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Research Article

Generalized Conformable Mean Value Theorems with Applications to Multivariable Calculus

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The conformable derivative and its properties have been recently introduced. In this research work, we propose and prove some new results on the conformable calculus. By using the definitions and results on conformable derivatives of higher order, we generalize the theorems of the mean value which follow the same argument as in the classical calculus. The value of conformable Taylor remainder is obtained through the generalized conformable theorem of the mean value. Finally, we introduce the conformable version of two interesting results of classical multivariable calculus via the conformable formula of finite increments.

1. Introduction

The history of fractional calculus goes back to the late seventeenth century when L'Hospital proposed the fractional-order derivative. With the introduction of fractional calculus, various newly proposed definitions have been introduced. Some of the common definitions are the Caputo, Riesz, Riesz-Caputo, and Riemann-Liouville fractional ones (refer to [1, 2] for more information about fractional definitions, and see [3, 4] for research studies on the mathematical analysis of fractional calculus). A new local-type fractional definition [5] of derivative and integral has been recently proposed by Khalil et al. in [6]. Conformable derivative is basically considered as a natural extension of the classical derivative that satisfies the properties of usual derivative. In addition, conformable derivative is a generalized version of q-derivative or fractal derivative (refer to the introduction section in [7] for discussion about this relationship). Almeida et al. (2016) discussed in [8] that conformable derivative is an interesting topic of research that deserves to be studied further. In addition, both Zhao & Luo (2017) and Khalil et al. (2019) presented the physical and geometrical meaning of conformable derivative in [9, 10], respectively. Tuan et al. [11] investigated the mild solutions' existence and regularity of the proposed initial value problem for time diffusion equation in the sense of conformable derivative. This main goal of this newly introduced definition is to overcome the difficulties associated with obtaining the solutions for the equations formulated in the sense of nonlocal fractional definitions [12]. Motivated by the introduction of this definition, several research works have been conducted on the mathematical analysis of functions of a real variable formulated in the sense of conformable definition such as chain rule, mean value theorem, Rolle's theorem, power series expansion, and integration by parts formulas [6, 12–14]. The conformable partial derivative of the order $\alpha \in (0, 1]$ of the real-valued functions of several variables and the conformable gradient vector has been defined as well as the conformable Clairaut's theorem for partial derivative has also been studied in [15]. The conformable Jacobian matrix has been proposed in [16], and the chain rule for multivariable conformable derivative has also been proposed. The conformable Euler's theorem on homogeneous has been successfully defined in [17].

Furthermore, many research studies have been conducted on the theoretical and practical elements of conformable differential equations shortly after the proposition of this new definition [5, 7, 12, 18–35]. Conformable derivative has also been applied in modeling and investigating phenomena in applied sciences and engineering [12] such as the nonlinear Boussinesq equation's travelling wave solutions [36], the coupled nonlinear Schrödinger equations [34] and regularized long wave Burgers equation [35] deterministic and stochastics forms, the approximate long water wave equation's exact solutions [37], the $(1+3)$-Zakharov-Kuznetsov equation with power-law nonlinearity analytical and numerical solutions [38], the $(2+1)$-dimensional Zoomeron equation [39, 40] and 3$^{\text{rd}}$-order modified KdV equation analytical solutions [39], and the exact solutions for Whitham-Broer-Kaup equation's three various models in shallow water [41].

The paper is organized as follows: The main concepts of the conformable calculus are presented in the next section. After that, with the help of the definitions and results on conformable derivatives of higher order, the theorems of the mean value are generalized which follow the same argument as in the classical calculus. We also introduce the value of conformable Taylor remainder via the generalized conformable theorems of the mean value. Finally, we characterize the functions of several variables in which one of their conformable partial derivatives is null, and we also obtain the first conformable formula of finite increments.

