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Research Article

A Diophantine Problem with Unlike Powers of Primes

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Let k be an integer with $4 \le k \le 6$ and η be any real number. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_5$ are nonzero real numbers, not all of them have the same sign, and λ_1/λ_2 is irrational. It is proved that the inequality $|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^k + \eta| < (\max_{1\le j\le 5} p_j)^{-\sigma(k)}$ has infinitely many solutions in prime variables p_1, p_2, p_3, p_4, and p_5, where $0 < \sigma(4) < 1/36, 0 < \sigma(5) < 4/189$, and $0 < \sigma(6) < 1/54$. This gives an improvement of the recent results.

1. Introduction

The determination of the minimal s such that the Diophantine equation

$$N = \sum_{i=1}^{s} x_i^{i+1}, \tag{1}$$

is solvable in positive integers $x_1, \ldots, x_s$, for all sufficiently large integers N is an interesting problem in additive number theory. In 1951, Roth [1] proved that $s = 50$ is acceptable. This result was subsequently improved by Thanigasalam et al. [2–4], Vaughan and Vaughan [5, 6], Brüdern and Brüdern [7, 8] and Ford and Ford [9, 10]. The best currently known result is due to Ford [10], with $s = 14$. Schwarz [11] suggested to analyze the related Diophantine inequality. The first result was obtained by Brüdern [12], who showed that the values of

$$\sum_{i=1}^{22} \lambda_{i+1} x_i^{i+1}, \tag{2}$$

at integer points $(x_1, \ldots, x_{22})$ are dense on the real line provided that $\lambda_2, \ldots, \lambda_{23}$ are nonzero real numbers and λ_2/λ_3 is irrational. Thanks to a pruning technique, Brüdern [13] proved that the values taken by

$$\sum_{i=1}^{16} \lambda_{i+1} x_i^{i+1}, \tag{3}$$

at integer points $(x_1, \ldots, x_{16})$ are dense on the real line if $\lambda_2, \ldots, \lambda_{17}$ are nonzero real numbers and at least one of the ratios λ_i/λ_j is irrational.

Suppose that $x_1, \ldots, x_s$ are prime variables, N is a sufficiently large integer, and $N + s$ is even. In 1969, Vaughan proved in his doctoral thesis that (1) is solvable if $s = 31$. Later, Vaughan [5] improved upon his own result by taking $s = 30$ in place of $s = 31$. By calculating, the exponential density more accurately, Shan [14] showed that $s = 23$ is acceptable. In addition, Prachar [15] established that each sufficiently large odd integer N can be represented as

$$N = p_1 + p_2^2 + p_3^3 + p_4^4 + p_5^5, \tag{4}$$

where $p_1, p_2, \ldots, p_5$ are prime numbers. As a corollary of [16] in Theorem 1, Ren and Tsang obtained the same result as Prachar. It is of some interest to consider the analogous form for Diophantine inequalities. Let $\lambda_1, \lambda_2, \ldots, \lambda_5$ be nonzero real numbers, not all of them have the same sign and λ_1/λ_2 as irrational. In 2016, Ge and Li [17] proved that, for any given real numbers η and $\sigma, 0 < \sigma < 1/720$, there exist infinitely many solutions in prime numbers p_j to the inequality

$$\left|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta\right| < \left(\max_{1\le j\le 5} p_j^j\right)^{-\sigma}. \tag{5}$$

Let $k \ge 4$ be an integer. The first author [18] investigated the solvability of more general Diophantine inequality

$$\left|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^k + \eta\right| < \left(\max_{1\le j\le 5} p_j\right)^{-\sigma(k)}, \tag{6}$$

and proved that (6) has infinitely many solutions in prime variables p_j for $0 < \sigma(4) < 5/288$ and $0 < \sigma(k) < 5/(6k^2(k+1))$ with $k \ge 5$. Subsequently, Liu [19] obtained $0 < \sigma(5) < 5/288$. In [20], the first author and Qu showed that $0 < \sigma(5) < 5/252$ is acceptable. Very recently, this result was improved by Zhu [21], who obtained $0 < \sigma(5) < 1/48$. In [22], Gao and Liu gave an improvement ([18] in Theorem 1.2) in case $k \ge 6$, and they proved $0 < \sigma(6) < 1/56$ particularly.

The main purpose of this paper is to sharpen the above results in case $4 \le k \le 6$. We obtain the following theorem.

Theorem 1. *Let k be an integer with $4 \le k \le 6$ and η be any given real number. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_5$ are nonzero real numbers, not all of them are same sign, and λ_1/λ_2 is irrational. Then, inequality (6) has infinitely many solutions in prime variables $p_1, p_2, \ldots, p_5$, where $0 < \sigma(4) < 1/36$, $0 < \sigma(5) < 4/189$, and $0 < \sigma(6) < 1/54$.*

The improvement derives not only from the use of the function $\rho(m)$ constructed by Harman and Kumchev (see Section 8 in [23] and Section 5 in [24], for details) but also from some ingredients in [21]. It is worth remarking that Ge et al. [25] obtained $0 < \sigma(5) < 1/32$, if the condition "λ_1/λ_2 is irrational" in Theorem 1 is replaced by "λ_1/λ_2 is irrational and λ_2/λ_4 and λ_3/λ_5 are rational."

