Research Article

Controlled $\ast$ -Operator Frames on Hilbert $C^\ast$ -Modules

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Received 24 February 2021; Accepted 27 March 2021; Published 9 April 2021

Academic Editor: Ngai-Ching Wong

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In this paper, we study the concept of controlled $\ast$ -operator frames for the space of all adjointable operators on a Hilbert $C^\ast$ -module H . Also, we discuss characterizations of controlled $\ast$ -operator frames and we give some properties. Some illustrative examples are provided to advocate the usability of our results.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [2] by Daubechies et al., frame theory began to be widely used, particularly, in the more specialized context of wavelet frames and Gabor frames [3]. Frames have been used in signal processing, image processing, data compression, and sampling theory.

Controlled frames in Hilbert spaces have been introduced by Balazs [4] to improve the numerical efficiency of iterative algorithms for inverting the frame operator.

Controlled frames in Hilbert $C^\ast$ -modules were introduced by Rashidi and Rahimi [5], and the authors showed that they share many useful properties with their corresponding notions in a Hilbert space. $\ast$ -operator frames for $\text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$ has been studied by Rossafi and Kabbaj [6]. For more details, see [7–9].

In this paper, we introduce the notion of controlled $\ast$ -operator frames for $\text{End}_{\mathcal{A}}^{\ast}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert $C^\ast$ -modules.

Let I be a countable index set. In this section, we briefly recall the definitions and basic properties of $C^\ast$ -algebra, Hilbert $C^\ast$ -modules, frame, and operator frame in Hilbert $C^\ast$ -modules. For information about frames in Hilbert spaces, we refer to [10]. Our references for $C^\ast$ -algebras are [11, 12].

For a $C^\ast$ -algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is positive ($a \geq 0$) if $a = a^\ast$ and $sp(a) \subset \mathbb{R}^+$. $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1 (see [13]). Let $\mathcal{A}$ be a unital $C^\ast$ -algebra and $\mathcal{H}$ be a left $\mathcal{A}$ -module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$ -module if $\mathcal{H}$ is equipped with an $\mathcal{A}$ -valued inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}: \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A}$, such that is sesquilinear, positive definite, and respects the module action. In the other words,

- (i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$, for all $x \in \mathcal{H}$ and $\langle x, x \rangle_{\mathcal{A}} = 0$, if and only if $x = 0$
- (ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a \langle x, z \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$, for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$
- (iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^\ast$, for all $x, y \in \mathcal{H}$

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x \rangle_{\mathcal{A}}\|^{1/2}$. If $\mathcal{H}$ is complete with $\|\cdot\|$, it is called a Hilbert $\mathcal{A}$ -module or a Hilbert $C^\ast$ -module over $\mathcal{A}$.

For every a in $C^\ast$ -algebra $\mathcal{A}$, we have $|a| = (a^\ast a)^{1/2}$ and the $\mathcal{A}$ -valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle_{\mathcal{A}}^{1/2}$ for $x \in \mathcal{H}$.

Example 1 (see [14]). If $\{\mathcal{H}_k\}_{k \in \mathbb{N}}$ is a countable set of Hilbert $\mathcal{A}$ -modules, then one can define their direct sum $\bigoplus_{k \in \mathbb{N}} \mathcal{H}_k$. On the $\mathcal{A}$ -module $\bigoplus_{k \in \mathbb{N}} \mathcal{H}_k$ of all sequences

$x = (x_k)_{k \in \mathbb{N}}$: $x_k \in \mathcal{H}_k$, such that the series $\sum_{k \in \mathbb{N}} \langle x_k, x_k \rangle_{\mathcal{A}}$ is norm-convergent in the $\mathcal{C}^*$ -algebra $\mathcal{A}$, we define the inner product by

$$\langle x, y \rangle := \sum_{k \in \mathbb{N}} \langle x_k, y_k \rangle_{\mathcal{A}}, \quad (1)$$

for $x, y \in \oplus_{k \in \mathbb{N}} \mathcal{H}_k$.

Hence, $\oplus_{k \in \mathbb{N}} \mathcal{H}_k$ is a Hilbert $\mathcal{A}$ -module.

The direct sum of a countable number of copies of a Hilbert $\mathcal{C}^*$ -module $\mathcal{H}$ is denoted by $l^2(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$ -modules. A map $T: \mathcal{H} \rightarrow \mathcal{K}$ is said to be adjointable if there exists a map $T^*: \mathcal{K} \rightarrow \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We also reserve the notation $\text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $\text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

Let $GL^+(\mathcal{H})$ be the set for all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse.

The following lemmas will be used to prove our main results.

