Research Article

On Some New Fixed Point Results with Applications to Matrix Difference Equations

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The aim of this article is to discuss the convergence of iterative sequences of the Prešić type involving new classes of operators satisfying Prešić type Θ -contractive condition in the context of metric spaces. Some examples are also provided to show the significance of the investigation of finding fixed points. Some convergence results for a class of matrix difference equations will be derived as application.

1. Introduction and Preliminaries

Banach's contraction principle [1] is one of the decisive results of fixed point theory. It states that if we have a self mapping $\mathcal{L}$ on a complete metric space $(\mathcal{S}, \sigma)$ and a constant $\lambda \in (0, 1)$ such that

$$\sigma(\mathcal{L}\mathbf{a}, \mathcal{L}\mathbf{b}) \leq \lambda \sigma(\mathbf{a}, \mathbf{b}), \quad (1)$$

holds, $\forall \mathbf{a}, \mathbf{b} \in \mathcal{S}$, then there exists a unique $\mathbf{a}^* \in \mathcal{S}$ such that $\mathbf{a}^* = \mathcal{L}\mathbf{a}^*$.

Because of its substance and accessibility, many authors have established various fascinating supplements and extensions of this principle (see [1–21] and references therein). From now to onward, we will consider k as positive integer and $(\mathcal{S}, \sigma)$ as complete metric space.

In 1965, Prešić [2] generalized the famous Banach contraction principle and applied the obtained results to secure the convergence of a specific type of sequences. Prešić established the following theorem.

Theorem 1 (see [2]). Let $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be a mapping satisfying the following contractive condition:

$$\begin{aligned} &\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_k, \mathbf{a}_{k+1})) \\ &\leq \lambda_1 \sigma(\mathbf{a}_1, \mathbf{a}_2) + \lambda_2 \sigma(\mathbf{a}_2, \mathbf{a}_3) + \dots + \lambda_k \sigma(\mathbf{a}_k, \mathbf{a}_{k+1}), \end{aligned} \quad (2)$$

for every $\mathbf{a}_1, \dots, \mathbf{a}_{k+1} \in \mathcal{S}$, where $\lambda_1, \lambda_2, \dots, \lambda_k$ are non-negative constants such that $\lambda_1 + \lambda_2 + \dots + \lambda_k < 1$. Then, $\forall \mathbf{a}^* \in \mathcal{S}$ such that $\mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*) = \mathbf{a}^*$ and it is unique. Moreover if $\mathbf{a}_1, \dots, \mathbf{a}_k$ are arbitrary points in $\mathcal{S}$ and for $n \in \mathbb{N}$,

$$\mathbf{a}_{n+k} = \mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}), \quad (3)$$

then the sequence $\{\mathbf{a}_n\}$ is convergent and $\mathbf{a}^* = \lim \mathbf{a}_n = \mathcal{L}(\lim \mathbf{a}_n, \lim \mathbf{a}_n, \dots, \lim \mathbf{a}_n)$.

A mapping $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ satisfying inequality (2) is said to be a Prešić operator. A point $\mathbf{a}^* \in \mathcal{S}$ is called a fixed point of $\mathcal{L}$ if $\mathbf{a}^* = \mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*)$. If $k = 1$ in Theorem 1, then we get the Banach contraction principle as a specific case.

Ćirić et al. [3] established the following theorem.

Theorem 2 (see [3]). Assume that $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ satisfies

$$\begin{aligned} &\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_k, \mathbf{a}_{k+1})) \\ &\leq \lambda \max\{\sigma(\mathbf{a}_1, \mathbf{a}_2), \sigma(\mathbf{a}_2, \mathbf{a}_3), \dots, \sigma(\mathbf{a}_k, \mathbf{a}_{k+1})\}, \end{aligned} \quad (4)$$

for any $\mathbf{a}_1, \dots, \mathbf{a}_{k+1} \in \mathcal{S}$, where $0 < \lambda < 1$. Then, $\exists \mathbf{a}^* \in \mathcal{S}$ such that $\mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*) = \mathbf{a}^*$. Moreover, for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence given in (3) is convergent and

$$\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathcal{L}\left(\lim_{n \rightarrow \infty} \mathbf{a}_n, \dots, \lim_{n \rightarrow \infty} \mathbf{a}_n\right). \quad (5)$$

If in addition,

$$\sigma(\mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*), \mathcal{L}(\mathbf{a}', \dots, \mathbf{a}')) < \sigma(\mathbf{a}^*, \mathbf{a}'), \quad (6)$$

holds for all $\mathbf{a}^*, \mathbf{a}' \in \mathcal{S}$, with $\mathbf{a}^* \neq \mathbf{a}'$, then the fixed point in $\mathcal{S}$ is unique.

Pwfâcurar [4] obtained a convergence theorem for Prešić–Kannan contraction in this way.

Definition 1 (see [4]). Assume that $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$. If $\exists \lambda \in \mathbb{R}$ with $0 < \lambda k(k+1) < 1$ such that

$$\begin{aligned} \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\ \leq \lambda \sum_{i=1}^{k+1} \sigma(\mathbf{a}_i, \mathcal{L}(\mathbf{a}_i, \dots, \mathbf{a}_i)), \end{aligned} \quad (7)$$

holds for all $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$. Then,

- (i) $\mathcal{L}$ has a unique fixed point $\mathbf{a}^* \in \mathcal{S}$.
- (ii) For any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to $\mathbf{a}^*$. Supplementary in this direction, we refer the readers to [5–8, 10].

Very recently, Jleli et al. [11] gave a new variety of contraction called Θ -contraction and established generalized results for these contractions.