2. Basic Definitions and Tools

Definition 1. Given a function $f\colon [0,\infty) \longrightarrow R$. Then, the conformable derivative of order α [6] is defined by

$$\left(T_\alpha f\right)(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{1}$$

for all $t > 0$, $0 < \alpha \leq 1$. If f is α differentiable in some $(0,a)$, $a > 0$, and $\lim_{t \longrightarrow 0^+} (T_\alpha f)(t)$ exists, then it is defined as

$$\left(T_\alpha f\right)(0) = \lim_{t \longrightarrow 0^+} \left(T_\alpha f\right)(t). \tag{2}$$

Theorem 1 (see [6]). *If a function $f\colon [0,\infty) \longrightarrow R$ is α-differentiable at $t_0 > 0$, $0 < \alpha \leq 1$, then f is continuous at t_0.*

Theorem 2 (see [6]). *Let $0 < \alpha \leq 1$, and let f, g be α differentiable at a point $t > 0$. Then, we have*

(i) $T_\alpha (af + bg) = a(T_\alpha f) + b(T_\alpha g)$, $\forall a, b \in R$.
(ii) $T_\alpha (t^p) = pt^{p-\alpha}$, $\forall p \in R$.
(iii) $T_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
(iv) $T_\alpha (fg) = f(T_\alpha g) + g(T_\alpha f)$.
(v) $T_\alpha (f/g) = (g(T_\alpha f) - f(T_\alpha g)/g^2)$.
(vi) If, in addition, f is differentiable, then $(T_\alpha f)(t) = t^{1-\alpha}(df/dt)(t)$.

The conformable derivative of certain functions using the above definition is given as follows:

(i) $T_\alpha(1) = 0$.
(ii) $T_\alpha(\sin(at)) = at^{1-\alpha}\cos(at)$.
(iii) $T_\alpha(\cos(at)) = -at^{1-\alpha}\sin(at)$.
(iv) $T_\alpha(e^{at}) = ae^{at}$, $a \in R$.

Definition 2. The (left) conformable derivative starting from a of a given function $f\colon [a,\infty) \longrightarrow R$ of order $0 < \alpha \leq 1$ [13] is defined by

$$\left(T_\alpha^a f\right)(t) = \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t + \varepsilon (t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{3}$$

When $a = 0$, it is expressed as $(T_\alpha f)(t)$. If f is α differentiable in some (a,b), then the following can be defined as

$$\left(T_\alpha^a f\right)(a) = \lim_{t \longrightarrow a^+} \left(T_\alpha^a f\right)(t). \tag{4}$$

Theorem 3 Chain Rule (see [13]). *Assume $f, g\colon (a,\infty) \longrightarrow R$ be (left) α differentiable functions, where $0 < \alpha \leq 1$. By letting $h(t) = f(g(t))$, $h(t)$ isα differentiable for all $t \neq a$ and $g(t) \neq 0$; therefore, we have the following:*

$$\left(T_\alpha^a h\right)(t) = \left(T_\alpha^a f\right)(g(t)) \cdot \left(T_\alpha^a g\right)(t) \cdot (g(t))^{\alpha-1}. \tag{5}$$

If $t = a$, then we obtain

$$\left(T_\alpha^a h\right)(a) = \lim_{t \longrightarrow a^+} \left(T_\alpha^a f\right)(g(t)) \cdot \left(T_\alpha^a g\right)(t) \cdot (g(t))^{\alpha-1}. \tag{6}$$

Theorem 4 Rolle's Theorem (see [6]). *Let $a > 0$, $\alpha \in (0,1]$, and $f\colon [a,b] \longrightarrow R$ be a given function that satisfies*

(i)- f is continuous on $[a,b]$.
(ii) f is α-differentiable on (a,b).
(iii)- $f(a) = f(b)$.

Then, there exists $c \in (a,b)$ such that $(T_\alpha f)(c) = 0$.

Corollary 1 (see [14]). *Let $I \subset [o,\infty)$, $\alpha \in (0,1]$, and $f\colon I \longrightarrow R$ be a given function that satisfies*

(i)- f is α differentiable on I.
(ii)- $f(a) = f(b) = 0$ for certain $a, b \in I$.

Then, there exists $c \in (a,b)$, such that $(T_\alpha f)(c) = 0$.

Theorem 5 *Mean Value Theorem* (see [6]). *Let $a > 0$, $\alpha \in (0,1]$, and $f\colon [a,b] \longrightarrow R$ be a given function that satisfies*

(i) f is continuous on $[a,b]$.
(ii)- f is α differentiable on (a,b).

Then, there exists $c \in (a,b)$, such that

$$\left(T_\alpha f\right)(c) = \frac{f(b) - f(a)}{\left(b^\alpha/\alpha\right) - \left(a^\alpha/\alpha\right)}. \tag{7}$$

Theorem 6 (see [14]). *Let $a > 0$, $\alpha \in (0, 1]$, and $f: [a, b] \longrightarrow R$ be a given function that satisfies*

(i) f is continuous on $[a, b]$.