Lemma 1 (see [15]). *If $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is a $*$ -homomorphism between $\mathcal{C}^*$ -algebras, then φ is increasing, that is, if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.*

Lemma 2 (see [15]). *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$ -modules and $T \in \text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$.*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*

$$\|(T^*T)^{-1}\|^{-1} I_{\mathcal{H}} \leq T^*T \leq \|T\|^2 I_{\mathcal{H}}. \quad (2)$$

(ii) *If T is surjective, then the adjointable map TT^* is invertible and*

$$\|(TT^*)^{-1}\|^{-1} I_{\mathcal{K}} \leq TT^* \leq \|T\|^2 I_{\mathcal{K}}. \quad (3)$$

Lemma 3 (see [16]). *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$ -modules. If $T \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$, then*

$$\langle Tx, Tx \rangle_{\mathcal{A}} \leq \|T\|^2 \langle x, x \rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \quad (4)$$

Lemma 4 (see [17]). *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$ -modules and $T \in \text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$. Then, the following statements are equivalent:*

- (i) T is surjective
- (ii) T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $m\|x\| \leq \|T^*x\|$ for all $x \in \mathcal{K}$
- (iii) T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $m'\langle x, x \rangle_{\mathcal{A}} \leq \langle T^*x, T^*x \rangle_{\mathcal{A}}$ for all $x \in \mathcal{K}$

2. Controlled $*$ -Operator Frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$

We begin this section with the following definition.

Definition 2 (see [6]). A family of adjointable operators $\{T_i\}_{i \in I}$ on a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ over a unital C^* -algebra $\mathcal{A}$ is said to be an operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$, if there exist two positive constants $A, B > 0$ such that

$$A\langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle T_i x, T_i x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \quad (5)$$

The numbers A and B are called lower and upper bound of the operator frames, respectively. If $A = B = \lambda$, the operator frame is λ -tight.

If $A = B = 1$, it is called a normalized tight operator frames or a Parseval operator frames.

If only upper inequality of (5) holds, then $\{T_i\}_{i \in I}$ is called an operator Bessel sequence for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

If the sum in the middle of (5) is convergent in norm, the operator frame is called standard.

Definition 3. Let $C, C' \in GL^+(\mathcal{H})$, and a family of adjointable operators $\{T_i\}_{i \in I}$ on a Hilbert $\mathcal{A}$ -module $\mathcal{H}$ over a unital C^* -algebra $\mathcal{A}$ is said to be a (C, C') -controlled $*$ -operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ if there exists two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle x, x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle T_i C x, T_i C' x \rangle_{\mathcal{A}} \leq B\langle x, x \rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}. \quad (6)$$

The elements A and B are called lower and upper bound of the (C, C') -controlled $*$ -operator frames, respectively.

If $A = B = \lambda$, the (C, C') -controlled $*$ -operator frame is called λ -tight.

If $A = B = 1_{\mathcal{A}}$, it is called a normalized tight (C, C') -controlled $*$ -operator frames or a Parseval (C, C') -controlled $*$ -operator frames.

If only upper inequality of (6) holds, then $\{T_i\}_{i \in I}$ is called an (C, C') -controlled $*$ -operator Bessel sequence for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

Example 2. Let $\mathcal{A} = l^\infty$ be the unitary C^* -algebra of all bounded complex-valued sequences and let $\mathcal{H} = C_0$, the set of all sequences converging to zero equipped with the $\mathcal{A} = l^\infty$ -inner product:

$$\langle x, y \rangle_{\mathcal{A}} = \langle (x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}} \rangle_{\mathcal{A}} = (x_i \bar{y}_i)_{i \in \mathbb{N}}, \quad (7)$$

for all $x = (x_i)_{i \in \mathbb{N}}$ and $y = (y_i)_{i \in \mathbb{N}} \in \mathcal{H}$.

It is clear to see that $\mathcal{H}$ is a Hilbert C^* -module over $\mathcal{A} = l^\infty$.

Let $j \in \mathbb{N}$ and $(a_i)_{i \in \mathbb{N}} = (1 + (1/i))_{i \in \mathbb{N}}$, and we define $T_j \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ by

$$T_j((x_i)_{i \in \mathbb{N}}) = (\delta_{ij} a_j x_j)_{i \in \mathbb{N}}, \quad (x_i)_{i \in \mathbb{N}} \in \mathcal{H}. \quad (8)$$

Let $\alpha \in \mathbb{R}^*$, and we define two operators C and C' on $\mathcal{H}$ by

$$\begin{aligned}
C: \mathcal{H} &\longrightarrow \mathcal{H}, \\
x &\longrightarrow Cx = \alpha x, \\
C': \mathcal{H} &\longrightarrow \mathcal{H}, \\
x &\longrightarrow C'x = \frac{1}{\alpha}x.
\end{aligned} \tag{9}$$

We have

$$\begin{aligned}
\sum_{j \in \mathbb{N}} \langle T_j Cx, T_j C'x \rangle &= \langle (\delta_{ij} a_j \alpha x_j)_{i \in \mathbb{N}}, (\delta_{ij} a_j \frac{1}{\alpha} x_j)_{i \in \mathbb{N}} \rangle_{\mathcal{A}} \\
&= \left\langle \left(1 + \frac{1}{i}\right)^2 x_i \bar{x}_i \right\rangle_{i \in \mathbb{N}} \\
&= \left(1 + \frac{1}{i}\right)_{i \in \mathbb{N}} \langle x, x \rangle_{\mathcal{A}} \left(1 + \frac{1}{i}\right)_{i \in \mathbb{N}}.
\end{aligned} \tag{10}$$

Therefore, $(T_j)_{j \in \mathbb{N}}$ is a (C, C') -controlled tight $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

Proposition 1. Every (C, C') -controlled operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ is a (C, C') -controlled $*$ -operator frames.

