Research Article

VL Reciprocal Status Index and Co-Index of Graphs

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For the vertex set $V(G)$ of a graph G , the sum of reciprocals of the breadth (distances) between the vertex $v \in V(G)$ and whole other remaining vertices of G is called reciprocal status of v . In this study, first of all, we introduced the VL reciprocal status index and VL reciprocal status co-index of a graph G . Later, we exposed some sharp bounds for these indices. Furthermore, we determined the VL reciprocal status (co-)index over some standard graphs. Finally, we presented correlations between VL reciprocal status index and some properties of Butane derivatives via a table and illustrated with a figure.

1. Introduction and Preliminaries

Topological indices are known as special graph invariants, and they are mainly used to calculate QSAR (quantitative structure-activity relationship) and QSPR (quantitative structure-property relationship) studies, cf. [15, 16]. Although the Wiener index ([17]) is the oldest topological index, later on some distance-based topological indices have defined as hyper Wiener index [13], Harary index [4, 8], and so on. The main idea in the study of topological indices is whether they are useful for identifying chemical properties. In general, octane isomers are the exceptional data for these studies.

If we suppose G is connected having order n also having size m , then it is quite well known that, while the vertex and edge sets are shown by $V(G)$ and $E(G)$, respectively, an edge connecting (joining) the vertices u_1 and u_2 is shown by u_1u_2 , and the degree d_u of $u \in E(G)$ is actually the number of edges incident to u . On the contrary, $d(u_1, u_2)$, which notates the distance between any two vertices u_1 and u_2 , is actually the length of the shortest path connecting u_1 and u_2 , and the maximum breadth between any pair of vertices of G is defined as the diameter $diam(G)$. For unmentioned graph theoretic terminologies, we may refer [1]. The

following reminders will be needed for the construction of results in this paper:

In [2], for G , the VL index has recently been exposed by

$$VL(G) = \frac{1}{2} \sum_{uv \in E(G)} [d_e + d_f + 4], \quad (1)$$

where $d_e = d(u_1) + d(u_2) - 2$ and $d_f = d(u_1) \cdot d(u_2) - 2$. According to the same reference, the VL index indicates a nice output in the meaning of polychlorinated biphenyl and octane isomers. Actually, $VL(G)$ index can also be written as

$$VL(G) = \frac{1}{2} \sum_{u_1u_2 \in E(G)} [d(u_1) + d(u_2) + d(u_1) \cdot d(u_2)]. \quad (2)$$

We will develop some new type of VL indices, see Definition 1 below.

For the vertex u_1 , Harary [3] defined a parameter called status (or, equivalently and transmission), which is the sum of u_1 's breadths to remaining vertices of G , and that is shown by

$$\sigma(u_1) = \sum_{u_2 \in V(G)} d(u_1, u_2). \quad (3)$$

The reciprocal status [11] of $u_1 \in V(G)$ is defined by the sum of reciprocals of u_1 's breadths (distances) to remaining vertices in G . This is shown by

$$rs(u_1) = \sum_{u_2 \in V(G)} \frac{1}{d(u_1, u_2)}. \quad (4)$$

As it mentioned at the beginning, the oldest index, which is namely the Wiener index [17], is given by

$$W(G) = \sum_{\{u_1, u_2\} \subseteq V(G)} d(u_1, u_2) = \frac{1}{2} \sum_{u_1 \in V(G)} \sigma(u_1). \quad (5)$$

In fact, it is also called as total status [3]. In literature knowledge, there are some studies about the different types of transmission topological indices, see, for instance, [5–7, 9, 10, 12, 14]. Additionally, the status and reciprocal status having base (topological) co-indices have recently been introduced in [12], and also, explicit formulae for them via order and size of G are obtained.

Based on the construction of different types of reciprocal status-based topological indices and co-indices until now, in here, we present two new topological based indices, namely, VL reciprocal status index and VL reciprocal status co-index of a connected graph G .