Notation. Throughout the paper, ε and δ are arbitrarily small, fixed positive real numbers. Any statement in which ε occurs holds for each positive ε. The implicit constants in O-term, $\ll$- and $\gg$-symbols depend at most on $\lambda_1, \lambda_2, \ldots, \lambda_5$ and ε. The letter p, with or without subscript, is reserved for a prime number. By $A \asymp B$, we mean that $A \ll B$ and $A \gg B$. For simplicity, we write $\mathscr{L} = \log X$ and $e(\alpha) = \exp(2\pi i\alpha)$.

2. Preliminaries

We apply the Davenport–Heilbronn circle method (see [26] and Chapter 11 in [27]) to prove Theorem 1. Since λ_1/λ_2 is irrational, there are infinitely many convergents to its continued fraction. Let q be any denominator of a convergent to λ_1/λ_2. As in [20], let X run through the sequences:

$$X = q^{(7/3)}. \tag{7}$$

We set

$$\begin{aligned} \mathscr{I} &= \left[\frac{1}{3}X^{(1/2)}, \frac{2}{3}X^{(1/2)}\right), \\ S_2^*(\alpha) &= \sum_{m\in\mathscr{I}} \rho(m) e\left(\alpha m^2\right), \end{aligned} \tag{8}$$

where the function $\rho(m)$ is defined by 5.2 in [24]. According to [24], $\rho(m)$ is a nontrivial lower bound for the characteristic function of the set of primes in $\mathscr{I}$, and it satisfies

$$\rho(m) \le \begin{cases} 1, & \text{if } m \text{ is a prime}, \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

For further properties of $\rho(m)$, see Lemma 1 and (4.2)–(4.4) in [24]. Let

$$\begin{aligned} I_j &= \left[(\delta X)^{(1/j)}, X^{(1/j)}\right], \\ S_j(\alpha) &= \sum_{p\in I_j} (\log p) e\left(\alpha p^j\right). \end{aligned} \tag{10}$$

By the prime number theorem, it is easy to show that $S_j(\alpha) \ll X^{(1/j)}$. For any fixed $\tau > 0$, set $K_\tau(\alpha) = (\pi\alpha)^{-2}\sin^2(\pi\tau\alpha)$ for $\alpha \ne 0$ and $K_\tau(0) = \tau^2$. Clearly, we have

$$K_\tau(\alpha) \ll \min\left(\left(\tau^2, |\alpha|^{-2}\right)\right). \tag{11}$$

A straightforward application of the Cauchy integral formula gives

$$\int_{-\infty}^{+\infty} e(x\alpha) K_\tau(\alpha) \mathrm{d}\alpha = \max(0, \tau - |x|). \tag{12}$$

Identity (12) is also a corollary of Lemma 4 in [26]. For $4 \le k \le 6$, put

$$G(\alpha) = S_1(\lambda_1\alpha) S_2^*(\lambda_2\alpha)\left(\prod_{j=3}^{4} S_j\left(\lambda_j\alpha\right)\right) S_k(\lambda_5\alpha) e(\alpha\eta) K_\tau(\alpha). \tag{13}$$

We write

$$I(\tau, \eta, \mathfrak{X}) = \int_{\mathfrak{X}} G(\alpha) \mathrm{d}\alpha, \tag{14}$$

for any measurable subset $\mathfrak{X}$ of $\mathbb{R}$. It follows from (9) and (12) that

$$\begin{aligned} I(\tau, \eta, \mathbb{R}) &= \sum_{\substack{p_j\in I_j, j=1,3,4, \\ p_5\in I_k, m_2\in\mathscr{I}}} \rho(m_2) \prod_{\substack{1\le j\le 5 \\ j\ne 2}} \log p_j \times \int_{-\infty}^{+\infty} e\left(\left(\lambda_1 p_1 + \lambda_2 m_2^2 + \sum_{j=3}^{4}\lambda_j p_j^j + \lambda_5 p_5^k + \eta\right)\alpha\right) K_\tau(\alpha)\mathrm{d}\alpha \\ &= \sum_{\substack{p_j\in I_j, j=1,3,4, \\ p_5\in I_k, m_2\in\mathscr{I}}} \rho(m_2) \prod_{\substack{1\le j\le 5 \\ j\ne 2}} \log p_j \times \max\left(0, \tau - \left|\lambda_1 p_1 + \lambda_2 m_2^2 + \sum_{j=3}^{4}\lambda_j p_j^j + \lambda_5 p_5^k + \eta\right|\right) \\ &\le \tau\mathscr{L}^4 \mathscr{N}(X), \end{aligned} \tag{15}$$

where $\mathcal{N}(X)$ denotes the number of solutions of the inequality

$$\left|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^k + \eta\right| < \tau, \tag{16}$$

with $p_2 \in \mathcal{I}$, $p_5 \in I_k$, and $p_j \in I_j$ for $j \in \{1, 3, 4\}$. In what follows, we take

$$\tau = \begin{cases} X^{-(1/36)+20\varepsilon} & \text{if } k = 4, \\ X^{-(4/189)+20\varepsilon} & \text{if } k = 5, \\ X^{-(1/54)+20\varepsilon} & \text{if } k = 6, \end{cases} \tag{17}$$

actually. We now divide the real line into three disjoint parts:

$$\begin{aligned} \mathfrak{M} &= \left\{\alpha\colon |\alpha| \le X^{-(1/8)}\right\}, \\ \mathfrak{m} &= \left\{\alpha\colon X^{-(1/8)} < |\alpha| \le \xi\right\}, \\ \mathfrak{t} &= \{\alpha\colon |\alpha| > \xi\}, \end{aligned} \tag{18}$$

where $\xi = \tau^{-2} X^{(1/16)-(1/4k)+10\varepsilon}$. These sets are called the major arc, the minor arcs, and the trivial regions, respectively.