Proof. Let $\{T_i\}_{i \in I}$ be a (C, C') -controlled $*$ -operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

Then, there exist two positives constants $A, B > 0$ such that

$$A \langle x, x \rangle_{\mathcal{A}} \leq \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \leq B \langle x, x \rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{11}$$

Hence,

$$\begin{aligned}
(\sqrt{A})1_{\mathcal{A}} \langle x, x \rangle_{\mathcal{A}} ((\sqrt{A})1_{\mathcal{A}})^* &\leq \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \\
&\leq (\sqrt{B})1_{\mathcal{A}} \langle x, x \rangle_{\mathcal{A}} ((\sqrt{B})1_{\mathcal{A}})^*, \quad x \in \mathcal{H}.
\end{aligned} \tag{12}$$

Therefore, $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{A}}$.

Let $\{T_i\}_{i \in I}$ be a (C, C') -controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

The bounded linear operator $T_{(C, C')}: \mathcal{H} \longrightarrow l^2(\{\mathcal{H}\})$ is given by

$$T_{(C, C')}(x) = \{T_i (C' C)^{1/2} x\}_{i \in I}, \quad x \in \mathcal{H}, \tag{13}$$

which is called the analysis operator for the (C, C') -controlled $*$ -operator frame $\{T_i\}_{i \in I}$.

The adjoint operator $T_{CC'}^*: l^2(\mathcal{H}) \longrightarrow \mathcal{H}$ is given by

$$T_{(C, C')}^*(\{y_i\}_{i \in I}) = \sum_{i \in I} (CC')^{1/2} T_i^* y_i, \quad \{y_i\}_{i \in I} \in l^2(\mathcal{H}), \tag{14}$$

which is called the synthesis operator for the (C, C') -controlled $*$ -operator frames $\{T_i\}_{i \in I}$.

When C and C' commute with each other and commute with the operator $T_i^* T_i$ for each $i \in I$, we define the (C, C') -controlled frame operator $S_{(C, C')}: \mathcal{H} \longrightarrow \mathcal{H}$ by $S_{(C, C')}x = T_{(C, C')}T_{(C, C')}^*x = \sum_{i \in I} C' T_i^* T_i Cx$.

From now, we assume that C and C' commute with each other and commute with the operator $T_i^* T_i$ for each $i \in I$. $\square$

Proposition 2. The (C, C') -controlled frame operator $S_{(C, C')}$ is bounded, positive, self-adjoint, and invertible.

Proof. Let $\{T_i\}_{i \in I}$ be a (C, C') -controlled $*$ -operator frames; then,

$$A \langle x, x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \leq B \langle x, x \rangle_{\mathcal{A}} B^*, \tag{15}$$

and hence,

$$\sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} = \langle \sum_{i \in I} C' T_i^* T_i Cx, x \rangle_{\mathcal{A}} = \langle S_{CC'} x, x \rangle_{\mathcal{A}}. \tag{16}$$

Then,

$$A \langle x, x \rangle_{\mathcal{A}} A^* \leq \langle S_{CC'} x, x \rangle_{\mathcal{A}} \leq B \langle x, x \rangle_{\mathcal{A}} B^*. \tag{17}$$

It is clear that $S_{CC'}$ is positive and bounded operator. We have

$$\begin{aligned}
\langle S_{(C, C')} x, y \rangle_{\mathcal{A}} &= \langle \sum_{i \in I} C' T_i^* T_i Cx, y \rangle_{\mathcal{A}} \\
&= \sum_{i \in I} \langle C' T_i^* T_i Cx, y \rangle_{\mathcal{A}} \\
&= \sum_{i \in I} \langle x, C T_i^* T_i C' y \rangle_{\mathcal{A}} \\
&= \langle x, \sum_{i \in I} C T_i^* T_i C' y \rangle_{\mathcal{A}} \\
&= \langle x, S_{(C', C)} y \rangle_{\mathcal{A}}.
\end{aligned} \tag{18}$$

Therefore, $S_{(C, C')}^* = S_{(C', C)}$. Since C and C' commute with each other and commute with $T_i^* T_i$, we have $S_{(C, C')}$ self-adjoint.