(ii)- f is α differentiable on (a, b).

If $(T_\alpha g)(t) = 0$ for all $t \in (a, b)$, then f is a constant on $[a, b]$.

Corollary 2 (see [14]). *Let $a > 0$, $\alpha \in (0, 1]$, and $F, G: [a, b] \longrightarrow R$ be functions such that $(T_\alpha F)(t) = (T_\alpha G)(t)$ for all $t \in (a, b)$. Then, there exists a constant C such that*

$$F(t) = G(t) + C. \tag{8}$$

Theorem 7 Extended Mean Value Theorem (see [14]). *Let $a > 0$, $\alpha \in (0, 1]$, and $f, g: [a, b] \longrightarrow R$ be functions that satisfy*

(i) f, g are continuous on $[a, b]$.

(ii) f, g are α differentiable on (a, b).

(iii) $(T_\alpha g)(t) \neq 0$ for all $t \in (a, b)$.

(iv) $g(b) \neq g(a)$.

(v) $(T_\alpha f)(t)$ and $(T_\alpha g)(t)$ not annulled simultaneously on $[a, b]$.

Then, there exists $c \in (a, b)$, such that

$$\frac{(T_\alpha f)(c)}{(T_\alpha g)(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{9}$$

Remark 1. Observe that Theorem 5 is a special case of this theorem for $g(t) = (t^\alpha/\alpha)$.

Theorem 8 (see [14]). *Let $a > 0$, $\alpha \in (0, 1]$, and $f: [a, b] \longrightarrow R$ be a given function that satisfies*

(i) f is continuous on $[a, b]$.

(ii) f is α-differentiable on (a, b).

Then, we have the following:

(i) If $(T_\alpha f)(t) > 0$ for all $t \in (a, b)$, then f is increasing on $[a, b]$.

(ii) If $(T_\alpha f)(t) < 0$ for all $t \in (a, b)$, then f is decreasing on $[a, b]$.

Theorem 9 (see [13]). *Assume f is infinitely α differentiable function, for some $0 < \alpha \leq 1$ at the neighborhood of a point t_0. Then, f has the following fractional power series expansion:*

$$f(t) = \sum_{k=0}^{\infty} \frac{\left({}^{(k)}T_\alpha^{t_0}\right)(t_0)}{\alpha^k k!}(t - t_0)^{k\alpha}, \quad t_0 < t < t_0 + R^{(1/\alpha)}. \tag{10}$$

Here, $({}^{(k)}T_\alpha^{t_0})(t_0)$ means the application of the conformable derivative k times.

Finally, the conformable partial derivative of a real-valued function with several variables is defined as follows.

Definition 3 (see [15, 16]). Let f be a real-valued function with n variables and $\mathbf{a} = (a_1, \ldots, a_n) \in R^n$ be a point whose i^{th} component is positive. Then, the limit can be expressed as follows

$$\lim_{\varepsilon \longrightarrow 0} \frac{f\left(a_1, \ldots, a_i + \varepsilon a_i^{1-\alpha}, \ldots, a_n\right) - f(a_1, \ldots, a_n)}{\varepsilon}, \tag{11}$$

if the above limit exists, then we have the i^{th} conformable partial derivative of f of the order $\alpha \in (0, 1]$ at $\mathbf{a}$, denoted by $(\partial^\alpha/\partial x_i^\alpha) f(\mathbf{a})$.

3. Main Results

From the definitions and results on conformable derivatives of higher order, the theorems of the mean value are easily generalized which follow the same argument as in the classical calculus [42].

Theorem 10. *Let $a > 0$, $\alpha \in (0, 1]$, and $f, g: [a, b] \longrightarrow R$ be functions that satisfy*

(i)- $f, g \in C^{(n-1)\alpha}([a, b])$.

(ii)- $({}^n T_\alpha f)(t)$ and $({}^n T_\alpha g)(t)$ exist for all t in $[a, b)$.