In the following sections, we shall prove that the dominant contribution to $I(\tau, \eta, \mathbb{R})$ is from the major arc, and the contribution from the minor arcs and the trivial region can be neglected.

3. The Major Arc

Our first goal is to show that

$$|I(\tau, \eta, \mathfrak{M})| \gg \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-1}. \tag{19}$$

The proof of (19) is quite similar to that given in Section 3 in [20]. For completeness of exposition, we briefly present the proof procedure below.

Let

$$\begin{aligned} \mathfrak{M}_1 &= \left\{\alpha\colon |\alpha| \le X^{-1+(5/12k)-\varepsilon}\right\}, \\ \mathfrak{M}_2 &= \left\{\alpha\colon X^{-1+(5/12k)-\varepsilon} < |\alpha| \le X^{-(7/8)}\right\}, \\ \mathfrak{M}_3 &= \left\{\alpha\colon X^{-(7/8)} < |\alpha| \le X^{-(1/8)}\right\}. \end{aligned} \tag{20}$$

Then, we have $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2 \cup \mathfrak{M}_3$ and

$$I(\tau, \eta, \mathfrak{M}) = I(\tau, \eta, \mathfrak{M}_1) + I(\tau, \eta, \mathfrak{M}_2) + I(\tau, \eta, \mathfrak{M}_3). \tag{21}$$

By a similar argument as that in pp. 1656–1657 in [20], we can obtain

$$\left|I(\tau, \eta, \mathfrak{M}_1)\right| \gg \tau^2 X^{(13/12)+(1/k)} \mathcal{L}^{-1}. \tag{22}$$

To estimate the integrals $I(\tau, \eta, \mathfrak{M}_2)$ and $I(\tau, \eta, \mathfrak{M}_3)$, we need the following two lemmas.

Lemma 1. *Let $j \ge 2$ be an integer. Then, for nonzero real number λ and any $\varepsilon > 0$, we have*

$$S_j(\lambda\alpha) \ll \begin{cases} X^{(1/j)\left(1-j\cdot 4^{1-j}\right)+\varepsilon} |\alpha|^{-4^{1-j}} & \text{for } X^{-1} \le |\alpha| \le X^{-1+(1/2j)}, \\ X^{\frac{1}{j}\left(1-(1/2)\cdot 4^{1-j}\right)+\varepsilon} & \text{for } X^{-1+(1/2j)} < |\alpha| \le X^{-(1/2j)}. \end{cases} \tag{23}$$

Proof. It follows from Theorem 1 in [28]. □

Lemma 2. *For $4 \le k \le 6$, suppose that*

$$\begin{aligned} F(\alpha) \in \Big\{ & S_1^2(\lambda_1\alpha), S_3^8(\lambda_3\alpha), \left(S_2^*(\lambda_2\alpha)\right)^2 S_3^4(\lambda_3\alpha), \\ & \left(S_2^*(\lambda_2\alpha)\right)^2 S_4^4(\lambda_4\alpha), \\ & \left(S_2^*(\lambda_2\alpha)\right)^2 S_5^6(\lambda_5\alpha), \left(S_2^*(\lambda_2\alpha)\right)^2 S_6^8(\lambda_5\alpha), \\ & \left(S_2^*(\lambda_2\alpha) S_3(\lambda_3\alpha) S_k(\lambda_5\alpha)\right)^2, \left(S_2^*(\lambda_2\alpha)\right)^2 S_4^2(\lambda_4\alpha) S_5^4(\lambda_5\alpha) \Big\}. \end{aligned} \tag{24}$$

Then, we have

$$\begin{aligned} \int_{-1}^{1} |F(\alpha)| d\alpha &\ll X^{-1} F(0)^{1+\varepsilon}, \\ \int_{\mathbb{R}} |F(\alpha)| K_\tau(\alpha) d\alpha &\ll \tau X^{-1} F(0)^{1+\varepsilon}. \end{aligned} \tag{25}$$

Proof. See Lemma 3.7 in [20]. □

When $\alpha \in \mathfrak{M}_2$, it follows from (23) that

$$S_4(\lambda_4\alpha) \ll X^{(15/64)+\varepsilon} |\alpha|^{-(1/64)} \ll X^{(1/4)-(5/768k)+2\varepsilon}. \tag{26}$$