From the definition of controlled $*$ -operator frame, we have

$$A \langle x, x \rangle_{\mathcal{A}} A^* \leq \langle S_{(C, C')} x, x \rangle_{\mathcal{A}} \leq B \langle x, x \rangle_{\mathcal{A}} B^*. \tag{19}$$

So,

$$A \cdot \text{Id}_{\mathcal{H}} \cdot A^* \leq S_{(C, C')} \leq B \cdot \text{Id}_{\mathcal{H}} \cdot B^*, \tag{20}$$

where $\text{Id}_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$. Thus, $S_{(C, C')}$ is invertible. $\square$

3. Some Characterizations of Controlled $*$ -Operator Frames

Theorem 1. Let $\{T_i\}_{i \in I} \subset \text{End}_{\mathcal{A}}^*(\mathcal{H})$ such that $\sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}}$ converge in norm $\mathcal{A}$. Then, $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frames if and only if

$$\|A^{-1}\|^{-2}\|\langle x, x \rangle_{\mathcal{A}}\| \leq \left\| \sum_{i \in I} \langle T_i C x, T_i C' x \rangle_{\mathcal{A}} \right\| \leq \|B\|^2 \|\langle x, x \rangle_{\mathcal{A}}\|, \quad (21)$$

for every $x \in \mathcal{H}$ and strictly nonzero elements A, B in $\mathcal{A}$.

Proof. Suppose that $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frames; then, we have

$$\begin{aligned} \langle x, x \rangle_{\mathcal{A}} &\leq A^{-1} \langle S_{(C, C')} x, x \rangle_{\mathcal{A}} (A^*)^{-1}, \quad x \in \mathcal{H}, \\ \langle S_{(C, C')} x, x \rangle_{\mathcal{A}} &\leq B \langle x, x \rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}. \end{aligned} \quad (22)$$

Hence,

$$\|A^{-1}\|^{-2}\|\langle x, x \rangle_{\mathcal{A}}\| \leq \left\| \sum_{i \in I} \langle T_i C x, T_i C' x \rangle_{\mathcal{A}} \right\| \leq \|B\|^2 \|\langle x, x \rangle_{\mathcal{A}}\|. \quad (23)$$

Conversely, assume that (21) holds. From (13), the (C, C') -controlled frame operator $S_{(C, C')}$ is positive, self-adjoint, and invertible. Then, we have, for all $x \in \mathcal{H}$,

$$\begin{aligned} \langle (S_{(C, C')})^{1/2} x, (S_{(C, C')})^{1/2} x \rangle_{\mathcal{A}} &= \langle S_{(C, C')} x, x \rangle_{\mathcal{A}} \\ &= \sum_{i \in I} \langle T_i C x, T_i C' x \rangle_{\mathcal{A}}. \end{aligned} \quad (24)$$

Using (21) and (24), we obtain

$$\|A^{-1}\|^{-2} \cdot \|x\| \leq \left\| (S_{(C, C')})^{1/2} x \right\| \leq \|B\|^2 \cdot \|x\|. \quad (25)$$

Using (25) and Lemma 4, we conclude that $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frames $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

The following theorem shows that any $*$ -operator frame is a C^2 -controlled $*$ -operator frames for $\mathcal{H}$ and vice versa. $\square$

Theorem 2. Let $C \in GL^+(\mathcal{H})$. The family $\{T_i\}_{i \in I} \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ is a $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ if and only if $\{T_i\}_{i \in I}$ is a C^2 -controlled $*$ -operator frames.

Proof. Let $\{T_i\}_{i \in I}$ be a C^2 -controlled $*$ -operator frames with bounds A and B . Then,

$$\sum_{i \in I} \langle T_i C x, T_i C x \rangle_{\mathcal{A}} \leq B \langle x, x \rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}. \quad (26)$$

On the one hand, for all $x \in \mathcal{H}$, we have

$$\begin{aligned} A \langle x, x \rangle_{\mathcal{A}} A^* &= A \langle C C^{-1} x, C C^{-1} x \rangle_{\mathcal{A}} A^* \\ &\leq A \|C\|^2 \langle C^{-1} x, C^{-1} x \rangle_{\mathcal{A}} A^* \\ &\leq \|C\|^2 \sum_{i \in I} \langle T_i C C^{-1} x, T_i C C^{-1} x \rangle_{\mathcal{A}} \\ &= \|C\|^2 \sum_{i \in I} \langle T_i x, T_i x \rangle_{\mathcal{A}}. \end{aligned} \quad (27)$$

Then,

$$A \|C\|^{-1} \langle x, x \rangle_{\mathcal{A}} A^* \|C\|^{-1} \leq \sum_{i \in I} \langle T_i x, T_i x \rangle_{\mathcal{A}}. \quad (28)$$

On the other hand, for any $x \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{i \in I} \langle T_i x, T_i x \rangle_{\mathcal{A}} &= \sum_{i \in I} \langle T_i C C^{-1} x, T_i C C^{-1} x \rangle_{\mathcal{A}} \\ &\leq B \langle C^{-1} x, C^{-1} x \rangle_{\mathcal{A}} B^* \\ &\leq B \|C^{-1}\|^2 \langle x, x \rangle_{\mathcal{A}} B^*. \end{aligned} \quad (29)$$

Then,

$$\begin{aligned} A \|C\|^{-1} \langle x, x \rangle_{\mathcal{A}} A^* \|C\|^{-1} &\leq \sum_{i \in I} \langle T_i x, T_i x \rangle_{\mathcal{A}} \\ &\leq B \|C\|^{-1} \langle x, x \rangle_{\mathcal{A}} B^* \|C\|^{-1}. \end{aligned} \quad (30)$$

Therefore, $\{T_i\}_{i \in I}$ is a $*$ -operator frames with bounds $A \|C\|^{-1}$ and $B \|C\|^{-1}$.

