Research Article

The Sigma Coindex of Graph Operations

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The sigma coindex is defined as the sum of the squares of the differences between the degrees of all nonadjacent vertex pairs. In this paper, we propose some mathematical properties of the sigma coindex. Later, we present precise results for the sigma coindices of various graph operations such as tensor product, Cartesian product, lexicographic product, disjunction, strong product, union, join, and corona product.

1. Introduction

Let G be a simple graph with a vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n_G$ and $|E(G)| = m_G$. The degree of a vertex g in G , denoted by $\deg_G(g)$, is defined as the number of incident edges to it. The complement of a G graph, denoted by $\overline{G}$, is the graph with the same vertex set. Here, any two vertices g_1 and g_2 are adjacent if and only if they are not adjacent in G . The number of edges of the graph $\overline{G}$ is denoted by $\overline{m}_G$, where $\overline{m}_G = \binom{n_G}{2} - m_G$. For other undefined notations and terminology from graph theory, the readers are referred to [1].

Chemical graph theory is the field of study of mathematical chemistry in relation to chemical graphs. The basic idea here is to reveal the properties of molecules using the information corresponding to chemical graphics. For this, topological indices are among the most used tools. Topological indices are constant numbers that reveal the structure of the graph. These constant numbers are used in the modeling of molecules in chemistry and biology. Until today, many topology indices have been defined and used as a tool in QSAR/QSPR studies.

The oldest degree-based topological indices are the first and second Zagreb indices being considered in [2]. These indices are defined as follows:

$$\begin{aligned} M_1(G) &= \sum_{g \in V(G)} d_G^2(g) = \sum_{g_1, g_2 \in E(G)} [d_G(g_1) + d_G(g_2)], \\ M_2(G) &= \sum_{g_1, g_2 \in E(G)} d_G(g_1)d_G(g_2). \end{aligned} \quad (1)$$

The first and second Zagreb coindices are defined as [3]

$$\begin{aligned} \overline{M}_1(G) &= \sum_{g_1, g_2 \notin E(G)} [d_G(g_1) + d_G(g_2)], \\ \overline{M}_2(G) &= \sum_{g_1, g_2 \notin E(G)} d_G(g_1)d_G(g_2), \end{aligned} \quad (2)$$

respectively.

The forgotten topological index was introduced by Furtula and Gutman [4], $F(G)$, as the sum of cubes of vertex degrees:

$$F(G) = \sum_{g \in V(G)} d_G^3(g) = \sum_{g_1, g_2 \in E(G)} [d_G^2(g_1) + d_G^2(g_2)]. \quad (3)$$

The forgotten coindex of a graph G is introduced as follows [5]:

$$\overline{F}(G) = \sum_{g_1, g_2 \notin E(G)} [d_G^2(g_1) + d_G^2(g_2)]. \quad (4)$$

The hyper-Zagreb index was first introduced by [6]. This index is defined as follows:

$$HM(G) = \sum_{g_1, g_2 \in E(G)} (d_G(g_1) + d_G(g_2))^2. \quad (5)$$

The hyper-Zagreb coindex was introduced by Veylaki et al. [7], as are defined as follows:

$$\overline{HM}(G) = \sum_{g_1, g_2 \notin E(G)} (d_G(g_1) + d_G(g_2))^2. \quad (6)$$

The sigma index of a graph is defined as [8]

$$\sigma(G) = \sum_{g_1, g_2 \in E(G)} (d_G(g_1) - d_G(g_2))^2 = F(G) - 2M_2(G). \quad (7)$$

In [8], the authors studied sigma index and its properties, especially the inverse problem for it. After that Jahanbani and Ediz [9] presented the properties of this index under various graph products.

With this motivation, we define the sigma coindex and the total sigma index as, respectively,

$$\begin{aligned} \bar{\sigma}(G) &= \sum_{g_1, g_2 \notin E(G)} (d_G(g_1) - d_G(g_2))^2, \\ \sigma_t(G) &= \frac{1}{2} \sum_{g_1, g_2 \in V(G)} (d_G(g_1) - d_G(g_2))^2. \end{aligned} \quad (8)$$

Graph operations are an important subject of graph theory. Many complex graphs can be obtained by applying graph operations to simpler graphs. Until today, many studies have been done on graph operations. In [10–12], the algebraic properties of tensor, lexicographic, and Cartesian products of monogenic semigroup graphs were presented. Azari [13] put forward some results on the eccentric connectivity coindex of several graph operations. In [14], the upper bounds on the multiplicative Zagreb indices of graph operations were given. Nacaroglu et al. [15] gave some bounds on the multiplicative Zagreb coindices of graph operations. Ascioğlu et al. [16] presented formulae for omega invariant of some graph operations. In [17], F index of different corona products of two given graphs was calculated. Das et al. [18] examined the Harary index of graph operations. We refer the reader to [19] for more properties and applications of graph products.