Combining this with the Cauchy–Schwarz inequality and Lemma 2 gives

$$\begin{aligned}
\left|I(\tau,\eta,\mathfrak{M}_2)\right| &\ll \tau^2 \sup_{\alpha\in\mathfrak{M}_2}\left|S_4(\lambda_4\alpha)\right|\int_{\mathfrak{M}_2}\left|S_1(\lambda_1\alpha)S_2^*(\lambda_2\alpha)S_3(\lambda_3\alpha)S_k(\lambda_5\alpha)\right|\mathrm{d}\alpha \\
&\ll \tau^2 \sup_{\alpha\in\mathfrak{M}_2}\left|S_4(\lambda_4\alpha)\right|\left(\int_{-1}^{1}\left|S_1(\lambda_1\alpha)\right|^2\mathrm{d}\alpha\right)^{(1/2)} \\
&\quad\times\left(\int_{-1}^{1}\left|S_2^*(\lambda_2\alpha)S_3(\lambda_3\alpha)S_k(\lambda_5\alpha)\right|^2 d\alpha\right)^{(1/2)} \\
&\ll \tau^2 X^{(1/4)-(5/768k)+2\varepsilon}\cdot\left(X^{1+2\varepsilon}\right)^{(1/2)}\cdot\left(X^{(2/3)+(2/k)+\varepsilon}\right)^{(1/2)} \\
&\ll \tau^2 X^{(13/12)+(1/k)-\varepsilon},
\end{aligned} \tag{27}$$

where (11) is used.

When $\alpha\in\mathfrak{M}_3$, (23) implies

$$S_4(\lambda_4\alpha)\ll X^{(1/4)-(1/512)+\varepsilon}. \tag{28}$$

Proceeding as in the proof of (27), we have

$$\left|I(\tau,\eta,\mathfrak{M}_3)\right|\ll\tau^2 X^{(13/12)+(1/k)-\varepsilon}. \tag{29}$$

This with (27), (22), and (21) yields (19).

4. The Minor Arcs

The next thing to do in the proof is to establish that

$$\left|I(\tau,\eta,\mathfrak{m})\right|\ll\tau^2 X^{(13/12)+(1/k)}\mathscr{L}^{-2}. \tag{30}$$

This work forms the bulk of the present paper. We subdivide $\mathfrak{m}$ into four disjoint parts: $\mathfrak{m}=\mathfrak{m}_1\cup\mathfrak{m}_2\cup\mathfrak{m}_3\cup\mathfrak{m}_4$, where

$$\begin{aligned}
\mathfrak{m}_1 &= \left\{\alpha\in\mathfrak{m}\colon \left|S_1(\lambda_1\alpha)\right|\le X^{(6/7)+2\varepsilon}\right\}, \\
\mathfrak{m}_2 &= \left\{\alpha\in\mathfrak{m}\colon \left|S_1(\lambda_1\alpha)\right|> X^{(6/7)+2\varepsilon}, \left|S_2^*(\lambda_2\alpha)\right|\le X^{(3/7)+2\varepsilon}, \left|S_3(\lambda_3\alpha)\right|> X^{(11/36)+2\varepsilon}\right\}, \\
\mathfrak{m}_3 &= \left\{\alpha\in\mathfrak{m}\colon \left|S_1(\lambda_1\alpha)\right|> X^{(6/7)+2\varepsilon}, \left|S_2^*(\lambda_2\alpha)\right|\le X^{(3/7)+2\varepsilon}, \left|S_3(\lambda_3\alpha)\right|\le X^{(11/36)+2\varepsilon}\right\}, \\
\mathfrak{m}_4 &= \left\{\alpha\in\mathfrak{m}\colon \left|S_1(\lambda_1\alpha)\right|> X^{(6/7)+2\varepsilon}, \left|S_2^*(\lambda_2\alpha)\right|> X^{(3/7)+2\varepsilon}\right\}.
\end{aligned} \tag{31}$$

Therefore,

$$I(\tau,\eta,\mathfrak{m})=\sum_{j=1}^{4}I\left(\tau,\eta,\mathfrak{m}_j\right). \tag{32}$$

To prove (30), it suffices to show that $\left|I(\tau,\eta,\mathfrak{m}_j)\right|\ll\tau^2X^{(13/12)+(1/k)}\mathscr{L}^{-2}$ holds for $1\le j\le 4$.

We apply Hölder's inequality and Lemma 2 to estimate $\left|I(\tau,\eta,\mathfrak{m}_1)\right|$. When $k=4$, we have

$$\begin{aligned}
&\left|I(\tau,\eta,\mathfrak{m}_1)\right| \\
&\quad\ll\left(\sup_{\alpha\in\mathfrak{m}_1}\left|S_1(\lambda_1\alpha)\right|\right)^{(1/4)}\left(\int_{\mathbb{R}}\left|S_1(\lambda_1\alpha)\right|^2K_\tau(\alpha)\mathrm{d}\alpha\right)^{(3/8)} \\
&\quad\times\left(\int_{\mathbb{R}}\left|S_3(\lambda_3\alpha)\right|^8K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)}\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_4(\lambda_4\alpha)\right|^4K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)} \\
&\quad\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_k(\lambda_5\alpha)\right|^4K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)} \\
&\ll\left(X^{(6/7)+2\varepsilon}\right)^{(1/4)}\left(\tau X^{1+\varepsilon}\right)^{(3/8)}\left(\tau X^{(5/3)+\varepsilon}\right)^{(1/8)}\left(\tau X^{1+\varepsilon}\right)^{(1/4)}\left(\tau X^{(4/k)+\varepsilon}\right)^{(1/4)} \\
&\ll\tau X^{(13/12)+(1/k)-(1/28)+2\varepsilon}.
\end{aligned} \tag{33}$$