Therefore, the following definition plays a key role for this study:

Definition 1. Let us consider a connected graph G . Therefore, the VL reciprocal status index over G is defined by

$$VLRS(G) = \frac{1}{2} \sum_{u_1 u_2 \in E(G)} [rs(u_1) + rs(u_2) + rs(u_1) \cdot rs(u_2)], \quad (6)$$

whereas the VL reciprocal status co-index is defined by

$$\overline{VLRS}(G) = \frac{1}{2} \sum_{u_1 u_2 \notin E(G)} [rs(u_1) + rs(u_2) + rs(u_1) \cdot rs(u_2)]. \quad (7)$$

For an example of $VLRS(G)$ and $\overline{VLRS}(G)$ indices, one may see the graph in Figure 1. Clearly, $VLRS(G) = 83.625$ and $\overline{VLRS}(G) = 17$.

This paper is organised as in the following. After this introductory part, we will present some “nice” bounds for the VL reciprocal status index, see equation (6) in Section 2. In Section 3, we will state and prove some results about VL reciprocal status index of standard graphs such as complete K_n , complete bipartite $K_{p,q}$, path P_n , cycle C_n , wheel W_{k+1} , windmill F_k , and star graph S_n . In Section 4 and 5, in parallel to Sections 2 and 3, we will compute some nice bounds for the VL reciprocal status co-index defined in equation (7) and present some results about this new co-index of standard graphs depicted above. In Section 6, the correlation of the VL reciprocal status index with some properties of Butane derivatives will be presented by means of a table and figure. Section 7 shows the importance of these new indices in terms of heavy atomic count and some future ideas.

FIGURE 1: A graph with five vertices.

2. Bounds for VL Reciprocal Status Index

In Section 2, we present some lower and upper bounds for $VLRS$ defined in equation (6) and then characterize the strictness conditions of these bounds.

The following is needed to obtain our first main result in this paper:

Lemma 1. Assume G is connected owning n vertices whereas $di\ am(G) = D$. Thus,

$$\frac{1}{8} \sum_{uv \in E(G)} 2s + t + (n-1)(s+n+3) \leq VLRS(G) \quad (8)$$

$$\leq \frac{1}{2} \sum_{uv \in E(G)} p(2+p) + (1+p)qs + q^2t,$$

where $s = d(u) + d(v)$, $t = d(u) \cdot d(v)$, $p = (n-1/D)$, and $q = 1 - (1/D)$. Both equalities hold if $di\ am(G) \leq 2$.

Proof. We first note that, for any $u \in V(G)$, there exist d_u vertices having distance one from u and the remaining $n-1-d_u$ vertices having distances at least 2.

Upper bound: for $u \in V(G)$,

$$rs(u) \geq d_u + \frac{1}{D} (n-1-d_u) = \frac{n-1}{D} + d_u \left(1 - \frac{1}{D}\right), \quad (9)$$

which implies that

$$VLRS(G) = \frac{1}{2} \sum_{uv \in E(G)} [rs(u) + rs(v) + rs(u) \cdot rs(v)] \leq \frac{1}{2} \sum_{uv \in E(G)} p(2+p) + (1+p)qs + q^2t, \quad (10)$$

where $s = d_u + d_v$, $t = d_u \cdot d_v$, $p = (n-1/D)$, and $q = 1 - (1/D)$.

Lower bound: again, for $u \in V(G)$, there is

$$rs(u) \leq d_u + \frac{1}{2} (n-1-d_u) = \frac{1}{2} [d_u + (n-1)]. \quad (11)$$

Therefore,

$$VLRS(G) = \frac{1}{2} \sum_{uv \in E(G)} [rs(u) + rs(v) + rs(u) \cdot rs(v)] \geq \frac{1}{8} \sum_{uv \in E(G)} 2s + t + (n-1)(s+n+3), \quad (12)$$

where $s = d_u + d_v$, $t = d_u \cdot d_v$, $p = (n - 1/D)$, and $q = 1 - (1/D)$ as required.

For the equality: suppose the diameter of G is given as 1 or 2. Thus, equality holds. For the converse part, let $\text{diam} \geq 3$, and let us assume that (Text translation failed), where $s = d_u + d_v$, $t = d_u \cdot d_v$, $p = (n - 1/D)$, and $q = 1 - (1/D)$. Thus, there exists one pair of vertices (at least) u_1, u_2 having $d(u_1, u_2) \geq 3$. So,

$$rs(u_1) \leq d_{u_1} + \frac{1}{3} + \frac{1}{2}(n - 2 - d_{u_1}) = \frac{n}{2} - \frac{2}{3} + \frac{d_{u_1}}{2}. \quad (13)$$

Similarly, for u_2 , we get $rs(u_2) \leq (n/2) - (2/3) + (d_{u_2}/2)$ but for remaining whole vertices $u \in V(G)$, it is obtained $rs(u) \leq (n/2) - (1/2) + (d_u/2)$.