In this study, we will calculate the sigma coindex of two graphs under some graph products as corona, join, union, lexicographic product, disjunction, tensor product, Cartesian product, and strong product.

2. Some Properties of Sigma Coindex

All operations examined in this section are binary. Therefore, we will consider graphs of G and H as two finite and simple graphs. Let us examine the sigma coindices of some special graphs before moving on to the basic results.

Proposition 1. $\bar{\sigma}(G) = \sigma(\bar{G})$.

Proof. From definition of the sigma coindex, we have

$$\begin{aligned} \bar{\sigma}(G) &= \sum_{g_1, g_2 \notin E(G)} (d_G(g_1) - d_G(g_2))^2 \\ &= \sum_{g_1, g_2 \in E(\bar{G})} ((n_G - 1 - d_{\bar{G}}(g_1)) - (n_G - 1 - d_{\bar{G}}(g_2)))^2 \\ &= \sum_{g_1, g_2 \in E(\bar{G})} (d_{\bar{G}}(g_1) - d_{\bar{G}}(g_2))^2 \\ &= \sigma(\bar{G}), \end{aligned} \quad (9)$$

as required. $\square$

Proposition 2. $\bar{\sigma}(\bar{G}) = \sigma(G)$.

Proof. From definition of the sigma coindex, we have as required

$$\begin{aligned} \bar{\sigma}(\bar{G}) &= \sum_{g_1, g_2 \notin E(\bar{G})} (d_{\bar{G}}(g_1) - d_{\bar{G}}(g_2))^2 \\ &= \sum_{g_1, g_2 \in E(G)} ((n_G - 1 - d_G(g_1)) - (n_G - 1 - d_G(g_2)))^2 \\ &= \sum_{g_1, g_2 \in E(G)} (d_G(g_1) - d_G(g_2))^2 \\ &= \sigma(G). \end{aligned} \quad (10)$$

$\square$

Proposition 3. $\bar{\sigma}(G) = n_G M_1(G) - \sigma(G) - 4m_G^2$.

Proof. The proof follows by the expression $\sigma(G) + \bar{\sigma}(G) = n_G M_1(G) - 4m_G^2$ from Lemma 4 of [20]. $\square$

By the following proposition, we can give the sigma coindices of the complete graphs, star graphs, cycles, and path graphs.

Proposition 4

$$\begin{aligned} \bar{\sigma}(K_n) &= 0, \\ \bar{\sigma}(S_n) &= 0, \\ \bar{\sigma}(C_n) &= 0, \\ \bar{\sigma}(P_n) &= 2(n-3). \end{aligned} \quad (11)$$

3. Sigma Coindex under Graph Operations

In this section, we give some formulae for the sigma coindices of some graph operations as union, join, corona product, tensor product, Cartesian product, lexicographic product, and strong product.

The tensor product of graphs G and H , denoted as $G \otimes H$, is the graph with $V(G \otimes H) = V(G) \times V(H)$. The vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if g_1 is adjacent to g_2 in G and h_1 is adjacent to h_2 in H . Also we know that $d_{G \otimes H}(g_1, h_1) = d_G(g_1)d_H(h_1)$ and $m(G \otimes H) = 2m(G)m(H)$.

Let us first formulate the sigma index of the tensor products of any two graphs as shown in Theorem 1.

Theorem 1. Let $\Gamma = G \otimes H$. Then

$$\sigma(\Gamma) = F(G)F(H) - 4M_2(G)M_2(H). \quad (12)$$

The proof follows from the expressions $M_2(G \otimes H) = 2M_2(G)M_2(H)$ in Theorem 2.1 of [21], $F(G \otimes H) = F(G)F(H)$ in Theorem 7 of [22], and (7).

Now we can express the sigma coindex under the tensor product of any two graphs using Theorem 1.