If $k=5$, then

$$\begin{aligned}
&\left|I(\tau,\eta,\mathfrak{m}_1)\right|\\
&\quad\ll\left(\sup_{\alpha\in\mathfrak{m}_1}\left|S_1(\lambda_1\alpha)\right|\right)^{(3/16)}\left(\int_{\mathbb{R}}\left|S_1(\lambda_1\alpha)\right|^2K_\tau(\alpha)\mathrm{d}\alpha\right)^{(13/32)}\\
&\quad\times\left(\int_{\mathbb{R}}\left|S_3(\lambda_3\alpha)\right|^8K_\tau(\alpha)\mathrm{d}\alpha\right)^{(3/32)}\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_4(\lambda_4\alpha)\right|^4K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)}\\
&\quad\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_k(\lambda_5\alpha)\right|^6K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)}\\
&\quad\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)S_3(\lambda_3\alpha)S_k(\lambda_5\alpha)\right|^2K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)}\\
&\ll\left(X^{(6/7)+2\varepsilon}\right)^{(3/16)}\left(\tau X^{1+\varepsilon}\right)^{(13/32)}\left(\tau X^{(5/3)+\varepsilon}\right)^{(3/32)}\left(\tau X^{1+\varepsilon}\right)^{(1/4)}\left(\tau X^{(6/k)+\varepsilon}\right)^{(1/8)}\left(\tau X^{(2/3)+(2/k)+\varepsilon}\right)^{(1/8)}\\
&\ll\tau X^{(13/12)+(1/k)-(3/112)+2\varepsilon}.
\end{aligned}\tag{34}$$

In case $k=6$, we obtain

$$\begin{aligned}
&\left|I(\tau,\eta,\mathfrak{m}_1)\right|\\
&\quad\ll\left(\sup_{\alpha\in\mathfrak{m}_1}\left|S_1(\lambda_1\alpha)\right|\right)^{(1/6)}\left(\int_{\mathbb{R}}\left|S_1(\lambda_1\alpha)\right|^2K_\tau(\alpha)\mathrm{d}\alpha\right)^{(5/12)}\\
&\quad\times\left(\int_{\mathbb{R}}\left|S_3(\lambda_3\alpha)\right|^8K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/12)}\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_4(\lambda_4\alpha)\right|^4K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)}\\
&\quad\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_k(\lambda_5\alpha)\right|^8K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/12)}\\
&\quad\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)S_3(\lambda_3\alpha)S_k(\lambda_5\alpha)\right|^2K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/6)}\\
&\ll\left(X^{(6/7)+2\varepsilon}\right)^{(1/6)}\left(\tau X^{1+\varepsilon}\right)^{(5/12)}\left(\tau X^{(5/3)+\varepsilon}\right)^{(1/12)}\left(\tau X^{1+\varepsilon}\right)^{(1/4)}\left(\tau X^{(8/k)+\varepsilon}\right)^{(1/12)}\left(\tau X^{(2/3)+(2/k)+\varepsilon}\right)^{(1/6)}\\
&\ll\tau X^{(13/12)+(1/k)-(1/42)+2\varepsilon}.
\end{aligned}\tag{35}$$

It follows from (33)–(35) and (17) that

$$\left|I(\tau,\eta,\mathfrak{m}_1)\right|\ll\tau^2X^{(13/12)+(1/k)-\varepsilon}\ll\tau^2X^{(13/12)+(1/k)}\mathscr{L}^{-2}.\tag{36}$$

In order to establish an upper bound for $|I(\tau,\eta,\mathfrak{m}_2)|$ as small as possible, we need the following lemma.

Lemma 3 (Lemma 3.4 in [21]). *Let*

$$\mathfrak{N}=\left\{\alpha\colon X^{(11/36)+2\varepsilon}<\left|S_3(\lambda_3\alpha)\right|\leq X^{(1/3)}\right\}.\tag{37}$$

Then, we have

$$\int_{\mathfrak{N}}\left|S_3(\lambda_3\alpha)\right|^2\left|S_4(\lambda_4\alpha)\right|^2K_\tau(\alpha)\mathrm{d}\alpha\ll\tau X^{(1/6)+4\varepsilon}.\tag{38}$$

For $4\leq k\leq 6$, by the Cauchy–Schwarz inequality, Lemmas 2 and 3, we obtain

$$\begin{aligned}
\left|I(\tau,\eta,\mathfrak{m}_2)\right|&\ll X^{(1/k)}\left(\sup_{\alpha\in\mathfrak{m}_2}\left|S_2^*(\lambda_2\alpha)\right|\right)\left(\int_{\mathfrak{N}}\left|S_3(\lambda_3\alpha)\right|^2\left|S_4(\lambda_4\alpha)\right|^2K_\tau(\alpha)d\alpha\right)^{(1/2)}\\
&\quad\times\left(\int_{\mathbb{R}}\left|S_1(\lambda_1\alpha)\right|^2K_\tau(\alpha)d\alpha\right)^{(1/2)}\\
&\ll\tau X^{(1/k)+(3/7)+(1/12)+(1/2)+4\varepsilon}\\
&\ll\tau X^{(13/12)+(1/k)-(1/14)+5\varepsilon},
\end{aligned}\tag{39}$$

where the trivial upper bound $S_k(\lambda_5\alpha) \ll X^{(1/k)}$ is used. It is easily derived from (17) that

$$\left|I(\tau,\eta,\mathfrak{m}_2)\right| \ll \tau^2 X^{(13/12)+(1/k)-\varepsilon} \ll \tau^2 X^{(13/12)+(1/k)}\mathscr{L}^{-2}. \tag{40}$$