For the converse, suppose that $\{T_i\}_{i \in I}$ is a $*$ -operator frames with bounds M and N .

On the one hand, we have, for any $x \in \mathcal{H}$,

$$M \langle x, x \rangle_{\mathcal{A}} M^* \leq \sum_{i \in I} \langle T_i x, T_i x \rangle_{\mathcal{A}} \leq N \langle x, x \rangle_{\mathcal{A}} N^*. \quad (31)$$

Thus, for all $x \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{i \in I} \langle T_i C x, T_i C x \rangle_{\mathcal{A}} &\leq N \|C\|^2 \langle x, x \rangle_{\mathcal{A}} N^* \\ &= N \|C\| \langle x, x \rangle_{\mathcal{A}} N^* \|C\|. \end{aligned} \quad (32)$$

On the other hand, we have

$$\begin{aligned} M \langle x, x \rangle_{\mathcal{A}} M^* &= M \langle C^{-1} C x, C^{-1} C x \rangle_{\mathcal{A}} M^* \\ &\leq \|C^{-1}\|^2 \sum_{i \in I} \langle T_i C x, T_i C x \rangle_{\mathcal{A}}. \end{aligned} \quad (33)$$

Therefore,

$$\begin{aligned} M \|C^{-1}\|^{-1} \langle x, x \rangle_{\mathcal{A}} M^* \|C^{-1}\|^{-1} &\leq \sum_{i \in I} \langle T_i C x, T_i C x \rangle_{\mathcal{A}} \\ &\leq N \|C\| \langle x, x \rangle_{\mathcal{A}} N^* \|C\|. \end{aligned} \quad (34)$$

This gives that $\{T_i\}_{i \in I}$ is a C^2 -controlled $*$ -operator frame with bounds $M \|C^{-1}\|^{-1}$ and $N^* \|C\|$. $\square$

Proposition 3. Let $\{T_i\}_{i \in I}$ be a $*$ -operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with frame operator S and let $C, C' \in GL^+(\mathcal{H})$. Suppose that C, C' commute with each other and commute with $T_i T_i^*$ for all $i \in I$; then, $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

Proof. Let $\{T_i\}_{i \in I}$ be an $*$ -operator frame with bounds A and B . Then, by (5), we have

$$\begin{aligned} \langle x, x \rangle_{\mathcal{A}} &\leq A^{-1} \langle S x, x \rangle_{\mathcal{A}} (A^*)^{-1}, \\ \langle S x, x \rangle_{\mathcal{A}} &\leq B \langle x, x \rangle_{\mathcal{A}} B^*. \end{aligned} \quad (35)$$

Hence,

$$\|A^{-1}\|^{-2} \|\langle x, x \rangle_{\mathcal{A}}\| \leq \left\| \sum_{i \in I} \langle T_i x, T_i x \rangle_{\mathcal{A}} \right\| \leq \|B\|^2 \|\langle x, x \rangle_{\mathcal{A}}\|. \quad (36)$$

We have

$$\left\| \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \right\| = \|\langle S_{(C,C')}x, x \rangle_{\mathcal{A}}\|, \quad (37)$$

$$\left\| \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \right\| = \|C\| \cdot \|C'\| \cdot \|\langle Sx, x \rangle_{\mathcal{A}}\|. \quad (38)$$

Using (36) and (38), we have

$$\begin{aligned} \|A^{-1}\|^{-2} \|C\| \cdot \|C'\| \|\langle x, x \rangle_{\mathcal{A}}\| &\leq \left\| \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \right\| \\ &\leq \|B\|^2 \|C\| \cdot \|C'\| \|\langle x, x \rangle_{\mathcal{A}}\|. \end{aligned} \quad (39)$$

Therefore, from Theorem 1, we conclude that $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frame with bounds $\|A^{-1}\|^{-1} \|C\|^{1/2} \cdot \|C'\|^{1/2}$ and $\|B\| \|C\|^{1/2} \cdot \|C'\|^{1/2}$. $\square$

Theorem 3. Let $C, C' \in GL^+(\mathcal{H})$ and $\{T_i\}_{i \in I} \subset \text{End}_{\mathcal{A}}^*(\mathcal{H})$. Suppose that C, C' commute with each other and commute with $T_i T_i^*$ for all $i \in I$. The family $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator Bessel sequence for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with bound B if and only if the operator $T_{CC'}: l^2(\mathcal{H}) \rightarrow \mathcal{H}$ given by

$$T_{CC'}(\{y_i\}_{i \in I}) = \sum_{i \in I} (CC')^{1/2} T_i^* y_i, \quad \forall \{y_i\}_{i \in I} \in l^2(\mathcal{H}), \quad (40)$$

is well defined, bounded, and $\|T_{CC'}\| \leq \|B\|$.