Theorem 2. Let $\Gamma = G \otimes H$. Then,

$$\begin{aligned} \bar{\sigma}(\Gamma) &= n_G n_H M_1(G)M_1(H) + 4M_2(G)M_2(H) \\ &- F(G)F(H) - 16m_G^2 m_H^2. \end{aligned} \quad (13)$$

Proof. From the expression $\sigma(\Gamma) + \bar{\sigma}(\Gamma) = n_\Gamma M_1(\Gamma) - 4m_\Gamma^2$ in Lemma 4 of [20], we get

$$\bar{\sigma}(\Gamma) = n_{G \otimes H} M_1(G \otimes H) - \sigma(G \otimes H) - 4m_{G \otimes H}^2. \quad (14)$$

The proof follows from the expressions $M_1(G_1 \otimes G_2) = M_1(G_1)M_1(G_2)$ in Theorem 2.1 of [21] and Theorem 1. $\square$

Example 1. Using Theorem 2, we have

$$\begin{aligned} (1) \quad \bar{\sigma}(P_n \otimes P_m) &= 8(n+m)(nm+2) - 16(n+m)^2 + 4nm + 44, \\ (2) \quad \bar{\sigma}(P_n \otimes C_m) &= 8m(nm-2m-2), \\ (3) \quad \bar{\sigma}(P_n \otimes K_m) &= m(m-1)^2(2nm-6m+2). \end{aligned}$$

The Cartesian product of G and H ; denoted by $G \times H$; is the graph with the vertex set $V(G) \times V(H)$. The vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1 = g_2$ and h_1 is adjacent to h_2 in H ; or $h_1 = h_2$ and g_1 is adjacent to g_2 in G . Also we know that $m_{G \times H} = n_G m_H + m_G n_H$ and $d_{G \times H}(g_1, h_1) = d_G(g_1) + d_H(h_1)$, respectively.

Theorem 3. Let $\Gamma = G \times H$ be the Cartesian product of two graphs G and H . Then

$$\bar{\sigma}(\Gamma) = n_G^2 \sigma_t(H) + n_H^2 \sigma_t(G) - n_G \sigma(H) - n_H \sigma(G). \quad (15)$$

Proof. In Theorem 1 of [23] and Theorem 1 of [9], the following formulae are given, respectively:

$$M_1(G \times H) = n_H M_1(G) + n_G M_1(H) + 8m_G m_H, \quad (16)$$

$$\sigma(G \times H) = n_H \sigma(G) + n_G \sigma(H), \quad (17)$$

respectively. So the proof is completed by applying (8), (16), and (17) in Proposition 3. $\square$

$P_n \times P_m$ and $P_n \times C_m$ are known as the rectangular grid and the C_4 nanotube (see [24]), respectively.

Example 2. By using Theorem 3, we have

$$\begin{aligned} (1) \quad \bar{\sigma}(P_n \times P_m) &= 2n^2 m + 2nm^2 - 4n^2 - 4m^2 - 2n - 2m, \\ (2) \quad \bar{\sigma}(P_n \times C_m) &= 2nm^2 - 4m^2 - 2m. \end{aligned}$$

The lexicographic product $\Gamma = G[H]$ of graphs G and H is a graph with the vertex set $V(G) \times V(H)$. Any two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G[H]$ if and only if either g_1 is adjacent to g_2 in G or $g_1 = g_2$ and h_1 is adjacent to h_2 in H . Also we know that $m_{G[H]} = n_G m_H + n_H^2 m_G$ and $d_{G[H]}(g_1, h_1) = n_H d_G(g_1) + d_H(h_1)$.

We need sigma index to calculate the sigma coindex of the lexicographic product of two graphs. But Theorem 2 in [9] is not true. The correct statement is shown in Theorem 4.

Theorem 4. Let $\Gamma = G[H]$ be the lexicographic product of two graphs G and H . Then

$$\sigma(\Gamma) = n_H^4 \sigma(G) + n_G \sigma(H) + 2m_G n_H M_1(H) - 8m_G m_H^2. \quad (18)$$

In Theorem 5, we present the sigma coindex of the lexicographic product of the two graphs depending on some topological indices of these graphs.