Now, our aim is to reach a contradiction by assuming $\text{diam} \geq 3$. So, let us separate the partition $E(G)$ into three sets E_1, E_2 , and E_3 such that

$$E_1 = \left\{ u_1 v \mid rs(u_1) \leq \frac{n}{2} - \frac{2}{3} + \frac{d_{u_1}}{2} \text{ and } rs(v) \leq \frac{n}{2} - \frac{1}{2} + \frac{d_v}{2} \right\},$$

$$E_2 = \left\{ u_1 v \mid rs(u_2) \leq \frac{n}{2} - \frac{2}{3} + \frac{d_{u_2}}{2} \text{ and } rs(v) \leq \frac{n}{2} - \frac{1}{2} + \frac{d_v}{2} \right\}, \quad (14)$$

$$E_3 = \left\{ uv \mid rs(u) \leq \frac{n}{2} - \frac{1}{2} + \frac{d_u}{2} \text{ and } rs(v) \leq \frac{n}{2} - \frac{1}{2} + \frac{d_v}{2} \right\}. \quad (15)$$

Clearly, $|E_1| = d_{u_1}$, $|E_2| = d_{u_2}$, and $|E_3| = m - d_{u_1} - d_{u_2}$. Therefore,

$$\begin{aligned} VLRS(G) &= \frac{1}{2} \sum_{uv \in E(G)} [rs(u) + rs(v) + rs(u).rs(v)] = \frac{1}{2} \sum_{u_1 v \in E_1} [rs(u_1) + rs(v) + rs(u_1).rs(v)] \\ &\quad + \frac{1}{2} \sum_{u_2 v \in E_2} [rs(u_2) + rs(v) + rs(u_2).rs(v)] \\ &\quad + \frac{1}{2} \sum_{uv \in E_3} [rs(u) + rs(v) + rs(u).rs(v)] > \frac{1}{8} \sum_{uv \in E(G)} 2s + t + (n - 1)(s + n + 3), \end{aligned} \quad (16)$$

which gives a contradiction. Further, we get $\text{diam}(G) \leq 2$. $\square$

By means of direct application of Lemma 1, we can give our first main result after expanding the sums:

Theorem 1. Suppose G is connected, having total n vertices, and finally, suppose $\text{diam}(G) = D$. Thus,

$$\frac{1}{8} \left\{ (n + 1)\|d\|^2 + \frac{1}{2} A(d) \cdot d + (n - 1)(n + 3) \right\} \leq VLRS(G), \quad (17)$$

and

$$VLRS(G) \leq \frac{1}{2} \left\{ mp(2 + p) + (1 + p)q\|d\|^2 + \frac{q^2}{2} A(d) \cdot d \right\}. \quad (18)$$

Above equalities again hold if $\text{diam}(G) = D$.

The next two consequences of Theorem 2 can be given.

Corollary 1. Let G be as in Theorem 2 having m edges. Let δ and Δ be the min. and max. degrees of the elements in $V(G)$, respectively. After that,

$$\frac{m}{8} [(\delta + n)(\delta + n + 2) - 3] \leq VLRS(G) \leq \frac{m}{2} [p(2 + p) + 2q\Delta(1 + p) + q^2\Delta^2], \quad (19)$$

where $p = (n - 1/D)$ and $q = 1 - (1/D)$.

Proof. For the vertex $u \in V(G)$, it is clear that $\delta \leq d_u \leq \Delta$. Therefore, by substituting $d_u + d_v \geq 2\delta$, $d_u \cdot d_v \geq \delta^2$ in the

lower bound, and $d_u + d_v \leq 2\Delta$ but $d_u \cdot d_v \leq \Delta^2$ in the upper bound of Theorem 2, we get the required result. $\square$

Corollary 2. Suppose that G is connected r -regular on n vertices and m edges. Also, suppose that $\text{diam}(G) = D$. Thus,

$$\frac{m}{8} [(r + n)(r + n + 2) - 3] \leq VLRS(G) \leq \frac{m}{2} [p(2 + p) + 2rq(1 + p) + q^2r^2], \quad (20)$$

where $p = (n - 1/D)$ and $q = 1 - (1/D)$. Moreover, equalities in above hold if $D \leq 2$.