Definition 2. Let $\Theta: (0, \infty) \rightarrow (1, \infty)$ be a mapping satisfying the following:

- (i) (Θ_1) Θ is nondecreasing.
- (ii) (Θ_2) $\forall \{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \Theta(t_n) = 1 \iff \lim_{n \rightarrow \infty} t_n = 0$.
- (iii) (Θ_3) $\exists 0 < h < 1$ and $l \in (0, \infty]$ such that $\lim_{t \rightarrow 0^+} ((\Theta(t) - 1)/t^h) = l$.

A mapping $\mathcal{L}: \mathcal{S} \rightarrow \mathcal{S}$ is called a Θ -contraction if there exist some $\lambda \in (0, 1)$ and a function Θ satisfying (Θ_1) – (Θ_3) such that

$$\mathcal{L}\mathbf{a} \neq \mathcal{L}\mathbf{b} \implies \Theta(\sigma(\mathcal{L}\mathbf{a}, \mathcal{L}\mathbf{b})) \leq [\Theta(\sigma(\mathbf{a}, \mathbf{b}))]^\lambda, \quad (8)$$

$\forall \mathbf{a}, \mathbf{b} \in \mathcal{S}$.

Theorem 3 (see [11]). Let $\mathcal{L}: \mathcal{S} \rightarrow \mathcal{S}$ be a Θ -contraction; then, $\mathcal{L}$ has a unique fixed point.

We represent by the Ω set of all mappings $\Theta: (0, \infty) \rightarrow (1, \infty)$ satisfying (Θ_1) – (Θ_3) consistent with Samet et al. [11]. For more details, we refer the following [12, 14–16, 18–20] to the readers.

In this article, we discuss the convergence of $\{\mathbf{a}_n\}$ given by (3), where $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ is a Prešić type Θ -contraction. The given results unify and generalize various existing results of the literature.

2. Main Results

Motivated by the work of Samet et al. [11], we give the following definition.

Definition 3. A mapping $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ is called a Prešić type Θ -contraction if there exists some $\lambda \in (0, 1)$ such that

$$\begin{aligned} \Theta(\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1}))) \\ \leq [\Theta(\max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\})]^\lambda, \end{aligned} \quad (9)$$

$\forall (\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ with $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$.

Note that for $\Theta(t) = e^{\sqrt{t}}$, Prešić type Θ -contractive condition is reduced to

$$\begin{aligned} \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\ \leq \lambda^2 \max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\}, \end{aligned} \quad (10)$$

$\forall (\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$, $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$.

Furthermore, for $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ such that $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) = \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$, inequality (4) is also satisfied, i.e., $\mathcal{L}$ is a Ćirić–Prešić contraction.

Remark 1. Every Prešić type Θ -contraction $\mathcal{L}$ is a Prešić operator by (Θ_1) and (7), that is,

$$\begin{aligned} \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\ < \max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\}, \end{aligned} \quad (11)$$

$\forall (\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$, $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$. Thus, each Prešić type Θ -contraction $\mathcal{L}$ is a continuous function.

Now we present our main result in this way.

Theorem 4. Assume that $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ a Prešić type Θ -contraction. Then, for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to $\mathbf{a}^*$ in $\mathcal{S}$ and $\mathbf{a}^*$ is a fixed point of $\mathcal{L}$. Additionally, if $\sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}) \neq \mathcal{L}(\mathbf{b}, \dots, \mathbf{b}))$ implies that

$$\Theta(\sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{b}, \dots, \mathbf{b}))) \leq [\sigma(\mathbf{a}, \mathbf{b})]^\lambda, \quad (12)$$

$\forall \mathbf{a}, \mathbf{b} \in \mathcal{S}$ with $\mathbf{a} \neq \mathbf{b}$, then there exists unique $\mathbf{a}^* \in \mathcal{S}$ such that $\mathcal{L}\mathbf{a}^* = \mathbf{a}^*$.

Proof. Let $\mathbf{a}_1, \dots, \mathbf{a}_k$ be arbitrary k elements in $\mathcal{S}$. Define $\{\mathbf{a}_n\}$ in $\mathcal{S}$ by

$$\mathbf{a}_{n+k} = \mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}), \quad (13)$$

for $n \in \mathbb{N}$. If for some $n_0 \in \{1, 2, 3, \dots, k\}$, we have $\mathbf{a}_{n_0} = \mathbf{a}_{n_0+1}$, then

$$\begin{aligned} \mathbf{a}_{n_0+k} &= \mathcal{L}(\mathbf{a}_{n_0}, \mathbf{a}_{n_0+1}, \dots, \mathbf{a}_{n_0+k-1}) \\ &= \mathcal{L}(\mathbf{a}_{n_0+k}, \mathbf{a}_{n_0+k}, \dots, \mathbf{a}_{n_0+k}), \end{aligned} \quad (14)$$

that is, $\mathbf{a}_{n_0+k}$ is a fixed point of $\mathcal{L}$ and we have nothing to prove. So, we assume that $\mathbf{a}_{n+k} \neq \mathbf{a}_{n+k+1}$ for all $n \in \mathbb{N}$. Represent $\beta_{n+k} = \sigma(\mathbf{a}_{n+k}, \mathbf{a}_{n+k+1})$ for $n \in \mathbb{N}$ and