Theorem 5. Let $\Gamma = G[H]$ be the lexicographic product of two graphs G and H . Then

$$\bar{\sigma}(\Gamma) = n_H^4 \bar{\sigma}(G) + (n_G^2 - 2m_G) \sigma_t(H) - n_G \sigma(H). \quad (19)$$

Proof. From Proposition 3, we have

$$\bar{\sigma}(\Gamma) = n_\Gamma M_1(\Gamma) - \sigma(\Gamma) - 4m_\Gamma^2. \quad (20)$$

In Theorem 3 of [23], the following formula is given as

$$M_1(\Gamma) = n_H^3 M_1(G) + n_G M_1(H) + 8n_H m_H m_G. \quad (21)$$

By applying (18) and (21) in (20), we get

$$\begin{aligned} \bar{\sigma}(\Gamma) &= n_H^4 (n_G M_1(G) - \sigma(G)) + (n_G^2 n_H - 2m_G n_H) M_1(H) \\ &- n_G \sigma(H) - 4(n_G^2 m_H^2 - 2m_G m_H^2 + n_H^4 m_G^2). \end{aligned} \quad (22)$$

Also we have

$$\sigma_t(G) = \sigma(G) + \bar{\sigma}(G) = n_G M_1(G) - 4m_G^2. \quad (23)$$

So the proof is completed by applying (23) in (22). $\square$

Example 3. The sigma coindices of the fence graph $P_n[P_m]$ and the closed fence graph $C_n[P_m]$ are given as follows:

$$\begin{aligned} (1) \quad \bar{\sigma}(P_n[P_m]) &= 2nm^4 + 2n^2 m - 6m^4 - 4nm + 4m - 4n^2 + 6n - 8, \\ (2) \quad \bar{\sigma}(C_n[P_m]) &= 2n^2 m - 4nm + 6n - 4n^2. \end{aligned}$$

The disjunction product of G and H ; denoted by $G \vee H$; is a graph with the vertex set $V(G) \times V(H)$. The vertices (g_1, h_1) and (g_2, h_2) are adjacent iff either g_1 is adjacent to

g_2 in G ; or h_1 is adjacent to h_2 in H . Also we know that $m_{G \vee H} = n_G^2 m_H + m_G n_H^2 - 2m_G m_H$ and $d_{G \vee H}(g_1, h_1) = n_H d_G(g_1) + n_G d_H(h_1) - d_G(g_1) d_H(h_1)$.

Theorem 6. Let $\Gamma = G \vee H$ be the disjunctive product of the graphs G and H . Then

$$\begin{aligned} \bar{\sigma}(\Gamma) &= \left[n_H^3 - 4n_H m_H + M_1(H) + 2n_H^2 \overline{m_H} - 2n_H \overline{M_1}(H) + \frac{1}{2} \overline{HM}(H) \right] \bar{\sigma}(G) \\ &\quad + \left[n_G^3 - 4n_G m_G + M_1(G) + 2n_G^2 \overline{m_G} - 2n_G \overline{M_1}(G) + \frac{1}{2} \overline{HM}(G) \right] \bar{\sigma}(H). \end{aligned} \quad (24)$$

Proof. From definition of the disjunctive product and the sigma coindex, we have

$$\begin{aligned} \bar{\sigma}(\Gamma) &= \sum_{xy \notin E(\Gamma)} [d_\Gamma(x) - d_\Gamma(y)]^2 \\ &= \sum_{g_1 \in V(G)} \sum_{h_1 h_2 \notin E(H)} (d_{G \vee H}(g_1, h_1) - d_{G \vee H}(g_1, h_2))^2 \\ &\quad + \sum_{h_1 \in V(H)} \sum_{g_1 g_2 \notin E(G)} (d_{G \vee H}(g_1, h_1) - d_{G \vee H}(g_2, h_1))^2 \\ &\quad + \sum_{g_1 g_2 \notin E(G)} \sum_{h_1 h_2 \notin E(H)} (d_{G \vee H}(g_1, h_1) - d_{G \vee H}(g_2, h_2))^2. \end{aligned} \quad (25)$$