3. VL Reciprocal Status Index of Some Standard Graphs

As we mentioned in Section 1, we will determine VL reciprocal status index defined in equation (6) for some special graphs.

In the following, the first result of this section is about complete graphs.

Proposition 1. For a complete graph K_n on n vertices and m edges, $VLRS(K_n) = (m(n^2 - 1)/2)$.

Proof. It is known that $rs(u) = n - 1$ for any $u \in V(K_n)$. Henceforth, by equation (6), we get the result. $\square$

$$\begin{aligned} VLRS(K_{p,q}) &= \frac{pq}{8} [2(q + p) + q \cdot p + (p + q - 1)(q + p + p + q + 3)] \\ &= \frac{pq}{8} [2(p + q) + pq + (p + q - 1)(2p + 2q + 3)]. \end{aligned} \quad (22)$$

Hence, we get the result on $VLRS(K_{p,q})$ as required. $\square$

Now, we can investigate the situation for cycle graphs as follows.

The next proposition exposes the result for VLRS on complete bipartite graphs.

Proposition 2. For a complete bipartite graph $K_{p,q}$, we have

$$VLRS(K_{p,q}) = \frac{pq}{8} [2(p + q) + pq + (p + q - 1)(2p + 2q + 3)]. \quad (21)$$

Proof. One can partition $V(K_{p,q})$ into two different subsets V_1 and V_2 with the condition every single edge uv of $K_{p,q}$, and we let $u \in V_1$ and $v \in V_2$. Therefore, $d_u = q$ and $d_v = p$ such that $|V_1| = p$ and $|V_2| = q$. Clearly, $K_{p,q}$ has $n = p + q$ vertices and $m = pq$ edges, and also, the diameter $di\ am(k_{p,q}) \leq 2$. Hence, by the equality part of Theorem 2, it is achieved

Proposition 3. For a cycle graph C_n on $n \geq 3$ vertices,

$$VLRS(C_n) = \begin{cases} \frac{n}{2} \left[\frac{4}{n} + 4 \sum_{i=1}^{((n-2)/2)} \frac{1}{i} + \left(\frac{2}{n} + 2 \sum_{i=1}^{((n-2)/2)} \frac{1}{i} \right)^2 \right], & \text{if } n \text{ is even,} \\ 2n \left[\sum_{i=1}^{((n-2)/2)} \frac{1}{i} \left(1 + \sum_{i=1}^{((n-2)/2)} \frac{1}{i} \right) \right], & \text{if } n \text{ is odd.} \end{cases} \quad (23)$$

Proof

Case (i): if n is an even number, then for any vertex $u \in V(C_n)$, we have

$$\begin{aligned} rs(u) &= 2 \left[1 + \frac{1}{2} + \cdots + \frac{1}{((n-2)/2)} \right] + \frac{1}{(n/2)} \\ &= \frac{2}{n} + 2 \sum_{i=1}^{((n-2)/2)} \frac{1}{i}. \end{aligned} \quad (24)$$

Therefore, by equation (6), we obtain

$$VLRS(C_n) = \frac{n}{2} \left[\frac{4}{n} + 4 \sum_{i=1}^{((n-2)/2)} \frac{1}{i} + \left(\frac{2}{n} + 2 \sum_{i=1}^{((n-2)/2)} \frac{1}{i} \right)^2 \right], \quad (25)$$

while n is even.

Case (ii): if n is an odd number, then for $u \in V(C_n)$, we get

$$rs(u) = 2 \left[1 + \frac{1}{2} + \cdots + \frac{1}{((n-2)/2)} \right] = 2 \sum_{i=1}^{((n-2)/2)} \frac{1}{i}, \quad (26)$$

and then, by equation (6), we reach to

$$VLRS(C_n) = 2n \left[\sum_{i=1}^{((n-2)/2)} \frac{1}{i} \left(1 + \sum_{i=1}^{((n-2)/2)} \frac{1}{i} \right) \right], \quad (27)$$

as required. $\square$

As an example for Proposition 3.3, we can give $VLRS(C_3) = 12$ and $VLRS(C_4) = 22, 5$.