Proof. Assume that $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator Bessel sequence for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with bound B . As a result of (36),

$$\left\| \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \right\| \leq \|B\|^2 \|\langle x, x \rangle_{\mathcal{A}}\|. \quad (41)$$

We have

$$\begin{aligned} \|T_{(C,C')}(\{y_i\}_{i \in I})\|^2 &= \sup_{x \in \mathcal{H}, \|x\|=1} \left\| \sum_{i \in I} (CC')^{1/2} T_i^* y_i, x \right\|_{\mathcal{A}}^2 \\ &= \sup_{x \in \mathcal{H}, \|x\|=1} \left\| \sum_{i \in I} \langle (CC')^{1/2} T_i^* y_i, x \rangle_{\mathcal{A}} \right\|^2 \\ &= \sup_{x \in \mathcal{H}, \|x\|=1} \left\| \sum_{i \in I} \langle y_i, T(CC')^{1/2} x \rangle_{\mathcal{A}} \right\|^2 \\ &\leq \sup_{x \in \mathcal{H}, \|x\|=1} \left\| \sum_{i \in I} \langle y_i, y_i \rangle_{\mathcal{A}} \right\| \left\| \sum_{i \in I} \langle T_i (CC')^{1/2} x, T_i (CC')^{1/2} x \rangle_{\mathcal{A}} \right\| \\ &= \sup_{x \in \mathcal{H}, \|x\|=1} \left\| \sum_{i \in I} \langle y_i, y_i \rangle_{\mathcal{A}} \right\| \left\| \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \right\| \\ &\leq \sup_{x \in \mathcal{H}, \|x\|=1} \left\| \sum_{i \in I} \langle y_i, y_i \rangle_{\mathcal{A}} \right\| \|B\|^2 \|x\|^2 = \|B\|^2 \|\{y_i\}_{i \in I}\|^2. \end{aligned} \quad (42)$$

Then, the sum $\sum_{i \in I} \langle (CC')^{1/2} T_i^* y_i \rangle$ is convergent, and we have

$$\|T_{(C,C')}(\{y_i\}_{i \in I})\|^2 \leq \|B\|^2 \|\{y_i\}_{i \in I}\|^2. \quad (43)$$

Hence,

$$\|T_{(C,C')}\|^2 \leq \|B\|^2. \quad (44)$$

Thus, the operator $T_{(C,C')}$ is well defined, bounded, and

$$\|T_{(C,C')}\| \leq \|B\|. \quad (45)$$

For the converse, suppose that the operator $T_{(C,C')}$ is well defined, bounded, and $\|T_{(C,C')}\| \leq \|B\|$. For all $x \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} &= \sum_{i \in I} \langle C' T_i^* T_i Cx, x \rangle_{\mathcal{A}} \\
&= \sum_{i \in I} \langle (CC')^{1/2} T_i^* T_i (CC')^{1/2} x, x \rangle_{\mathcal{A}} \\
&= \langle T_{(C,C')} (\{y_i\}_{i \in I}), x \rangle_{\mathcal{A}} \\
&\leq \|T_{(C,C')}\| \left(\sum_{i \in I} \|y_i\| \right) \|x\| \\
&\leq \|T_{(C,C')}\| \left(\sum_{i \in I} \|T_i (CC')^{1/2} x\|^2 \right)^{1/2} \|x\| \\
&= \|T_{(C,C')}\| \left(\sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \right)^{1/2} \|x\|,
\end{aligned} \tag{46}$$

where $y_i = T_i (CC')^{1/2} x$.

Therefore,

$$\sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \leq \|T_{(C,C')}\|^2 \|x\|^2. \tag{47}$$

Hence,

$$\sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \leq \|B\|^2 \|x\|^2, \quad \text{as } \|T_{(C,C')}\| \leq \|B\|. \tag{48}$$

This give that $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator Bessel sequence for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$. $\square$

Theorem 4. Let $\{T_i\}_{i \in I} \subset \text{End}_{\mathcal{A}}^*(\mathcal{H})$ be a (C, C') -controlled $*$ -operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with bounds A and B , with

operator frame $S_{(C,C')}$. Let $\theta \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ be injective and has a closed range. Suppose that θ commute with C and C' . Then, $\{T_i \theta\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with frame operator $\theta^* S_{(C,C')} \theta$ with bounds $\|(\theta^* \theta)^{-1}\|^{(-1/2)} A$ and $\|\theta\| B$.