In other world, we have

$$\begin{aligned} \bar{\sigma}(\Gamma) &= \sum_{g_1 \in V(G)} \sum_{h_1 h_2 \notin E(H)} (n_G - d_G(g_1))^2 (d_H(h_1) - d_H(h_2))^2 \\ &\quad + \sum_{h_1 \in V(H)} \sum_{g_1 g_2 \notin E(G)} (n_H - d_H(h_1))^2 (d_G(g_1) - d_G(g_2))^2 \\ &\quad + \sum_{g_1 g_2 \notin E(G)} \sum_{h_1 h_2 \notin E(H)} \left[\left(n_H - \frac{1}{2} (d_H(h_1) + d_H(h_2)) \right) (d_G(g_1) - d_G(g_2)) + \left(n_G - \frac{1}{2} (d_G(g_1) + d_G(g_2)) \right) (d_H(h_1) - d_H(h_2)) \right]^2. \end{aligned} \quad (26)$$

The proof is completed by applying (1), (2), (6), and (8) in (26). $\square$

Let G and H be two graphs. The strong product of G and H is a graph with the vertex set of $V(G) \times V(H)$, denoted by $G \boxtimes H$. The vertices (g_1, h_1) and (g_2, h_2) are adjacent iff either $g_1 = g_2$ and $h_1 h_2 \in E(H)$; or $h_1 = h_2$ and $g_1 g_2 \in E(G)$; or

$g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. Also we know that $m(G \boxtimes H) = n(G)m(H) + m(G)n(H) + 2m(G)m(H)$ and $d_{G \boxtimes H}(g_1, h_1) = d_G(g_1) + d_H(h_1) + d_G(g_1)d_H(h_1)$.

Theorem 7. Let $\Gamma = G \boxtimes H$ be the strong product of two graphs G and H . Then

$$\begin{aligned} \bar{\sigma}(\Gamma) &= \left[2\overline{m_H} + 2\overline{M_1}(H) + \frac{1}{2} \overline{HM}(H) \right] \bar{\sigma}(G) + \left[2\overline{m_G} + 2\overline{M_1}(G) + \frac{1}{2} \overline{HM}(G) \right] \bar{\sigma}(H) \\ &\quad + \left[\frac{1}{2} n_H M_1(H) + 4m_H n_H + 2m_H^2 + n_H^2 \right] \bar{\sigma}(G) + \left[\frac{1}{2} n_G M_1(G) + 4m_G n_G + 2m_G^2 + n_G^2 \right] \bar{\sigma}(H). \end{aligned} \quad (27)$$

Proof. From definition of the strong product and the sigma coindex, we have

$$\bar{\sigma}(G \boxtimes H) = \sum_{(g_1, h_1) (g_2, h_2) \notin E(G \boxtimes H)} [d_{G \boxtimes H}(g_1, h_1) - d_{G \boxtimes H}(g_2, h_2)]^2. \quad (28)$$

The sum can be split into five parts:

$$\begin{aligned} & \sum_{g_1=g_2 \in V(G)} \sum_{h_1, h_2 \notin E(H)}, \\ & \sum_{h_1=h_2 \in V(H)} \sum_{h_1, h_2 \notin E(H)}, \\ & \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)}, \\ & \sum_{g_1, g_2 \notin E(G)} \sum_{h_1, h_2 \in E(H)}, \\ & \sum_{g_1, g_2 \notin E(G)} \sum_{h_1, h_2 \notin E(H)}. \end{aligned} \quad (29)$$

which we denote by S_1, S_2, S_3, S_4 , and S_5 , respectively. We have

$$\begin{aligned} S_1 &= \sum_{g_1 \in V(G)} \sum_{h_1, h_2 \notin E(H)} [d_{G \boxtimes H}(g_1, h_1) - d_{G \boxtimes H}(g_1, h_2)]^2 \\ &= \sum_{g_1 \in V(G)} \sum_{h_1, h_2 \notin E(H)} (d_G(g_1) + 1)^2 (d_H(h_1) - d_H(h_2))^2 \\ &= [M_1(G) + 4m_G + n_G] \bar{\sigma}(H). \end{aligned} \quad (30)$$

Similarly, we get

$$\begin{aligned} S_2 &= \sum_{h_1 \in V(H)} \sum_{g_1, g_2 \notin E(G)} [d_{G \boxtimes H}(g_1, h_1) - d_{G \boxtimes H}(g_2, h_1)]^2 \\ &= [M_1(H) + 4m_H + n_H] \bar{\sigma}(G). \end{aligned} \quad (31)$$