$\mu = \max\{\sigma(\mathbf{a}_1, \mathbf{a}_2), \sigma(\mathbf{a}_2, \mathbf{a}_3), \dots, \sigma(\mathbf{a}_k, \mathbf{a}_{k+1})\}$; then, we obtain $\beta_{n+k} > 0$, $\forall n \in \mathbb{N}$, and $\mu > 0$. Thus, for $n \leq k$, we get

$$\begin{aligned}
 1 < \Theta(\beta_{k+1}) &= \Theta(\sigma(\mathbf{a}_{k+1}, \mathbf{a}_{k+2})) \\
 &= \Theta(\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_{k+1}))) \\
 &\leq [\Theta(\max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\})]^\lambda \\
 &= [\Theta(\mu)]^\lambda, \\
 1 < \Theta(\beta_{k+2}) &= \Theta(\sigma(\mathbf{a}_{k+2}, \mathbf{a}_{k+3})) \\
 &= \Theta(\sigma(\mathcal{L}(\mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_{k+1}), \mathcal{L}(\mathbf{a}_3, \mathbf{a}_4, \dots, \mathbf{a}_{k+2}))) \\
 &\leq [\Theta(\max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 2 \leq i \leq k+1\})]^\lambda \\
 &\leq [\Theta(\mu)]^{\lambda^2} \\
 &\vdots,
 \end{aligned} \tag{15}$$

and so on. Hence,

$$\begin{aligned}
 1 < \Theta(\beta_{k+n}) &= \Theta(\sigma(\mathbf{a}_{n+k}, \mathbf{a}_{n+k+1})) \\
 &= \Theta(\sigma(\mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}), \mathcal{L}(\mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \dots, \mathbf{a}_{n+k}))) \\
 &\leq [\Theta(\mu)]^{\lambda^n},
 \end{aligned} \tag{16}$$

for $n \geq 1$. Letting $n \rightarrow \infty$, we get by (Θ_2) that

$$\lim_{n \rightarrow \infty} \Theta(\beta_{k+n}) = 1 \iff \lim_{n \rightarrow \infty} \beta_{k+n} = 0. \tag{17}$$

By (Θ_3) , $\exists 0 < h < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n \rightarrow \infty} \frac{\Theta(\beta_{k+n}) - 1}{\beta_{k+n}^h} = l. \tag{18}$$

Let $l < \infty$ and $\omega = (l/2) > 0$. By definition of the limit, $\exists n_1 \in \mathbb{N}$ such that

$$\left| \frac{\Theta(\beta_{k+n}) - 1}{\beta_{k+n}^h} - l \right| \leq \omega, \tag{19}$$

$\forall n > n_1$. It shows that

$$\frac{\Theta(\beta_{k+n}) - 1}{\beta_{k+n}^h} \geq l - \omega = \frac{l}{2} = \omega, \tag{20}$$

$\forall n > n_1$. Then,

$$n\beta_{k+n}^h \leq \omega n [\Theta(\beta_{k+n}) - 1], \tag{21}$$

$\forall n > n_1$, where $\omega = (1/\omega)$. Now we assume that $l = \infty$. Let $\omega > 0$. By definition of the limit, $\exists n_1 \in \mathbb{N}$ such that

$$\omega \leq \frac{\Theta(\beta_{k+n}) - 1}{\beta_{k+n}^h}, \tag{22}$$

$\forall n > n_1$. This implies that

$$n\beta_{k+n}^h \leq \omega n [\Theta(\beta_{k+n}) - 1], \tag{23}$$

$\forall n > n_1$, where $\omega = (1/\omega)$. Thus, in all cases, there exist $\omega > 0$ and $n_1 \in \mathbb{N}$ such that

$$n\beta_{k+n}^h \leq \omega n [\Theta(\beta_{k+n}) - 1], \tag{24}$$

for all $n > n_1$. Thus, by (10), we get

$$n\beta_{k+n}^h \leq \omega n ([\Theta(\mu)]^{\lambda^n} - 1). \tag{25}$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} n\beta_{k+n}^h = 0. \tag{26}$$

Thus, $\exists n_2 \in \mathbb{N}$ such that

$$n\beta_{k+n}^h \leq 1, \tag{27}$$

$\forall n > n_2$. Therefore, we get

$$\beta_{k+n} \leq \frac{1}{n^{1/h}}, \tag{28}$$

for all $n > n_2$. Now we show that $\{\mathbf{a}_n\}$ is a Cauchy sequence. For $m > n > n_2$, we have

$$\begin{aligned}
 \sigma(\mathbf{a}_{k+n}, \mathbf{a}_{k+m}) &= \sigma(\mathcal{L}(\mathbf{a}_n, \dots, \mathbf{a}_{k+n-1}), \mathcal{L}(\mathbf{a}_m, \dots, \mathbf{a}_{k+m-1})) \\
 &\leq \sigma(\mathcal{L}(\mathbf{a}_n, \dots, \mathbf{a}_{k+n-1}), \mathcal{L}(\mathbf{a}_{n+1}, \dots, \mathbf{a}_{k+n})) \\
 &\quad + \sigma(\mathcal{L}(\mathbf{a}_{n+1}, \dots, \mathbf{a}_{k+n}), \mathcal{L}(\mathbf{a}_{n+2}, \dots, \mathbf{a}_{k+n+1})) \\
 &\quad + \dots + \sigma(\mathcal{L}(\mathbf{a}_{m-1}, \dots, \mathbf{a}_{k+m-2}), \mathcal{L}(\mathbf{a}_m, \dots, \mathbf{a}_{k+m-1})) \\
 &= \sigma(\mathbf{a}_{n+k}, \mathbf{a}_{n+k+1}) + \sigma(\mathbf{a}_{n+k+1}, \mathbf{a}_{n+k+2}) + \dots + \sigma(\mathbf{a}_{m+k-1}, \mathbf{a}_{m+k}) \\
 &= \beta_{n+k} + \beta_{n+k+1} + \dots + \beta_{m+k-1} \\
 &= \sum_{i=n}^{m-1} \beta_{i+k} < \sum_{i=n}^{\infty} \beta_{i+k} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}} < \infty.
 \end{aligned} \tag{29}$$