Proof. Let $\{T_i\}_{i \in I} \subset \text{End}_{\mathcal{A}}^*(\mathcal{H})$ be a (C, C') -controlled $*$ -operator frames for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with bounds A and B ; then,

$$A \langle \theta x, \theta x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle T_i C \theta x, T_i C' \theta x \rangle_{\mathcal{A}} \leq B \langle \theta x, \theta x \rangle_{\mathcal{A}} B^*. \tag{49}$$

From Lemma 2, we have

$$\|(\theta^* \theta)^{-1}\|^{-1} \langle x, x \rangle_{\mathcal{A}} \leq \langle \theta x, \theta x \rangle_{\mathcal{A}}, \quad x \in \mathcal{H}. \tag{50}$$

Hence,

$$\begin{aligned}
\|(\theta^* \theta)^{-1}\|^{(-1/2)} A \langle x, x \rangle_{\mathcal{A}} \left(\|(\theta^* \theta)^{-1}\|^{(-1/2)} A \right)^* \\
\leq A \langle \theta x, \theta x \rangle_{\mathcal{A}} A^*.
\end{aligned} \tag{51}$$

Since

$$\langle \theta x, \theta x \rangle_{\mathcal{A}} \leq \|\theta\|^2 \langle x, x \rangle_{\mathcal{A}}, \tag{52}$$

we have

$$B \langle \theta x, \theta x \rangle_{\mathcal{A}} B^* \leq \|\theta\| B \langle x, x \rangle_{\mathcal{A}} (\|\theta\| B)^*, \quad x \in \mathcal{H}. \tag{53}$$

Using (49), (51), and (53), we have

$$\begin{aligned}
\|(\theta^* \theta)^{-1}\|^{(-1/2)} A \langle x, x \rangle_{\mathcal{A}} \left(\|(\theta^* \theta)^{-1}\|^{(-1/2)} A \right)^* &\leq \sum_{i \in I} \langle T_i C \theta x, T_i C' \theta x \rangle_{\mathcal{A}} \\
&\leq \|\theta\| B \langle x, x \rangle_{\mathcal{A}} (\|\theta\| B)^*, \quad x \in \mathcal{H}.
\end{aligned} \tag{54}$$

Therefore, $\{T_i \theta\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$. Moreover, for every $x \in \mathcal{H}$, we have

$$\begin{aligned}
\theta^* S_{(C,C')} \theta &= \theta^* \sum_{i \in I} C' T_i^* T_i C \theta x = \sum_{i \in I} \theta^* C' T_i^* T_i C \theta x \\
&= \sum_{i \in I} C' (T_i \theta)^* (T_i \theta) C x.
\end{aligned} \tag{55}$$

This completes the proof. $\square$

Corollary 1. Let $\{T_i\}_{i \in I} \subset \text{End}_{\mathcal{A}}^*(\mathcal{H})$ be a (C, C') -controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$, with frame operator $S_{(C,C')}$. Then, $\{T_i S_{(C,C')}^{-1}\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

Proof. The proof is a result of (53) for $\theta = S^{-1}$. $\square$

Theorem 5. Let $\{T_i\}_{i \in I} \subset \text{End}_{\mathcal{A}}^*(\mathcal{H})$ be a (C, C') -controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with bounds A and B . Let

$\theta \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$ be surjective. Then, $\{\theta T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with bounds $A \|(\theta \theta^*)^{-1}\|^{(-1/2)}$ and $B \|\theta\|$.

Proof. From the definition of (C, C') -controlled $*$ -operator frame, we have

$$A \langle x, x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} \leq B \langle x, x \rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}. \tag{56}$$

Using Lemma 2, we have

$$\begin{aligned}
\|(\theta \theta^*)^{-1}\|^{-1} \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}} &\leq \langle \theta T_i Cx, \theta T_i C'x \rangle_{\mathcal{A}} \\
&\leq \|\theta\|^2 \langle T_i Cx, T_i C'x \rangle_{\mathcal{A}}.
\end{aligned} \tag{57}$$

From (56) and (57), we have

$$\begin{aligned}
& A \left\| (\theta\theta^*)^{-1} \right\|^{(-1/2)} \langle x, x \rangle_{\mathcal{A}} \left(A \left\| (\theta\theta^*)^{-1} \right\|^{(-1/2)} \right)^* \\
& \leq \sum_{i \in I} \langle \theta T_i C x, \theta T_i C' x \rangle \leq B \|\theta\| \langle x, x \rangle_{\mathcal{A}} (B \|\theta\|)^*, \quad x \in \mathcal{H}.
\end{aligned} \tag{58}$$

Hence, $\{\theta T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

Under those conditions, a controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with $\mathcal{H}$ a C^* -module over a unital C^* -algebras $\mathcal{A}$ is also a controlled $*$ -operator frame for $\text{End}_{\mathcal{A}}^*(\mathcal{H})$ with $\mathcal{H}$ a C^* -module over a unital C^* -algebras $\mathcal{B}$. The following theorem answers these questions. $\square$

Theorem 6. Let $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$ and $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ be two hilbert C^* -modules and let $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$ -homomorphism and θ be a map on $\mathcal{H}$ such that $\langle \theta x, \theta y \rangle_{\mathcal{B}} = \varphi(\langle x, y \rangle_{\mathcal{A}})$ for all $x, y \in \mathcal{H}$. Suppose $\{T_i\}_{i \in I} \subset \text{End}_{\mathcal{A}}^*(\mathcal{H})$ is a (C, C') -controlled $*$ -operator frame for $(\mathcal{H}, \mathcal{A}, \langle \cdot, \cdot \rangle_{\mathcal{A}})$ with frame operator $S_{\mathcal{A}}$ and lower and upper bounds A and B , respectively. If θ is surjective such that $\theta T_i = T_i \theta$ for each $i \in I$ and $\theta C = C \theta$ and $\theta C' = C' \theta$, then $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frame for $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ with frame operator $S_{\mathcal{B}}$ and lower and upper bounds $\varphi(A)$ and $\varphi(B)$, respectively, and $\langle S_{\mathcal{B}} \theta x, \theta y \rangle_{\mathcal{B}} = \varphi(\langle S_{\mathcal{A}} x, y \rangle_{\mathcal{A}})$.

