty$.

Let $U \subset V(K_{k+2})$ with $|U| = k$, $G' = G - U = K_2 \vee (3K_2 \cup W)$, $e \in E(W)$ be the edge which connects two big suns, and $G'' = G' - e = K_2 \vee (3K_2 \cup H_1 \cup H_2)$. Let $S = V(K_2)$ (the first K_2 which belongs to K_{k+2} in G) and $G'' - S = 3K_2 \cup H_1 \cup H_2$. We have

$$\text{sun}(G'' - S) = 5 > 4 = 2|S|. \quad (57)$$

Hence, by means of Lemma 2, G'' has no P_3 -factor, G' is not a $P_{\geq 3}$ -factor avoidable graph, and G is not a $(P_{\geq 3}, k)$ -factor critical avoidable graph. Therefore, the $s(G)$ bound for $k = 0$ or 1 and $s'(G)$ bound for $k = 0$ in Theorem 4 are tight.

Last, we consider graph $G = K_{k+2} \vee (4K_2)$. Clearly, we have $s(G) = ((k+2)/4)$ and $s'(G) = ((k+2)/3)$. Let $U \subset V(K_{k+2})$ with $|U| = k$, $G' = G - U = K_2 \vee (4K_2)$, $e \in E(4K_2)$, and $G'' = G' - e = K_2 \vee (3K_2 \cup 2K_1)$. Let $S = V(K_2)$ (the first K_2 which belongs to K_{k+2} in G) and $G'' - S = 3K_2 \cup 2K_1$. We obtain

$$\text{sun}(G'' - S) = 5 > 4 = 2|S|. \quad (58)$$

Thus, according to Lemma 2, G'' has no P_3 -factor, G' is not a $P_{\geq 3}$ -factor avoidable graph, and G is not a $(P_{\geq 3}, k)$ -factor critical avoidable graph. That is to say, the $s(G)$ bound for $k \geq 2$ and $s'(G)$ bound for $k \geq 1$ in Theorem 4 are tight.

4. Conclusion

In our article, the relationship between sun toughness and the existence of path factor in particular settings is studied. The theorems yielded in this paper belong to existence theorems, i.e., under what kind of conditions the path factor exists. However, in a specific computer network, it needs to use a certain algorithm to determine the value of sun toughness of the fix network graph and give the eligible path factor on the graph from the algorithm point of view. The problems of such algorithms are worthy of consideration in the future research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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