This proves that $\{\mathbf{a}_n\}$ is Cauchy in $(\mathcal{S}, \sigma)$. As $(\mathcal{S}, \sigma)$ is complete, $\exists \mathbf{a}^*$ in $\mathcal{S}$ such that

$$\lim_{n, m \rightarrow \infty} \sigma(\mathbf{a}_n, \mathbf{a}_m) = \lim_{n \rightarrow \infty} \sigma(\mathbf{a}_n, \mathbf{a}^*) = 0. \quad (30)$$

Now as $\mathcal{L}$ is continuous, we get

$$\begin{aligned} \mathbf{a}^* &= \lim_{n \rightarrow \infty} \mathbf{a}_{n+k} = \lim_{n \rightarrow \infty} \mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}) \\ &= \mathcal{L}\left(\lim_{n \rightarrow \infty} \mathbf{a}_n, \lim_{n \rightarrow \infty} \mathbf{a}_{n+1}, \dots, \lim_{n \rightarrow \infty} \mathbf{a}_{n+k-1}\right) \\ &= \mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*). \end{aligned} \quad (31)$$

Now we show the uniqueness of fixed point of mapping $\mathcal{L}$. We suppose on the contrary that $\exists \mathbf{a}^*, \mathbf{a}' \in \mathcal{S}$ so that $\mathbf{a}^* = \mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*)$ and $\mathbf{a}' = \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')$ with $\mathbf{a}^* \neq \mathbf{a}'$. Thus, $\mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*) \neq \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')$. Hence, by given assumption, we have

$$\begin{aligned} \Theta(\sigma(\mathbf{a}^*, \mathbf{a}')) &= \Theta(\sigma(\mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*), \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')))) \\ &\leq [\Theta(\sigma(\mathbf{a}^*, \mathbf{a}'))]^\lambda, \end{aligned} \quad (32)$$

a contradiction as $\lambda \in (0, 1)$. Therefore, $\mathbf{a}^* = \mathbf{a}'$. $\square$

Example 1. Consider the sequence $\{\mathbf{a}_n\}$ as follows:

$$\begin{aligned} \mathbf{a}_1 &= 1 \\ \mathbf{a}_2 &= 1 + 5 \\ &\vdots \\ \mathbf{a}_n &= 1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1). \end{aligned} \quad (33)$$

Let $\mathcal{S} = \{\mathbf{a}_n : n \in \mathbb{N}\}$ and $\sigma(\mathbf{a}^*, \mathbf{a}') = |\mathbf{a}^* - \mathbf{a}'|$. Then, $(\mathcal{S}, \sigma)$ becomes a complete metric space. Consider $\mathcal{L}: \mathcal{S}^2 \rightarrow \mathcal{S}$ by

$$\mathcal{L}(\mathbf{a}_n^*, \mathbf{a}_n') = \begin{cases} \frac{\mathbf{a}_{n-1}^* + \mathbf{a}_{n-1}'}{2}, & \text{for } n > 1, \\ \frac{\mathbf{a}_1^* + \mathbf{a}_1'}{2}, & \text{otherwise.} \end{cases} \quad (34)$$

For $n > 3$, we get

$$\begin{aligned} &\sigma(\mathcal{L}(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \mathcal{L}(\mathbf{a}_{n-1}, \mathbf{a}_n)) \\ &= \sigma\left(\frac{\mathbf{a}_{n-3} + \mathbf{a}_{n-2}}{2}, \frac{\mathbf{a}_{n-2} + \mathbf{a}_{n-1}}{2}\right) \\ &= \frac{1}{2} \left| ((n-3)(2n-7) + (n-2)(2n-5)) - ((n-2)(2n-5) + (n-1)(2n-3)) \right| \\ &= \frac{1}{2} \left| 2n^2 - 13n + 21 + 2n^2 - 9n + 10 - (2n^2 - 9n + 10 + 2n^2 - 5n + 3) \right| \\ &= \frac{1}{2} (8n - 18) = 4n - 9, \\ &\max\{\sigma(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \sigma(\mathbf{a}_{n-1}, \mathbf{a}_n)\} \\ &= \max\{|(n-2)(2n-5) - (n-1)(2n-3)|, |(n-1)(2n-3) - n(2n-1)|\} \\ &= \max\{4n-7, 4n-3\} = 4n-3. \end{aligned} \quad (35)$$

Now

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{\sigma(\mathcal{L}(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \mathcal{L}(\mathbf{a}_{n-1}, \mathbf{a}_n))}{\max\{\sigma(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \sigma(\mathbf{a}_{n-1}, \mathbf{a}_n)\}} \\ &= \lim_{n \rightarrow \infty} \frac{4n-9}{4n-3} = 1. \end{aligned} \quad (36)$$

Thus,

$$\begin{aligned} &\sigma(\mathcal{L}(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \mathcal{L}(\mathbf{a}_{n-1}, \mathbf{a}_n)) \\ &\leq \lambda \max\{\sigma(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \sigma(\mathbf{a}_{n-1}, \mathbf{a}_n)\} \end{aligned} \quad (37)$$

does not hold for $\lambda \in (0, 1)$. Thus, the main hypothesis of Theorem 2 in [3] is not satisfied. Now, by considering the mapping $\Theta: (0, \infty) \rightarrow (1, \infty)$ defined by

$$\Theta(t) = e^{te^t}, \quad (38)$$

we can easily show that $\Theta \in \Omega$ and $\mathcal{L}$ is Prešić type Θ -contraction. Indeed, the following holds:

$$\begin{aligned} & e^{\sqrt{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})} e^{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})}} \\ & \leq e^{\lambda \sqrt{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), (\mathbf{a}_{i+1}, \mathbf{a}_{i+2})} e^{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), (\mathbf{a}_{i+1}, \mathbf{a}_{i+2})}}, \end{aligned} \quad (39)$$

for $\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1}) \neq \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})$, $i = 1, 2, \dots$, and for some $\lambda \in (0, 1)$. The above condition is equivalent to

$$\begin{aligned} & \sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2}) \\ & \cdot e^{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})} \\ & \leq \lambda^2 \max\{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), \sigma(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})\} \\ & \cdot e^{\max\{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), \sigma(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})\}}. \end{aligned} \quad (40)$$