Another result in special graphs can be presented for a path as in the following.

Proposition 4. For a path graph P_n on n vertices,

$$\begin{aligned} VLRS(P_n) = & \frac{1}{2} \left[\frac{n}{n-1} + 2 \sum_{i=1}^{n-2} \frac{1}{i} + \sum_{i=1}^{n-1} \frac{1}{i} + \sum_{i=1}^{n-1} \frac{1}{i} \cdot \sum_{j=1}^{n-2} \frac{1}{j} \right] \\ & + \frac{1}{2} \sum_{i=2}^{n-2} \left[\frac{n}{i(n-i)} + 2 \left(\sum_{j=1}^{i-1} \frac{1}{j} + \sum_{j=1}^{n-i-1} \frac{1}{j} \right) \right] + \left(\sum_{j=1}^{i-1} \frac{1}{j} + \sum_{j=1}^{n-i-1} \frac{1}{j} \right) \left(\sum_{j=1}^i \frac{1}{j} + \sum_{j=1}^{n-i-1} \frac{1}{j} \right). \end{aligned} \quad (28)$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of P_n where v_i is connected to v_{i+1} for $i = 1, 2, \dots, n-1$. Therefore, for each $1 \leq i \leq n$, we get

Therefore,

$$\begin{aligned} rs(v_1) &= \sum_{i=1}^{n-1} \frac{1}{i} = rs(v_n) \text{ and } rs(v_i) = \sum_{j=1}^{i-1} \frac{1}{j} + \sum_{j=1}^{n-i} \frac{1}{j}, \\ (2 \leq i \leq n-1). \end{aligned} \quad (29)$$

$$\begin{aligned} VLRS(P_n) &= \frac{1}{2} \sum_{uv \in E(P_n)} [rs(u) + rs(v) + rs(u) \cdot rs(v)] \\ &= \frac{1}{2} [rs(v_1) + rs(v_2) + rs(v_1) \cdot rs(v_2)] \\ &\quad + \frac{1}{2} \sum_{i=2}^{n-2} [rs(v_i) + rs(v_{i+1}) + rs(v_i) \cdot rs(v_{i+1})] + [rs(v_{n-1}) + rs(v_n) + rs(v_{n-1}) \cdot rs(v_n)], \end{aligned} \quad (30)$$

which implies the required result on $VLRS(P_n)$. $\square$

$$VLRS(W_{k+1}) = \frac{k}{8} (3k^2 + 22k + 27). \quad (31)$$

Remaining three propositions will be clarified, $VLRS$ indices for wheel, windmill, and star graphs, respectively.

Proposition 5. Let us consider a wheel graph W_{k+1} with $(k \geq 3)$. Then,

Proof. If we partition the edge set $E(W_{k+1})$ into two sets $E_1 = \{uv | d_u = k, d_v = 3\}$ and $E_2 = \{uv | d_u = 3, d_v = 3\}$, then it is easy to see that $|E_1| = |E_2| = k$ such that $di\ am(W_{k+1}) = 2$. Therefore, by the equality part of Theorem 2,

$$\begin{aligned} VLRS(W_{k+1}) &= \frac{1}{8} \sum_{uv \in E(W_{k+1})} 2[d_u + d_v] + d_u \cdot d_v + k[d_u + d_v + k + 4] \\ &= \frac{1}{8} \sum_{uv \in E_1} 2[d_u + d_v] + d_u \cdot d_v + k[d_u + d_v + k + 4] \\ &\quad + \frac{1}{8} \sum_{uv \in E_2} 2[d_u + d_v] + d_u \cdot d_v + k[d_u + d_v + k + 4] \\ &= \frac{k}{8} (3k^2 + 22k + 27). \end{aligned} \quad (32)$$