In addition, the following $n - 1$ equations are assumed:

$$\left({}^k T_\alpha f\right)(a)[g(b) - g(a)] = \left({}^k T_\alpha g\right)(a)[f(b) - f(a)], \quad \text{for } k = 1, 2, \ldots, n - 1. \tag{12}$$

Then, there exists $c \in (a, b)$, such that

$$({}^n T_\alpha f)(c)[g(b) - g(a)] = ({}^n T_\alpha g)(c)[f(b) - f(a)]. \tag{13}$$

Proof. Consider the following function:

$$F(t) = f(t)[g(b) - g(a)] - g(t)[f(b) - f(a)], \quad \forall t \in [a, b]. \tag{14}$$

Since F is continuous on $[a, b]$, α differentiable on (a, b), and $F(a) = F(b)$, then by Theorem 4, there exists $c_1 \in (a, b)$ such that

$$(T_\alpha f)(c_1)[g(b)-g(a)]-(T_\alpha g)(c_1)[f(b)-f(a)]=0. \tag{15}$$

Let us now consider the following function:

$$\begin{aligned}(T_\alpha F)(t) &= (T_\alpha f)(t)[g(b)-g(a)] \\ &\quad -(T_\alpha g)(t)[f(b)-f(a)], \quad \forall t \in [a,c_1],\end{aligned} \tag{16}$$

which is continuous on $[a,c_1]$, α differentiable on (a,c_1), and it is null at the extremes of interval $[a,c_1]$, by virtue of the above equation and hypothesis. Then, by Theorem 4, there exists $c_2 \in (a,c_1)$ such that

$$\left({}^2T_\alpha f\right)(c_2)[g(b)-g(a)]-\left({}^2T_\alpha g\right)(c_2)[f(b)-f(a)]=0. \tag{17}$$

So, we reiterate this process until we obtain the following equality:

$$\left({}^{n-1}T_\alpha f\right)(c_{n-1})[g(b)-g(a)]-\left({}^{n-1}T_\alpha h\right)(c_{n-1})[f(b)-f(a)]=0. \tag{18}$$

Then, we consider functions: ${}^{n-1}T_\alpha f$ and ${}^{n-1}T_\alpha g$, that are continuous on $[a,c_{n-1}]$, and α differentiable on (a,c_{n-1}). So, by Theorem 7, there exists $c \in (a,c_{n-1}) \subset (a,b)$ with

$$\left({}^nT_\alpha f\right)(c)[g(b)-g(a)]=\left({}^nT_\alpha g\right)(c)[f(b)-f(a)]. \tag{19}$$

This completes the proof of the theorem. □

Remark 2. The generalized conformable formula of extended mean value theorem is derived from previous theorem by taking $g(t)=(t^\alpha - a^\alpha)^n$.

Theorem 11. *Let $a_0>0$, $a \in (a_0,b)$ $\alpha \in (0,1]$, and $f\colon (a_0,b] \longrightarrow R$ be a function that satisfies*

(i) f is continuous on $[a,b]$.

(ii)- f is $n-1$ times α differentiable on (a,b).

(iii)- $({}^nT_\alpha f)(t)$ exist for all t in $[a,b)$.

In addition, the following $n-1$ equations are assumed:

$$(T_\alpha f)(a)=\left({}^2T_\alpha f\right)(a)=\cdots=\left({}^{n-1}T_\alpha f\right)(a)=0. \tag{20}$$

Then, there exists $c \in (a,b)$, such that

$$f(b)-f(a)=\frac{({}^nT_\alpha f)(c)}{\alpha^n \cdot n!}(t^\alpha - a^\alpha)^n. \tag{21}$$

Remark 3. A generalization of the conformable formula of mean value of Cauchy is also obtained.

Theorem 12. *Let $a_0>0$, $a \in (a_0,b)$, $\alpha \in (0,1]$, and $f,g\colon (a_0,b] \longrightarrow R$ be functions that satisfy*

(i) f,g are continuous on $[a,b]$.

(ii) f,g are $n-1$ times α differentiable on (a,b).

(iii) $({}^nT_\alpha f)(t)$ and $({}^nT_\alpha g)(t)$ exist for all t in $[a,b)$.

(iv)- $({}^nT_\alpha g)(t)\neq 0$ $\forall t \in (a,b)$.