So, we have to check that

$$\begin{aligned} & \frac{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2}) \cdot e^{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})}}{\max\{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), \sigma(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})\} \cdot e^{\max\{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), \sigma(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})\}}} \\ & \leq \lambda^2, \end{aligned} \quad (41)$$

for some $\lambda \in (0, 1)$. We discuss these two cases.

Case 1. For $i = n = 1$, we have

$$\begin{aligned} & \frac{\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2)), \mathcal{L}(\mathbf{a}_2, \mathbf{a}_3) e^{\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2)), \mathcal{L}(\mathbf{a}_2, \mathbf{a}_3)}}{\max\{\sigma((\mathbf{a}_1, \mathbf{a}_2)), \sigma(\mathbf{a}_2, \mathbf{a}_3)\} e^{\max\{\sigma((\mathbf{a}_1, \mathbf{a}_1)), \sigma(\mathbf{a}_2, \mathbf{a}_3)\}}} \\ & = \frac{\sigma(((\mathbf{a}_1 + \mathbf{a}_1)/2), ((\mathbf{a}_1 + \mathbf{a}_2)/2)) e^{\sigma(((\mathbf{a}_1 + \mathbf{a}_1)/2), ((\mathbf{a}_1 + \mathbf{a}_2)/2))}}{\max\{\sigma((\mathbf{a}_1, \mathbf{a}_2)), \sigma(\mathbf{a}_2, \mathbf{a}_3)\} e^{\max\{\sigma((\mathbf{a}_1, \mathbf{a}_2)), \sigma(\mathbf{a}_2, \mathbf{a}_3)\}}} \\ & = \frac{\sigma(1, (7/2)) e^{\sigma(1, (7/2))}}{\max\{\sigma((1, 6)), \sigma(6, 15)\} e^{\max\{\sigma((1, 6)), \sigma(6, 15)\}}} \\ & = \frac{5}{18} e^{(-13/2)} \\ & < e^{-2}. \end{aligned} \quad (42)$$

Case 2. For $i = n > 1$, we have

$$\begin{aligned} & \frac{\sigma(\mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1})), \mathcal{L}(\mathbf{a}_{n+1}, \mathbf{a}_{n+2}) \cdot e^{\sigma(\mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1})), \mathcal{L}(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})}}{\max\{\sigma((\mathbf{a}_n, \mathbf{a}_{n+1})), \sigma(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})\} \cdot e^{\max\{\sigma((\mathbf{a}_n, \mathbf{a}_{n+1})), \sigma(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})\}}} \\ & = \frac{\sigma(((\mathbf{a}_{n-1} + \mathbf{a}_n)/2), ((\mathbf{a}_n + \mathbf{a}_{n+1})/2)) \cdot e^{\sigma(((\mathbf{a}_{n-1} + \mathbf{a}_n)/2), ((\mathbf{a}_n + \mathbf{a}_{n+1})/2))}}{\max\{\sigma((\mathbf{a}_n, \mathbf{a}_{n+1})), \sigma(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})\} \cdot e^{\max\{\sigma((\mathbf{a}_n, \mathbf{a}_{n+1})), \sigma(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})\}}} \\ & = \frac{(1/2) \{ (4n^2 - 6n + 3) - (4n^2 + 2n + 1) \} \cdot e^{(1/2) \{ (4n^2 - 6n + 3) - (4n^2 + 2n + 1) \}}}{\max\{4n + 1, 2n + 2\} \cdot e^{\max\{4n + 1, 2n + 2\}}} \\ & = \frac{-4n + 1}{4n + 1} e^{-8n} \\ & < e^{-2}, \end{aligned} \quad (43)$$

with $\lambda = e^{-1}$.

Thus, all the conditions of Theorem 4 are satisfied and $(1, 1)$ is the unique fixed point of $\mathcal{L}$.

Example 2. Let $\mathcal{S} = [0, 1]$ and σ be such that

$$\sigma(\mathbf{a}^*, \mathbf{a}') = |\mathbf{a}^* - \mathbf{a}'|, \quad (44)$$

and $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be defined by

$$\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) = \frac{\mathbf{a}_1 + \mathbf{a}_k}{4k} \quad \text{for all } \mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}. \quad (45)$$