$\square$

Proposition 6. For a windmill graph F_k with $k \geq 2$, we have

$$VLRS(F_k) = \frac{k}{2}(5k^2 + 14k + 5). \quad (33)$$

Proof. In the proof, we will follow the same way as in the proof of Proposition 3. So, let us partition $E(F_k)$ into subsets $E_1 = \{uv | d_u = 2k, d_v = 2\}$ and $E_2 = \{uv | d_u = 2, d_v = 2\}$ which gives $|E_1| = 2k$ and $|E_2| = k$. Also, $di\ am(F_k) = 2$. After that, by considering the equality part of Theorem 2, we obtain the result on $VLRS(F_k)$. $\square$

Since a star graph is actually a tree on n nodes and one node's degree is $n - 1$ and the remaining $n - 1$ vertices having degree one, so next result easily follows by Theorem 2:

Proposition 7. For a star graph S_n , $VLRS(S_n) = (m/4)(n^2 + 2n - 2)$.

4. Bounds for the VL Reciprocal Status Co-Index

Now, we present some bounds for VL reciprocal status co-index of graphs defined in equation (7) and characterize the equality conditions on them.

$$\begin{aligned} & \frac{1}{16} [n(n-1) - 2m][(\delta + n)(\delta + n + 2) - 3] \\ & \leq \overline{VLRS}(G) \text{ and } \overline{VLRS}(G) \leq \frac{1}{4} [n(n-1) - 2m] \left[p(2+p) + 2q\delta(1+p) + q^2\Delta^2 \right], \end{aligned} \quad (35)$$

where $p = (n - 1/D)$ and $q = 1 - (1/D)$.

Proof. For $u \in V(G)$, we know that $\delta \leq d_u \leq \Delta$ which implies $2\delta \leq d_u + d_v \leq 2\Delta$. It is also known that the graph G under these assuming conditions has $n(n-1)/2 - m$ pair of nonadjacent vertices. Substituting $d_u + d_v \geq 2\delta$ and $d_u \cdot d_v \geq \delta^2$ in the lower bound and $d_u + d_v \leq 2\Delta$ and $d(u) \cdot d(v) \leq \Delta^2$ in the upper bound of Theorem 2, we achieve the result. $\square$

Corollary 4. Let G be connected r -regular and has n vertices, and also, let $di\ am(G) = D$. Then,

$$\begin{aligned} & \frac{m}{8} [(r+n)(r+n+2) - 3] \leq \overline{VLRS}(G) \\ & \leq \frac{m}{2} \left[p(2+p) + 2rq(1+p) + r^2q^2 \right], \end{aligned} \quad (36)$$

where $p = (n - 1/D)$ and $q = 1 - (1/D)$. Equalities hold if $di\ am(G) = D$.

Proof. $\forall u \in V(G)$, and substituting $d_u = r$ in Theorem 2, we get the result. $\square$

Proof of the next result is very similar as in the proof of Theorem 2 and, therefore, will be skipped.

Theorem 2. Let G be connected and has n vertices and also $di\ am(G) = D$. Thus,

$$\begin{aligned} & \frac{1}{8} \sum_{uv \in E(G)} 2s + t + (n-1)(s+n+3) \leq \overline{VLRS}(G) \\ & \leq \frac{1}{2} \sum_{uv \in E(G)} p(2+p) + (1+p)qs + q^2t, \end{aligned} \quad (34)$$

where $s = d_u + d_v$, $t = d_u \cdot d_v$, $p = (n - 1/D)$, and $q = 1 - (1/D)$. Moreover, equality holds on both sides if $D \leq 2$.

The following consequences are as important as Theorem 2

Corollary 3. Assume that G is connected and has n vertices and m edges and also $di\ am(G) = D$. Assume also that δ and Δ is the min. and max. degrees of vertices in $V(G)$, respectively. Therefore,

5. VL Reciprocal Status Co-Index of Some Standard Graphs

With a similar approach as in Section 3, we will determine VL reciprocal status co-index given in equation (7) of some special graph structures such as complete, complete bipartite, cycle, path, wheel, and windmill graphs.

Since the details of the following proposition are clear by considering the definitions of complete graphs and VL reciprocal status co-index, the proof will be omitted.

Proposition 8. For K_n , $\overline{VLRS}(K_n) = 0$.