The upper bound estimation of $|I(\tau,\eta,\mathfrak{m}_3)|$ plays a crucial role in the proof. The parameter τ, which is given by (17), is determined in this step. When $k = 4$, by Hölder's inequality and Lemma 2, we have

$$\begin{aligned}\left|I(\tau,\eta,\mathfrak{m}_3)\right| &\ll \left(\sup_{\alpha\in\mathfrak{m}_3}\left|S_3(\lambda_3\alpha)\right|\right)\left(\int_{\mathbb{R}}\left|S_1(\lambda_1\alpha)\right|^2 K_\tau(\alpha)d\alpha\right)^{(1/2)}\\ &\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right||^2\left|S_4(\lambda_4\alpha)\right|^4 K_\tau(\alpha)d\alpha\right)^{(1/4)}\\ &\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_k(\lambda_5\alpha)\right|^4 K_\tau(\alpha)d\alpha\right)^{(1/4)}\\ &\ll \left(X^{(11/36)+2\varepsilon}\right)\left(\tau X^{1+\varepsilon}\right)^{(1/2)}\left(\tau X^{1+\varepsilon}\right)^{(1/4)}\left(\tau X^{(4/k)+\varepsilon}\right)^{(1/4)}\\ &\ll \tau X^{(13/12)+(1/k)-(1/36)+4\varepsilon}.\end{aligned} \tag{41}$$

In the case of $k = 5$, we obtain

$$\begin{aligned}\left|I(\tau,\eta,\mathfrak{m}_3)\right| &\ll \left(\sup_{\alpha\in\mathfrak{m}_3}\left|S_2^*(\lambda_2\alpha)\right|\right)^{(1/6)}\left(\sup_{\alpha\in\mathfrak{m}_3}\left|S_3(\lambda_3\alpha)\right|\right)^{(1/3)}\\ &\times\left(\int_{\mathbb{R}}\left|S_1(\lambda_1\alpha)\right|^2 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/2)}\left(\int_{\mathbb{R}}\left|S_3(\lambda_3\alpha)\right|^8 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/12)}\\ &\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_4(\lambda_4\alpha)\right|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)}\\ &\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_k(\lambda_5\alpha)\right|^6 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/6)}\\ &\ll \left(X^{(3/7)+2\varepsilon}\right)^{(1/6)}\left(X^{(11/36)+2\varepsilon}\right)^{(1/3)}\left(\tau X^{1+\varepsilon}\right)^{(1/2)}\left(\tau X^{(5/3)+\varepsilon}\right)^{(1/12)}\\ &\times\left(\tau X^{1+\varepsilon}\right)^{(1/4)}\left(\tau X^{(6/k)+\varepsilon}\right)^{(1/6)}\\ &\ll \tau X^{(13/12)+(1/k)-(4/189)+4\varepsilon}.\end{aligned} \tag{42}$$

If $k = 6$, we deduce that

$$\begin{aligned}\left|I(\tau,\eta,\mathfrak{m}_3)\right| &\ll \left(\sup_{\alpha\in\mathfrak{m}_3}\left|S_3(\lambda_3\alpha)\right|\right)^{(2/3)}\left(\int_{\mathbb{R}}\left|S_1(\lambda_1\alpha)\right|^2 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/2)}\\ &\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_4(\lambda_4\alpha)\right|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)}\\ &\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)\right|^2\left|S_k(\lambda_5\alpha)\right|^8 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/12)}\\ &\times\left(\int_{\mathbb{R}}\left|S_2^*(\lambda_2\alpha)S_3(\lambda_3\alpha)S_k(\lambda_5\alpha)\right|^2 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/6)}\\ &\ll \left(X^{(11/36)+2\varepsilon}\right)^{(2/3)}\left(\tau X^{1+\varepsilon}\right)^{(1/2)}\left(\tau X^{1+\varepsilon}\right)^{(1/4)}\left(\tau X^{(8/k)+\varepsilon}\right)^{(1/12)}\left(\tau X^{(2/3)+(2/k)+\varepsilon}\right)^{(1/6)}\\ &\ll \tau X^{\frac{13}{12}+\frac{1}{k}-\frac{1}{54}+4\varepsilon}.\end{aligned} \tag{43}$$

Inequalities (41)–(43) and (17) together give

$$\left|I(\tau,\eta,\mathfrak{m}_3)\right| \ll \tau^2 X^{(13/12)+(1/k)-\varepsilon} \ll \tau^2 X^{(13/12)+(1/k)}\mathscr{L}^{-2}. \tag{44}$$

In the remainder of this section, we shall be trying to estimate $|I(\tau,\eta,\mathfrak{m}_4)|$. By a familiar dyadic dissection argument, we divide $\mathfrak{m}_4$ into at most $\ll\mathscr{L}^3$ disjoint sets $E(Z_1,Z_2,y)$. For $\alpha\in E(Z_1,Z_2,y)$, we have

$$Z_1 < \left|S_1(\lambda_1\alpha)\right| \le 2Z_1,\ Z_2 < \left|S_2^*(\lambda_2\alpha)\right| \le 2Z_2,\ y < |\alpha| \le 2y, \tag{45}$$

where $Z_1 = 2^{k_1}X^{(6/7)+2\varepsilon}$, $Z_2 = 2^{k_2}X^{(3/7)+2\varepsilon}$, and $y = 2^{k_3}X^{-(1/8)}$ for some nonnegative integers k_1, k_2, and k_3. For the sake of convenience, we take the notation $\mathscr{A}$ as a shortcut for $E(Z_1,Z_2,y)$, and let $m(\mathscr{A})$ stand for the Lebesgue measure of $\mathscr{A}$.