On the contrary, we have

$$\begin{aligned} S_3 &= \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)} [d_{G \boxtimes H}(g_1, h_1) - d_{G \boxtimes H}(g_2, h_2)]^2 \\ &= \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)} \left(\left[1 + \frac{1}{2} (d_H(h_1) + d_H(h_2)) \right] (d_G(g_1) - d_G(g_2)) + \left[1 + \frac{1}{2} (d_G(g_1) + d_G(g_2)) \right] (d_H(h_1) - d_H(h_2)) \right)^2 \\ &= 2 \sum_{g_1, g_2 \in E(G)} \sum_{h_1, h_2 \notin E(H)} \left(\left[1 + d_H(h_1) + d_H(h_2) + \frac{1}{4} (d_H(h_1) + d_H(h_2))^2 \right] (d_G(g_1) - d_G(g_2))^2 \right. \\ &\quad \left. + \left[1 + d_G(g_1) + d_G(g_2) + \frac{1}{4} (d_G(g_1) + d_G(g_2))^2 \right] (d_H(h_1) - d_H(h_2))^2 \right) \\ &= \left[2\overline{m_H} + 2\overline{M_1}(H) + \frac{1}{2}\overline{HM}(H) \right] \sigma(G) + \left[2m_G + 2M_1(G) + \frac{1}{2}HM(G) \right] \bar{\sigma}(H). \end{aligned} \quad (32)$$

Finally, similar to (32), we get

$$\begin{aligned} S_4 &= \sum_{g_1, g_2 \notin E(G)} \sum_{h_1, h_2 \in E(H)} [d_{G \boxtimes H}(g_1, h_1) - d_{G \boxtimes H}(g_2, h_2)]^2 \\ &= \left[2\overline{m_G} + 2\overline{M_1}(G) + \frac{1}{2}\overline{HM}(G) \right] \sigma(H) + \left[2m_H + 2M_1(H) + \frac{1}{2}HM(H) \right] \bar{\sigma}(G), \end{aligned} \quad (33)$$

$$\begin{aligned} S_5 &= \sum_{g_1, g_2 \notin E(G)} \sum_{h_1, h_2 \notin E(H)} [d_{G \boxtimes H}(g_1, h_1) - d_{G \boxtimes H}(g_2, h_2)]^2 \\ &= \left[2\overline{m_H} + 2\overline{M_1}(H) + \frac{1}{2}\overline{HM}(H) \right] \bar{\sigma}(G) + \left[2\overline{m_G} + 2\overline{M_1}(G) + \frac{1}{2}\overline{HM}(G) \right] \bar{\sigma}(H). \end{aligned} \quad (34)$$

The proof is completed by using (30)–(34) and the relations $HM(G) + \overline{HM}(G) = (n_G - 2)M_1(G) + 4m_G^2$ in

Theorem 2.1 of [25], $M_1(G) + \overline{M_1}(G) = 2m_G(n_G - 1)$ in Theorem 1 of [26]. $\square$

As an application of Theorem 7, we give below the sigma coindices of $P_n \boxtimes P_m$ and $P_n \boxtimes C_m$.

Example 4

$$\begin{aligned} P_n \boxtimes P_m &= 6(n+m)(3nm - 6n - 6m - 1) + 28nm + 140, \\ P_n \boxtimes C_m &= 18m^2 + 18nm^2 - 108m. \end{aligned} \quad (35)$$

Let G and H be vertex-disjoint graphs. The union of graphs G and H , denoted $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Theorem 8. Let $\Gamma = G \cup H$. Then

$$\bar{\sigma}(\Gamma) = \bar{\sigma}(G) + \bar{\sigma}(H) + n_H M_1(G) + n_G M_1(H) - 8m_G m_H. \quad (36)$$

Proof. From the definition of the sigma coindex of a graph, we have

$$\begin{aligned} \bar{\sigma}(\Gamma) &= \sum_{xy \notin E(G)} (d_\Gamma(x) - d_\Gamma(y))^2 \\ &= \sum_{g_1 g_2 \notin E(G)} (d_G(g_1) - d_G(g_2))^2 + \sum_{h_1 h_2 \notin E(H)} (d_H(h_1) - d_H(h_2))^2 \\ &\quad + \sum_{g \in V(G)} \sum_{h \in V(H)} (d_G(g) - d_H(h))^2 \\ &= \bar{\sigma}(G) + \bar{\sigma}(H) + \sum_{g \in V(G)} \sum_{h \in V(H)} (d_G^2(g) + d_H^2(h) - 2d_G(g)d_H(h)) \\ &= \bar{\sigma}(G) + \bar{\sigma}(H) + n_H M_1(G) + n_G M_1(H) - 8m_G m_H. \end{aligned} \quad (37)$$

□

Example 5. $\bar{\sigma}(K_{n_1} \cup K_{n_2}) = n_1 n_2 (n_1 - n_2)^2$.