In addition, the following $n-1$ equations are assumed:

$$\left({}^kT_\alpha f\right)(a)[g(b)-g(a)]=\left({}^kT_\alpha g\right)(a)[f(b)-f(a)], \quad \text{for } k=1,2,\ldots,n-1. \tag{22}$$

Then, there exists $c \in (a,b)$, such that

$$\frac{({}^nT_\alpha f)(c)}{({}^nT_\alpha g)(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}. \tag{23}$$

Proof. Using the formula (21) and the fact that $({}^nT_\alpha g)(t)\neq 0$ $\forall t \in (a,b)$, it follows that $g(b)-g(a)\neq 0$.

Dividing the two members of equality

$$\left({}^nT_\alpha f\right)(c)[g(b)-g(a)]=\left({}^nT_\alpha g\right)(c)[f(b)-f(a)], \tag{24}$$

by the product $({}^kT_\alpha g)(c)[g(b)-g(a)]$, the desired result is obtained.

We end this section by obtaining the value of conformable Taylor remainder through the generalized conformable theorems of the mean value. □

Definition 4. Let an open set $X \subset R$, $a \in X$, $\alpha \in (0,1]$, and $f\colon X \longrightarrow R$ be a function that satisfies

(i)- f is $n-1$ times α differentiable on a neighborhood of a point a.

(ii)- $({}^nT_\alpha f)(a)$ exists.

Then, the conformable Taylor remainder is defined by

$$R(t)=f(t)-p_n(x)=f(t)-\sum_{k=0}^{n}\left({}^kT_\alpha f\right)(a)\cdot\frac{(t^\alpha-a^\alpha)^k}{\alpha^k\cdot k!}, \quad \forall t \in X. \tag{25}$$

Theorem 13. *Let an open set $X \subset R$, $a \in X$, $\alpha \in (0,1]$, and $f\colon X \longrightarrow R$. If f is $n+1$ times α differentiable on $[a,t] \subset X$, Then, there exists $c \in (a,t)$, such that*

$$R(t)=\left({}^{n+1}T_\alpha f\right)(c)\cdot\frac{(t^\alpha-a^\alpha)^{n+1}}{\alpha^{n+1}\cdot(n+1)!}, \tag{26}$$

where R is called the conformable Lagrange form of the remainder.

Proof. By applying Theorem 12 to a function $R = f - p_n$ and using the fact that $({}^{n+1}T_\alpha R)(t) = ({}^{n+1}T_\alpha f)(t)$ and $({}^{k}T_\alpha R)(a) = 0$ for $k = 1, 2, \ldots, n$, our result is followed. □

4. Applications to Multivariable Calculus

In this section, we will introduce the conformable version of two interesting classical results on functions of several variables [42]. Using the conformable formula of finite increments [16], these results will be proven.

Theorem 14. *Let $\alpha \in (0, 1]$, $f: X \longrightarrow R$ be a real-valued function defined in an open and convex set $X \subset R^n$, such that for all $\mathbf{x} = (x_1, \ldots, x_n) \in X$, each $x_i > 0$. If the conformable partial derivative of f with respect to x_i exists and is null on X, then $f(\mathbf{x}) = f(x')$ for any points $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n), x' = (x_1, \ldots, x_i', \ldots, x_n) \in X$, for $i = 1, 2, \ldots, n$.*

Proof. Since x' is a convex set and $\mathbf{x} = (x_1, \ldots x_i, \ldots, x_n), x' = (x_1, \ldots, x_i', \ldots, x_n) \in X$, all points of the line segment $[\mathbf{x}, x']$ are also in X, so the function g is defined in the interval of endpoints x_i and x_i':

$$t \mapsto g(t) = f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n). \tag{27}$$

This function is α differentiable on the above interval, and its derivative at a point t is given by

$$(T_\alpha g)(t) = \frac{\partial^\alpha f(x_1, \ldots, t, \ldots, x_n)}{\partial x_i^\alpha}. \tag{28}$$

Therefore, by applying Theorem 5, there is a point c_i between x_i and x_i', such that

$$g(x_i') - g(x_i) = \left(\frac{x_i'^{\alpha}}{\alpha} - \frac{x_i^\alpha}{\alpha}\right) \cdot (T_\alpha g)(c_i). \tag{29}$$

Since point $\mathbf{c} = (x_1, \ldots, c_i, \ldots, x_n) \in X$; therefore, $(\partial^\alpha f(\mathbf{c})/\partial x_i^\alpha) = 0$, and the above equality leads to

$$f(x') - f(\mathbf{x}) = (x_i' - x_i) \cdot \frac{\partial^\alpha f(c)}{\partial x_i^\alpha}, \tag{30}$$

then $f(\mathbf{x}) = f(x')$, for $i = 1, 2, \ldots, n$, as we wanted to prove.