Define the mapping $\Theta: (0, \infty) \rightarrow (1, \infty)$ given by

$$\Theta(t) = e^{\sqrt{t}}. \quad (46)$$

It is given in [11] that $\Theta \in \Omega$. Now for $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{k+1} \in \mathcal{S}$, we have

$$\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) > 0. \quad (47)$$

We have

$$\begin{aligned}
 & \Theta(\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1}))) \\
 &= e^{\sqrt{\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})))}} \\
 &= e^{\sqrt{(1/4k)[(\mathbf{a}_1 - \mathbf{a}_2) + (\mathbf{a}_k - \mathbf{a}_{k+1})]}} \\
 &= e^{(1/2\sqrt{k})\sqrt{[(\mathbf{a}_1 - \mathbf{a}_2) + (\mathbf{a}_k - \mathbf{a}_{k+1})]}} \\
 &\leq e^{(1/2)\sqrt{\max\{\sigma(\mathbf{a}_1, \mathbf{a}_2), \sigma(\mathbf{a}_k, \mathbf{a}_{k+1})\}}} \\
 &\leq e^{(1/\sqrt{2})\sqrt{\max\{\sigma(\mathbf{a}_j, \mathbf{a}_{j+1}): 1 \leq j \leq k\}}} \\
 &= [\Theta(\max\{\sigma(\mathbf{a}_j, \mathbf{a}_{j+1}): 1 \leq j \leq k\})]^\lambda.
 \end{aligned} \tag{48}$$

with $\lambda = 1/\sqrt{2}$. Moreover, for all $\mathbf{a}^*, \mathbf{a}' \in \mathcal{S}$ with $\mathbf{a}^* \neq \mathbf{a}'$, we have

$$\begin{aligned}
 & \sigma(\mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*), \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')) \\
 &= \frac{|\mathbf{a}^* - \mathbf{a}'|}{2k} > 0, \\
 & \Theta(\sigma(\mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*), \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')))) \\
 &= \Theta\left(\frac{|\mathbf{a}^* - \mathbf{a}'|}{2k}\right) \\
 &= e^{\sqrt{(|\mathbf{a}^* - \mathbf{a}'|/2k)}} = e^{(1/\sqrt{2k})\sqrt{|\mathbf{a}^* - \mathbf{a}'|}} \\
 &\leq e^{(1/\sqrt{2})\sqrt{|\mathbf{a}^* - \mathbf{a}'|}} \\
 &= [\Theta(\sigma(\mathbf{a}^*, \mathbf{a}'))]^\lambda,
 \end{aligned} \tag{49}$$

with $\lambda = 1/\sqrt{2}$. Hence, all the hypotheses of Theorem 4 are satisfied. Furthermore, for some arbitrary $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ defined by (3) converges to $\mathbf{a}^* = 0$, which is the unique fixed point of mapping $\mathcal{L}$.

The upcoming result is an instant consequence of Theorem 4 by taking $\Theta(t) = e^{\sqrt{t}}$.

Corollary 1. Let $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be a given mapping. Assume that there exists some $\lambda \in (0, 1)$ such that

$$\begin{aligned}
 & \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\
 &\leq \lambda^2 \max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\},
 \end{aligned} \tag{50}$$

for all $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ with $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$. Then, for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to $\mathbf{a}^*$, and $\mathbf{a}^*$ is a fixed point of $\mathcal{L}$, that is, $\mathbf{a}^* = \mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*)$. Moreover, if

$$\sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{b}, \dots, \mathbf{b})) \leq \lambda^2 \sigma(\mathbf{a}, \mathbf{b}), \tag{51}$$

holds for all $\mathbf{a}, \mathbf{b} \in \mathcal{S}$ with $\mathbf{a} \neq \mathbf{b}$, then $\mathbf{a}^*$ is the unique fixed point of $\mathcal{L}$.

Corollary 2. Let $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be a given mapping. Suppose that there exist $\lambda_1, \lambda_2, \dots, \lambda_k$ non-negative constants with $\lambda_1 + \lambda_2 + \dots + \lambda_k < 1$ such that

$$\begin{aligned}
 & \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\
 &\leq \lambda_1 \sigma(\mathbf{a}_1, \mathbf{a}_2) + \lambda_2 \sigma(\mathbf{a}_2, \mathbf{a}_3) \\
 &\quad + \dots + \lambda_k \sigma(\mathbf{a}_k, \mathbf{a}_{k+1}),
 \end{aligned} \tag{52}$$

for all $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ with $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$. Then, for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to $\mathbf{a}^*$, where $\mathbf{a}^*$ is the unique fixed point of $\mathcal{L}$.

Proof. Evidently, (16) $\implies$ (15) with $\lambda^2 = \lambda_1 + \lambda_2 + \dots + \lambda_k$. Now, let $\mathbf{a}, \mathbf{b} \in \mathcal{S}$ with $\mathbf{a} \neq \mathbf{b}$. From (16), we have

$$\begin{aligned}
 & \sigma(\mathcal{L}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{b}, \mathbf{b}, \dots, \mathbf{b})) \\
 &\leq \sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{a}, \dots, \mathbf{a}, \mathbf{b})) \\
 &\quad + \sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}, \mathbf{b}), \mathcal{L}(\mathbf{a}, \dots, \mathbf{a}, \mathbf{b}, \mathbf{b})) \\
 &\quad + \dots + \sigma(\mathcal{L}(\mathbf{a}, \mathbf{b}, \dots, \mathbf{b}), \mathcal{L}(\mathbf{b}, \mathbf{b}, \dots, \mathbf{b})) \\
 &\leq (\lambda_k + \lambda_{k-1} + \dots + \lambda_1) \sigma(\mathbf{a}, \mathbf{b}) \\
 &= \lambda^2 \sigma(\mathbf{a}, \mathbf{b}),
 \end{aligned} \tag{53}$$

where $\lambda^2 = \lambda_k + \lambda_{k-1} + \dots + \lambda_1 \in (0, 1)$. Hence, all the hypotheses of Corollary 1 are satisfied.