Proof. Since θ is surjective, then, for every $y \in \mathcal{H}$, there exists $x \in \mathcal{H}$ such that $\theta x = y$. Using the definition of (C, C') -controlled $*$ -operator frame, we have

$$A \langle x, x \rangle_{\mathcal{A}} A^* \leq \sum_{i \in I} \langle T_i C x, T_i C' x \rangle \leq B \langle x, x \rangle_{\mathcal{A}} B^*, \quad x \in \mathcal{H}. \tag{59}$$

By Lemma 1, we have

$$\begin{aligned}
\varphi(A \langle x, x \rangle_{\mathcal{A}} A^*) & \leq \varphi \left(\sum_{i \in I} \langle T_i C x, T_i C' x \rangle_{\mathcal{A}} \right) \leq \varphi(B \langle x, x \rangle_{\mathcal{A}} B^*), \\
& x \in \mathcal{H}.
\end{aligned} \tag{60}$$

From the definition of $*$ -homomorphism, we have

$$\begin{aligned}
\varphi(A) \varphi(\langle x, x \rangle_{\mathcal{A}}) \varphi(A^*) & \leq \varphi \left(\sum_{i \in I} \langle T_i C x, T_i C' x \rangle_{\mathcal{A}} \right) \\
& \leq \varphi(B) \varphi(\langle x, x \rangle_{\mathcal{A}}) \varphi(B^*), \quad x \in \mathcal{H}.
\end{aligned} \tag{61}$$

Using the relation between θ and φ , we obtain

$$\begin{aligned}
\varphi(A) \langle \theta x, \theta x \rangle_{\mathcal{B}} (\varphi(A))^* & \leq \sum_{i \in I} \langle \theta T_i C x, \theta T_i C' x \rangle_{\mathcal{B}} \\
& \leq \varphi(B) \langle \theta x, \theta x \rangle_{\mathcal{B}} (\varphi(B))^*, \quad x \in \mathcal{H}.
\end{aligned} \tag{62}$$

Since $\theta T_i = T_i \theta$, $\theta C = C \theta$, and $\theta C' = C' \theta$, we have

$$\begin{aligned}
\varphi(A) \langle \theta x, \theta x \rangle_{\mathcal{B}} (\varphi(A))^* & \leq \sum_{i \in I} \langle T_i C \theta x, T_i C' \theta x \rangle_{\mathcal{B}} \\
& \leq \varphi(B) \langle \theta x, \theta x \rangle_{\mathcal{B}} (\varphi(B))^*, \quad x \in \mathcal{H}.
\end{aligned} \tag{63}$$

Therefore,

$$\begin{aligned}
\varphi(A) \langle y, y \rangle_{\mathcal{B}} (\varphi(A))^* & \leq \sum_{i \in I} \langle T_i C y, T_i C' y \rangle_{\mathcal{B}} \\
& \leq \varphi(B) \langle y, y \rangle_{\mathcal{B}} (\varphi(B))^*, \quad y \in \mathcal{H}.
\end{aligned} \tag{64}$$

This implies that $\{T_i\}_{i \in I}$ is a (C, C') -controlled $*$ -operator frame for $(\mathcal{H}, \mathcal{B}, \langle \cdot, \cdot \rangle_{\mathcal{B}})$ with bounds $\varphi(A)$ and $\varphi(B)$. Moreover, we have

$$\begin{aligned}
\varphi(\langle S_{\mathcal{A}} x, y \rangle_{\mathcal{A}}) & = \varphi \left(\left\langle \sum_{i \in I} T_i C x, T_i C' y \right\rangle_{\mathcal{A}} \right) \\
& = \sum_{i \in I} \varphi(\langle T_i C x, T_i C' y \rangle_{\mathcal{A}}) \\
& = \sum_{i \in I} \langle \theta T_i C x, \theta T_i C' y \rangle_{\mathcal{B}} \\
& = \sum_{i \in I} \langle T_i C \theta x, T_i C' \theta y \rangle_{\mathcal{B}} \\
& = \left\langle \sum_{i \in I} C' T_i^* T_i C \theta x, \theta y \right\rangle_{\mathcal{B}} \\
& = \langle S_{\mathcal{B}} \theta x, \theta y \rangle_{\mathcal{B}},
\end{aligned} \tag{65}$$

which completes the proof [18]. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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