Lemma 4 (Lemma 4.3 in [20]). *We have*

$$m(\mathscr{A}) \ll yX^{(18/7)+9\varepsilon}Z_1^{-2}Z_2^{-4}. \tag{46}$$

When $k = 4$, it follows from (11) and Hölder's inequality that

$$
\begin{aligned}
&|I(\tau,\eta,\mathscr{A})| \\
&\quad \ll \left(\int_{\mathscr{A}} |S_1(\lambda_1\alpha)|^2 |S_2^*(\lambda_2\alpha)|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/12)} \left(\int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(5/12)} \\
&\quad \times \left(\int_{\mathbb{R}} |S_3(\lambda_3\alpha)|^8 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)} \left(\int_{\mathbb{R}} |S_4(\lambda_4\alpha)|^{16} K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/48)} \\
&\quad \times \left(\int_{\mathbb{R}} |S_k(\lambda_5\alpha)|^{16} K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/48)} \left(\int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_4(\lambda_4\alpha)|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/6)} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_k(\lambda_5\alpha)|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/6)} \\
&\ll \left(Z_1^2 Z_2^4 \cdot m(\mathscr{A}) \cdot \min\left(\tau^2, y^{-2}\right)\right)^{(1/12)} \left(\tau X^{1+\varepsilon}\right)^{(5/12)} \left(\tau X^{(5/3)+\varepsilon}\right)^{(1/8)} \\
&\quad \times \left(\tau X^{3+\varepsilon}\right)^{(1/48)} \left(\tau X^{(12/k)+\varepsilon}\right)^{(1/48)} \left(\tau X^{1+\varepsilon}\right)^{(1/6)} \left(\tau X^{(4/k)+\varepsilon}\right)^{(1/6)} \\
&\ll \left(y X^{(18/7)+9\varepsilon} \cdot \min\left(\tau^2, y^{-2}\right)\right)^{(1/12)} \tau^{(11/12)} X^{(41/48)+(11/12k)+\varepsilon} \\
&\ll \tau X^{(13/12)+(1/k)-(1/28)+2\varepsilon},
\end{aligned} \tag{47}
$$

where Lemmas 2 and 3 are used.

When $k = 5$, by the similar argument as in the proof of (47), we obtain

$$
\begin{aligned}
&|I(\tau,\eta,\mathscr{A})| \\
&\quad \ll \left(\int_{\mathscr{A}} |S_1(\lambda_1\alpha)|^2 |S_2^*(\lambda_2\alpha)|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/16)} \left(\int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(7/16)} \\
&\quad \times \left(\int_{\mathbb{R}} |S_3(\lambda_3\alpha)|^8 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)} \left(\int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_4(\lambda_4\alpha)|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)} \\
&\quad \times \left(\int_{\mathbb{R}} | |S_2^*(\lambda_2\alpha)|^2 |S_4(\lambda_4\alpha)|^2 |S_k(\lambda_5\alpha)|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)} \\
&\ll \left(Z_1^2 Z_2^4 \cdot m(\mathscr{A}) \cdot \min\left(\tau^2, y^{-2}\right)\right)^{(1/16)} \left(\tau X^{1+\varepsilon}\right)^{(7/16)} \\
&\quad \times \left(\tau X^{(5/3)+\varepsilon}\right)^{(1/8)} \left(\tau X^{1+\varepsilon}\right)^{(1/8)} \left(\tau X^{(1/2)+(4/k)+\varepsilon}\right)^{(1/4)} \\
&\ll \left(y X^{(18/7)+9\varepsilon} \cdot \min\left(\tau^2, y^{-2}\right)\right)^{(1/16)} \tau^{(15/16)} X^{(43/48)+(1/k)+\varepsilon} \\
&\ll \tau X^{(13/12)+(1/k)-(3/112)+2\varepsilon}.
\end{aligned} \tag{48}
$$

When $k = 6$, we have

$$
\begin{aligned}
&|I(\tau,\eta,\mathscr{A})| \\
&\quad \ll \left(\int_{\mathscr{A}} |S_1(\lambda_1\alpha)|^2 |S_2^*(\lambda_2\alpha)|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/16)} \left(\int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(7/16)} \\
&\quad \times \left(\int_{\mathbb{R}} |S_3(\lambda_3\alpha)|^8 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)} \left(\int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_4(\lambda_4\alpha)|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2^*(\lambda_2\alpha)|^2 |S_k(\lambda_5\alpha)|^8 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)} \\
&\ll \left(Z_1^2 Z_2^4 \cdot m(\mathscr{A}) \cdot \min\left(\tau^2, y^{-2}\right)\right)^{(1/16)} \left(\tau X^{1+\varepsilon}\right)^{(7/16)} \left(\tau X^{(5/3)+\varepsilon}\right)^{(1/8)} \left(\tau X^{1+\varepsilon}\right)^{(1/4)} \left(\tau X^{(8/k)+\varepsilon}\right)^{(1/8)} \\
&\ll \left(y X^{(18/7)+9\varepsilon} \cdot \min\left(\tau^2, y^{-2}\right)\right)^{(1/16)} \tau^{(15/16)} X^{(43/48)+(1/k)+\varepsilon} \\
&\ll \tau X^{(13/12)+(1/k)-(3/112)+2\varepsilon}.
\end{aligned} \tag{49}
$$