Proposition 9. For $K_{p,q}$, there exists

$$\begin{aligned} \overline{VLRS}(K_{p,q}) = & \frac{p(p-1)}{4} \left[2q + p - 1 + \left(q + \frac{p-1}{2} \right)^2 \right] \\ & + \frac{q(q-1)}{4} \left[2p + q - 1 + \left(p + \frac{q-1}{2} \right)^2 \right]. \end{aligned} \quad (37)$$

Proof. For $|V_1| = p$ and $|V_2| = q$, let us assume V_1 and V_2 are the partite sets of $V(K_{p,q})$ with every single edge of $K_{p,q}$

owning one end in V_1 and other end in V_2 . If $u \in V_1$, then $rs(u) = q + (1/2)(p - 1)$, and also, if $u \in V_2$, then $rs(u) = p + (1/2)(q - 1)$. Therefore, for $u, v \in V_1$, $rs(u) + rs(v) =$

$2q + (p - 1)$, and for $u, v \in V_2$, $rs(u) + rs(v) = 2p + (q - 1)$. Hence, by equation (7), we obtain

$$\begin{aligned}\overline{VLR\overline{S}}(G) &= \frac{1}{2} \sum_{\{u,v\} \subseteq V_1} [rs(u) + rs(v) + rs(u).rs(v)] + \frac{1}{2} \sum_{\{u,v\} \subseteq V_2} [rs(u) + rs(v) + rs(u).rs(v)] \\ &= \frac{p(p-1)}{4} \left[2q + p - 1 + \left(q + \frac{p-1}{2} \right)^2 \right] + \frac{q(q-1)}{4} \left[2p + q - 1 + \left(p + \frac{q-1}{2} \right)^2 \right],\end{aligned}\quad (38)$$

as required. $\square$

Proposition 10. For a cycle graph C_n on $n \geq 3$ vertices, we have

$$\overline{VLR\overline{S}}(C_n) = \begin{cases} \frac{n(n-1) - 2n}{4} \left[\frac{4}{n} + 4 \sum_{i=1}^{((n-2)/2)} \frac{1}{i} + \left(\frac{2}{n} + 2 \sum_{i=1}^{((n-2)/2)} \frac{1}{i} \right)^2 \right], & \text{if } n \text{ is even,} \\ [n(n-1) - 2n] \left[\sum_{i=1}^{((n-2)/2)} \frac{1}{i} \left(1 + \sum_{i=1}^{((n-2)/2)} \frac{1}{i} \right) \right], & \text{if } n \text{ is odd.} \end{cases} \quad (39)$$

Proof

Case (i): as obtained in Proposition 3 in Case (i), if n is an even number, for any vertex u of C_n , we get $rs(u) = (2/n) + 2 \sum_{i=1}^{((n-2)/2)} (1/i)$. Therefore,

$$\begin{aligned}\overline{VLR\overline{S}}(C_n) &= \frac{1}{2} \sum_{uv \notin E(C_n)} [rs(u) + rs(v) + rs(u).rs(v)] \\ &= \frac{n(n-1) - 2n}{4} \left[\frac{4}{n} + 4 \sum_{i=1}^{((n-2)/2)} \frac{1}{i} + \left(\frac{2}{n} + 2 \sum_{i=1}^{((n-2)/2)} \frac{1}{i} \right)^2 \right].\end{aligned}\quad (40)$$

Case (ii): similarly, as we obtained in Proposition 3 in Case (ii), if n is odd, then for $u \in V(C_n)$, we have $rs(u) = 2 \sum_{i=1}^{((n-2)/2)} (1/i)$. Therefore, we get

$$\begin{aligned}\overline{VLR\overline{S}}(C_n) &= \frac{1}{2} \sum_{uv \notin E(C_n)} [rs(u) + rs(v) + rs(u).rs(v)] \\ &= [n(n-1) - 2n] \left[\sum_{i=1}^{((n-2)/2)} \frac{1}{i} + \left(1 + \sum_{i=1}^{((n-2)/2)} \frac{1}{i} \right) \right].\end{aligned}\quad (41)$$

Proposition 11. For a wheel graph W_{k+1} ($k \geq 4$), $\overline{VLR\overline{S}}(W_{k+1}) = (1/16)(k^2 - 3k)(k^2 + 10k + 21)$.

Proof. It is known that the nonadjacent pairs of vertices of the wheel graph W_{k+1} have degree 3, and there are actually $\square$

TABLE 1: Correlation of VLRS index and some properties of Butane derivatives.