Now consider the family Ω which contains large class of functions. For example, if

$$\Theta(t) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{t^\beta}\right), \tag{54}$$

where $0 < \beta < 1$ and $t > 0$, we can obtain the following theorem from our main theorem. $\square$

Theorem 5. Let $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be a given mapping. If there exist a mapping $\Theta \in \Omega$ and a constant $\beta, \lambda \in (0, 1)$ such that

$$\begin{aligned}
 & 2 - \frac{2}{\pi} \arctan\left(\frac{1}{(\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})))^\beta}\right) \\
 &\leq \left[2 - \frac{2}{\pi} \arctan\left(\frac{1}{[\max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\}]^\beta}\right)\right]^\lambda,
 \end{aligned} \tag{55}$$

for all $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ with $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$, then for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to $\mathbf{a}^*$, and $\mathbf{a}^*$ is a fixed point of $\mathcal{L}$, that is, $\mathbf{a}^* = \mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*)$. Moreover, if

$$2 - \frac{2}{\pi} \arctan \frac{1}{(\sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{b}, \dots, \mathbf{b}))^\beta)} \leq \left[2 - \frac{2}{\pi} \arctan \left(\frac{1}{(\sigma(\mathbf{a}, \mathbf{b}))^\beta} \right) \right]^\lambda, \quad (56)$$

holds $\forall \mathbf{a}, \mathbf{b} \in \mathcal{S}$ with $\mathbf{a} \neq \mathbf{b}$, then $\mathbf{a}^*$ is the unique fixed point of $\mathcal{L}$.

- (1) Theorem 1.3 in [3] and Theorem 1.2 in [2] are direct generalizations of Theorem 4.
- (2) Corollary 1 of Samet et al. in [11] can be deduced by putting $k = 1$ in Theorem 4.
- (3) Banach contraction principle [1] can be deduced from Corollaries 1 and 2 by taking $k = 1$.

3. Applications

We start this section with the definition of equilibrium point as follows.

Definition 6. Let $\mathcal{L}: \mathcal{S}^k \longrightarrow \mathcal{S}$. For given $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k \in \mathcal{S}$, consider the recursive sequence $\{\mathbf{a}_n\} \subset \mathcal{S}$ defined by

$$\mathbf{a}_{n+k} = \mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}), \quad (57)$$

for all $n \in \mathbb{N}$. A point $\bar{\mathbf{a}}$ is said to be an equilibrium point of equation (17) if the following condition is satisfied:

$$\bar{\mathbf{a}} = \mathcal{L}(\bar{\mathbf{a}}, \dots, \bar{\mathbf{a}}). \quad (58)$$

Definition 7. An equilibrium point $\bar{\mathbf{a}}$ is said to be global attractor if for all $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k \in \mathcal{S}$, we have $\sigma(\mathbf{a}_n, \bar{\mathbf{a}}) \longrightarrow 0$ as $n \longrightarrow \infty$.

If we have the recursive sequence $\{\mathcal{S}_n\} \subset P(N)$ defined by

$$\mathcal{S}_{n+k} = Q + \frac{1}{k} \sum_{i=0}^{k-1} E^* \psi(\mathcal{S}_{n+i}) E, \quad (59)$$

for all $n \in \mathbb{N}$, then we explore the global attractivity of (59) in this application. In this application, we explore the global attractivity of (18). Here $P(N)$ (for $N \geq 2$) denotes family of $N \times N$ Hermitian positive definite matrices, Q is an $N \times N$ Hermitian positive semidefinite matrix, E is an $N \times N$ nonsingular matrix, E^* is the conjugate transpose of E , and ψ is a function from $P(N)$ to $P(N)$.

We first analyze the Thompson metric σ on $P(N)$, which is defined by

$$\sigma(E_1, E_2) = \max \left\{ \log M \left(\frac{E_1}{E_2} \right), \log M \left(\frac{E_2}{E_1} \right) \right\}, \quad (60)$$

for $E_1, E_2 \in P(N)$, where $M(E_1/E_2) = \inf \{ \delta > 0: E_1 \leq \delta E_2 \} = \delta^+ (E_2^{-1/2} E_1^{-1/2} E_2)$, the maximal eigenvalue of $E_2^{-1/2} E_1^{-1/2} E_2$. Here, $E_1 \leq E_2$ means that $E_2 - E_1$ is positive semidefinite and $E_1 < E_2$ means that $E_2 - E_1$ is positive definite. Nussbaum [21] proved that $P(N)$ is a complete metric space regarding σ and

$$\sigma(E_1, E_2) = \left\| \ln(E_1^{-1/2} E_2 E_1^{-1/2}) \right\|, \quad (61)$$

where $\|\cdot\|$ is a spectral norm [21, 22]. Now we directly initiate the graceful properties of σ , i.e.,

$$\sigma(E_1, E_2) = \sigma(E_1^{-1}, E_2^{-1}) = \sigma(M^* E_1 M, M^* E_2 M), \quad (62)$$

for any nonsingular matrix M . The second valuable result is the nonpositive curvature property of σ in this way.

$$\sigma(E_1^r, E_2^r) \leq c \sigma(E_1, E_2), \quad c \in [0, 1]. \quad (63)$$

In accordance with (62) and (63), we get

$$\sigma(M^* X^r M, M^* Y^r M) \leq |c| \sigma(X, Y), \quad c \in [-1, 1], \quad (64)$$

for all $E_1, E_2 \in P(N)$.

Lemma 1 (see [23]). For all $E_1, E_2, E_3, E_4 \in P(N)$, we have

$$\sigma(E_1 + E_2, E_3 + E_4) \leq \max \{ \sigma(E_1, E_3), \sigma(E_2, E_4) \}. \quad (65)$$

Moreover, for all positive semidefinite E_1 and $E_2, E_3 \in P(N)$, we have

$$\sigma(E_1 + E_2, E_1 + E_3) \leq \sigma(E_2, E_3). \quad (66)$$

Let $\psi: P(N) \longrightarrow P(N)$ be a Θ -contraction regarding σ . For $X_1, X_2, \dots, X_k \in P(N)$, consider $\{X_n\} \subset P(N)$ defined by (59).