Finally, we introduce the first formula of finite increments for functions of several variables, involving conformable partial derivatives. □

Theorem 15. *Let $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in R^n$, $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\mathbf{n}$ be points $\mathbf{x}_\mathbf{i} = (b_1, \ldots, b_i, a_{i+1}, \ldots, a_n)$ (note that $\mathbf{x}_0 = \mathbf{a}$ and $\mathbf{x}_\mathbf{n} = \mathbf{b}$), and line segment $S_i = [\mathbf{x}_{\mathbf{i}-1}, \mathbf{x}_\mathbf{i}]$, for $i = 1, 2, \ldots, n$. Let $\alpha \in (0, 1]$, and $f: X \longrightarrow R$ be a real-valued function defined in an open set $X \subset R^n$ containing line segments $S_1, S_2, \ldots, S_n$, such that for all $\mathbf{x} = (x_1, \ldots, x_n) \in X$, each $x_i > 0$. If the conformable partial derivative of f with respect to x_i exists on X, then there is a point c_i between a_i and b_i, for $i = 1, 2, \ldots, n$, such that*

$$f(b_1, b_2, \ldots b_n) - f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} \left(\frac{b_i^\alpha}{\alpha} - \frac{a_i^\alpha}{\alpha}\right) \cdot \frac{\partial^\alpha f(b_1, \ldots, b_{i-1}, c_i, a_{i+1} \ldots, a_n)}{\partial x_i^\alpha}. \tag{31}$$

Proof. First, we will express the difference $f(\mathbf{b}) - f(\mathbf{a})$ as follows:

$$f(\mathbf{b}) - f(\mathbf{a}) = f(\mathbf{x}_\mathbf{n}) - f(\mathbf{x}_0) = \sum_{i=1}^{n} [f(\mathbf{x}_\mathbf{i}) - f(\mathbf{x}_{\mathbf{i}-1})]. \tag{32}$$

Let us now consider, for $i = 1, 2, \ldots, n$, the real function g_i of the real variable t, defined on the closed interval of endpoints a_i and b_i, by

$$t \longmapsto g_i(t) = f(b_1, \ldots, b_{i-1}, t, a_{i+1}, \ldots, a_n). \tag{33}$$

Since the conformable partial derivative of f with respect to x_i exists on X and $S_i \subset X$, then g_i is α differentiable on the above interval, and its derivative at a point t, is given by

$$(T_\alpha g_i)(t) = \frac{\partial^\alpha f(b_1, \ldots, b_{i-1}, t, a_{i+1}, \ldots, a_n)}{\partial x_i^\alpha}. \tag{34}$$

Therefore, by applying Theorem 5, there is a point c_i between a_i and b_i, such that

$$g_i(b_i) - g_i(a_i) = \left(\frac{b_i^\alpha}{\alpha} - \frac{a_i^\alpha}{\alpha}\right) \cdot (T_\alpha g_i)(c_i). \tag{35}$$

Then, it is verified that

$$f(\mathbf{x}_\mathbf{i}) - f(\mathbf{x}_{\mathbf{i}-1}) = \left(\frac{b_i^\alpha}{\alpha} - \frac{a_i^\alpha}{\alpha}\right) \cdot \frac{\partial^\alpha f(b_1, \ldots, b_{i-1}, t, a_{i+1}, \ldots, a_n)}{\partial x_i^\alpha}. \tag{36}$$

By taking the above expression to equation (32), our result is followed. □

5. Conclusion

In this research work, some new results regarding the conformable mean value theorems have been proposed. As in classical calculus, higher-order derivatives have been applied to generalize the mean value theorems. Likewise, the Lagrange expression has been established for the Taylor conformable remainder. In the context of the calculus of functions of several variables, according to the conformable mean value theorem, the functions in which one of its conformable partial derivatives is null have been characterized, and the first conformable formula of finite increments has been obtained. The findings of this investigation indicate that the results obtained in the sense of the conformable derivative coincide with the results obtained in the classical case of integer order. Finally, our obtained results, in addition to a theoretical interest, show great potential to be applied in a future research work concerning various applications in the field of natural sciences and engineering.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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