Name of the compound	Surface tension	Complexity	Heavy atomic count	VLRS
1,4-Butanedithiol	31.1	17.5	6	37.9902
2-Butanone	22.9	38.5	5	28.3333
1,3-Butanediol	34.9	28.7	6	42.0138
Butane dinitrile	40.7	92	6	80.0276
Butanediamide	53	96.6	8	79.8916
Butane-1-sulfonamide	41.9	133	8	85.0402
1-Butanethiol	24.8	13.1	5	25.1528
1,4-Diaminobutane	35.8	17.5	6	37.9902
Butane-1,4-disulfonic acid	77.9	266	12	191.8427
Butyraldehyde	23.1	24.8	5	25.1528
2,3-Butanedione	27.3	71.5	6	46.6667
Butanedihydrazide	59	119	10	137.2034
1-Butanesulfonyl chloride	36.4	133	8	85.0402

FIGURE 2: Correlation of VL reciprocal status index with physical properties of Butane derivatives.

total $(k(k+1)/2) - 2k$ pairs of nonadjacent vertices. Also, $di\ am(W_{k+1}) = 2$. Therefore, by the equality part of Theorem 2,

$$\begin{aligned}
 \overline{VLRS}(W_{k+1}) &= \frac{1}{8} \sum_{uv \notin E(W_{k+1})} 2[d(u) + d(v)] + d(u) \cdot d(v) + k[d(u) + d(v) + k + 4] \\
 &= \frac{1}{16} (k^2 - 3k)(k^2 + 10k + 21).
 \end{aligned} \tag{42}$$

Hence, the result. $\square$

Proposition 12. For a windmill graph F_k with $(k \geq 2)$, $\overline{VLRS}(F_k) = (1/16)(4k^2 - 4k)(4k^2 + 16k + 12)$.

$$\begin{aligned}\overline{VLRS}(F_k) &= \frac{1}{8} \sum_{uv \notin E(F_k)} 2[d(u) + d(v)] + d(u) \cdot d(v) + 2k[d(u) + d(v) + 2k + 4] \\ &= \frac{1}{16}(4k^2 - 4k)(4k^2 + 16k + 12).\end{aligned}\tag{43}$$

as required. $\square$

6. Correlation of VL Reciprocal Status Index and Some Properties of Butane Derivatives

QSPR studies have taken attention from both mathematicians and chemists since it opens new ways to new inventions. In particular, obtaining the properties of compounds without any big effort is time and money saving. Here, we see that VL reciprocal status index VLRS has a nice correlation including all three physical properties surface tension, complexity, and heavy atomic count, see Table 1.

Correlation coefficient value of VLRS with surface tension, complexity, and heavy atomic count is 0.95, 0.98, and 0.97, respectively, see Figure 2.

Using the data of Table 1, the scatter plot between the surface tension, complexity, heavy atomic count, and VLRS index of Butane derivatives is shown in Figure 2:

7. Conclusion

In the meaning of invariants over graphs, graph theoretical indices are used for the computation QSAR and QSPR. A large number indices have been exposed in literature which some of them have placed the application areas such as model physical and molecules in chemistry and pharmacy fields. By introducing the VL reciprocal status index and VL reciprocal status co-index of connected graphs as new indices in mathematical part of chemistry, in here, we stated and proved some upper and lower bounds on these new indices. Further, VLRS index and $\overline{VLRS}$ co-index of certain standard graphs are obtained. As an important conclusion, it is observed that VLRS index has a nice correlation on heavy atomic count 0.98.

Motivating this work, in forthcoming papers, one may sketch the results related to VL status (co-)index in terms of different parameters and indices. Also, it may be planned to design on the results about VL status index and co-index of some transmission regular graphs, nanostructures, and certain Archimedean lattice as well as may be planned to create a platform to study different structured VL indices in provisions of chemical/biological aspects. We are sure this paper will be useful into QSPR/QSAR studies.

Proof. It is already known that the nonadjacent pairs of vertices of the windmill graph F_k have degrees 2, and there are total $(2k(2k+1)/2) - 3k$ such pairs in F_k . Further, $di\ am(F_k) = 2$. Therefore, again by the equality part of Theorem 2,

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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