Theorem 8. Equation (59) has a unique equilibrium point $\bar{X} \in P(N)$. Furthermore, $\bar{X}$ is global attractor.

Proof. Define $\mathcal{L}: P(N)^k \longrightarrow P(N)$ by

$$\begin{aligned} \mathcal{L}(M_1, M_2, \dots, M_k) \\ = Q + \frac{1}{k} [E^* \psi(M_1) E + E^* \psi(M_2) E + \dots + E^* \psi(M_k) E], \end{aligned} \quad (67)$$

for all $M_1, M_2, \dots, M_k \in P(N)$.

Let $M_1, M_2, \dots, M_{k+1} \in P(N)$. By Lemma 1, we get

$$\begin{aligned}
& \sigma \mathcal{L}(M_1, M_2, \dots, M_k), \mathcal{L}(M_2, M_3, \dots, M_{k+1}) \\
&= \sigma \left(Q + \frac{1}{k} \sum_{i=1}^k E^* \psi(M_i) E, Q + \frac{1}{k} \sum_{j=2}^{k+1} E^* \psi(M_j) E \right) \\
&\leq \sigma \left(\frac{1}{k} \sum_{i=1}^k E^* \psi(M_i) E, \frac{1}{k} \sum_{j=2}^{k+1} E^* \psi(M_j) E \right) \\
&= \sigma \left(\sum_{i=1}^k ((1/\sqrt{k})E)^* \psi(M_i) ((1/\sqrt{k})E), \sum_{j=2}^{k+1} ((1/\sqrt{k})E)^* \psi(M_j) ((1/\sqrt{k})E) \right).
\end{aligned} \tag{68}$$

Denote $O = (1/\sqrt{k})E$. Then, using again Lemma 1, we have

$$\begin{aligned}
& \sigma(\mathcal{L}(M_1, M_2, \dots, M_k), \mathcal{L}(M_2, M_3, \dots, M_{k+1})) \\
&\leq \sigma \left(\sum_{i=1}^k O^* \psi(M_i) O, \sum_{j=2}^{k+1} O^* \psi(M_j) O \right) \\
&= \sigma \left(\begin{matrix} O^* \psi(M_1) O + O^* \psi(M_2) O + \dots + \\ O^* \psi(M_k) O, O^* \psi(M_2) O + \\ O^* \psi(M_3) O + \dots + O^* \psi(M_{k+1}) O \end{matrix} \right) \\
&\leq \max \left\{ \begin{matrix} \sigma(O^* \psi(M_1) O, O^* \psi(M_2) O), \\ \sigma(O^* \psi(M_2) O, O^* \psi(M_3) O), \dots, \\ \sigma(O^* \psi(M_k) O, O^* \psi(M_{k+1}) O) \end{matrix} \right\} \\
&= \max \{ \sigma(O^* \psi(M_j) O, O^* \psi(M_{j+1}) O) \},
\end{aligned} \tag{69}$$

for $j = 1, 2, \dots, k$. As A is nonsingular, O is also nonsingular. By (19), $\forall j = 1, 2, \dots, k$, we get

$$\sigma(O^* \psi(M_j) O, O^* \psi(M_{j+1}) O) = \sigma(\psi(M_j), \psi(M_{j+1})). \tag{70}$$

Now as ψ is a Θ -contraction, for all $j = 1, 2, \dots, k$, we have

$$\Theta(\sigma(O^* \psi(M_j) O, O^* \psi(M_{j+1}) O)) \leq [\Theta(\sigma(M_j, M_{j+1}))]^\lambda, \tag{71}$$

for some $\lambda \in (0, 1)$. Thus, we have

$$\begin{aligned}
& \Theta(\sigma(\mathcal{L}(M_1, M_2, \dots, M_k), \mathcal{L}(M_2, M_3, \dots, M_{k+1}))) \\
&\leq [\Theta(\max\{\sigma(M_i, M_{i+1}): i = 1, 2, \dots, k\})]^\lambda,
\end{aligned} \tag{72}$$

for all $M_1, M_2, \dots, M_{k+1} \in P(N)$. Thus, by Theorem 4, there exists a global attractor equilibrium point $\bar{X} \in P(N)$.

Now for $M_1, M_2 \in P(N)$ such that $\mathcal{L}(M_1, M_1, \dots, M_1) \neq \mathcal{L}(M_2, M_2, \dots, M_2)$, we have

$$\begin{aligned}
& \Theta(\sigma(\mathcal{L}(M_1, M_1, \dots, M_1), \mathcal{L}(M_2, M_2, \dots, M_2))) \\
&= \Theta(\sigma(Q + E^* \psi(M_1) E, Q + E^* \psi(M_2) E)) \\
&\leq \Theta(\sigma(E^* \psi(M_1) E, E^* \psi(M_2) E)) \\
&= \Theta(\sigma(\psi(M_1), \psi(M_2))) \\
&\leq [\Theta(\sigma(M_1, M_2))]^\lambda.
\end{aligned} \tag{73}$$

Again, by Theorem 4, we get the unique equilibrium point. $\square$

4. Conclusion

Jleli and Samet [11] very recently exploited the idea of Θ -contraction and established some generalized theorems for these contractions in generalized metric spaces. We continued their investigations and defined Prešić type Θ -contraction. In this paper, we studied the convergence of iterative sequences of the Prešić type involving new classes of operators satisfying Prešić type Θ -contractive condition in the context of metric spaces. Some examples are also provided to show the significance of the investigation of finding fixed points. As application, we derived some convergence results for a class of matrix difference equations.

Data Availability

The data supporting the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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