Thanks to (17) and (47)–(49), we are led to the conclusion that

$$\left|I(\tau,\eta,\mathfrak{m}_4)\right| \ll \mathscr{L}^3 \cdot \max_{\mathscr{A}} |I(\tau,\eta,\mathscr{A})| \ll \tau^2 X^{(13/12)+(1/k)-\varepsilon} \ll \tau^2 X^{(13/12)+(1/k)} \mathscr{L}^{-2}. \tag{50}$$

This together with (36), (40), (44), and (32) gives (30).

5. The Trivial Regions

Finally, it only remains to treat $I(\tau,\eta,\mathfrak{t})$. Suppose that r and j are positive integers with $r \le j$. For any $\xi \in [X^{\varepsilon}, +\infty)$ and nonzero real λ, we have (see (5.1) and (5.2) in [20])

$$\begin{aligned} &\int_{\xi}^{+\infty} \left|S_j(\lambda\alpha)\right|^{2^r} K_\tau(\alpha)\mathrm{d}\alpha \ll \xi^{-1} X^{((2^r-r)/j)+\varepsilon}, \\ &\int_{\xi}^{+\infty} \left|S_2^*(\lambda_2\alpha)\right|^4 K_\tau(\alpha)\mathrm{d}\alpha \ll \xi^{-1} X^{1+\varepsilon}. \end{aligned} \tag{51}$$

It follows from Hölder's inequality that

$$\begin{aligned} |I(\tau,\eta,\mathfrak{t})| &\ll \int_{\xi}^{+\infty} \left| S_k(\lambda_5\alpha) S_2^*(\lambda_2\alpha) \prod_{\substack{1\le j\le 4 \\ j\ne 2}} S_j(\lambda_j\alpha) \right| K_\tau(\alpha)\mathrm{d}\alpha \\ &\ll \left(\int_{\xi}^{+\infty} \left|S_1(\lambda_1\alpha)\right|^2 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/2)} \left(\int_{\xi}^{+\infty} \left|S_2^*(\lambda_2\alpha)\right|^4 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/4)} \\ &\quad \times \left(\int_{\xi}^{+\infty} \left|S_3(\lambda_3\alpha)\right|^8 K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/8)} \left(\int_{\xi}^{+\infty} \left|S_4(\lambda_4\alpha)\right|^{16} K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/16)} \\ &\quad \times \left(\int_{\xi}^{+\infty} \left|S_k(\lambda_5\alpha)\right|^{16} K_\tau(\alpha)\mathrm{d}\alpha\right)^{(1/16)} \\ &\ll \left(\xi^{-1} X^{1+\varepsilon}\right)^{(1/2)} \left(\xi^{-1} X^{1+\varepsilon}\right)^{(1/4)} \left(\xi^{-1} X^{(5/3)+\varepsilon}\right)^{(1/8)} \left(\xi^{-1} X^{3+\varepsilon}\right)^{(1/16)} \left(\xi^{-1} X^{(12/k)+\varepsilon}\right)^{(1/16)} \\ &\ll \xi^{-1} X^{(55/48)+(3/4k)+2\varepsilon}. \end{aligned} \tag{52}$$

Recalling that $\xi = \tau^{-2} X^{(1/16)-(1/4k)+10\varepsilon}$ and inserting this expression into (52) yields

$$|I(\tau,\eta,\mathfrak{t})| \ll \tau^2 X^{(13/12)+(1/k)-\varepsilon} \ll \tau^2 X^{(13/12)+(1/k)} \mathscr{L}^{-2}. \tag{53}$$

6. Completion of the Proof

We are now in a position to get the desired conclusion. It should be noted that

$$I(\tau,\eta,\mathbb{R}) = I(\tau,\eta,\mathfrak{M}) + I(\tau,\eta,\mathfrak{m}) + I(\tau,\eta,\mathfrak{t}). \tag{54}$$

From this and (19), (30), and (53), we infer that $I(\tau,\eta,\mathbb{R}) \gg \tau^2 X^{(13/12)+(1/k)} \mathscr{L}^{-1}$. Hence, by (15),

$$\mathscr{N}(X) \gg \tau X^{(13/12)+(1/k)} \mathscr{L}^{-5}. \tag{55}$$

This implies inequality (16) has $\gg \tau X^{(13/12)+(1/k)} \mathscr{L}^{-5}$ solutions in quintuples of primes $(p_1, p_2, \ldots, p_5)$ with $p_2 \in \mathscr{I}$, $p_5 \in I_k$, and $p_j \in I_j$, for $j \in \{1,3,4\}$. Notice that λ_1/λ_2 is irrational, q is any denominator of a convergent to λ_1/λ_2 and $X = q^{7/3}$. By substituting (17) into (55), we deduce that $\mathscr{N}(X) \longrightarrow +\infty$ as $q \longrightarrow +\infty$. In view of

$$\max_{1\le j\le 5} p_j \asymp X, \tag{56}$$

and (17), we obtain the required range of $\sigma(k)$ in Theorem 1. This completes the proof of Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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