Let G and H be vertex-disjoint graphs. Then the join, $G + H$, of G and H , is the supergraph of $G \cup H$ in which each vertex of G is adjacent to every vertex of H . The join of two graphs is also known as their sum. Thus, for example, the complete bipartite graph is $K_n + K_m = K_{n,m}$. The degree of a vertex x of $G + H$ is defined by

$$d_{G+H}(x) = \begin{cases} d_G(x) + n_H, & \text{if } x \in V(G), \\ d_H(x) + n_G, & \text{if } x \in V(H). \end{cases} \quad (38)$$

Theorem 10. Let $\Gamma = G + H$. Then

$$\bar{\sigma}(\Gamma) = \bar{\sigma}(G) + \bar{\sigma}(H). \quad (39)$$

Proof. From the definition of the sigma coindex of a graph, we have

$$\begin{aligned} \bar{\sigma}(\Gamma) &= \sum_{xy \notin E(\Gamma)} (d_\Gamma(x) - d_\Gamma(y))^2 \\ &= \sum_{g_1 g_2 \notin E(G)} ((d_G(g_1) + n_H) - (d_G(g_2) + n_H))^2 \\ &\quad + \sum_{h_1 h_2 \notin E(H)} ((d_H(h_1) + n_G) - (d_H(h_2) + n_G))^2 \\ &= \bar{\sigma}(G) + \bar{\sigma}(H). \end{aligned} \quad (40)$$

□

Let $G = G_1 + G_2 + \dots + G_p$. Thus $\overline{n_{G_i}} = n - n_{G_i}$ ($1 \leq i \leq p$). Also we have

$$d_G(u) = d_{G_i}(u) + n_{G_i}, \quad u \in V(G_i). \quad (41)$$

From Theorem 10, we have the following result.

Corollary 1. Let $G_1, G_2, \dots, G_p$ be vertex-disjoint graphs. If $G = G_1 + G_2 + \dots + G_p$, then

$$\bar{\sigma}(G) = \sum_{i=1}^p \bar{\sigma}(G_i). \quad (42)$$

The graph $G + K_1$ is called suspension of G (see [27]). By using Theorem 10, we get Example 6.

Example 6. $\bar{\sigma}(G + K_1) = \bar{\sigma}(G)$.

The corona product of graphs G and H , denoted $G \circ H$, is the graph obtained by taking one copy of G and n_H copies of H and then joining the i -th vertex of G to every vertex in the i -th copy of H for $1 \leq i \leq n_G$. Corona product operation is closed with identity (see [28]).

Theorem 11. Let $\Gamma = G \circ H$. Then

$$\bar{\sigma}(\Gamma) = \bar{\sigma}(G) + n_G \bar{\sigma}(H) + 2 \binom{n_G}{2} \sigma_t(H) + (n_G - 1)K, \quad (43)$$

where

$$K = n_H M_1(G) + n_G M_1(H) + 4m_G n_H (n_H - 1) - 4m_H n_G (n_H - 1) - 8m_G m_H + n_G n_H (n_H - 1)^2. \quad (44)$$

Proof. From definition of the sigma coindex, we have

$$\begin{aligned} \bar{\sigma}(\Gamma) &= \sum_{xy \notin E(\Gamma)} (d_\Gamma(x) - d_\Gamma(y))^2 \\ &= \sum_{g_1, g_2 \notin E(G)} ((d_G(g_1) + n_H) - (d_G(g_2) + n_H))^2 \\ &\quad + n_G \sum_{h_1, h_2 \notin E(H)} ((d_H(h_1) + 1) - (d_H(h_2) + 1))^2 \\ &\quad + (n_G - 1) \sum_{g_1 \in V(G)} \sum_{h_1 \in V(H)} ((d_G(g_1) + n_H) - (d_H(h_1) + 1))^2 \\ &\quad + \binom{n_G}{2} \sum_{h_1, h_2 \in V(H)} ((d_H(h_1) + 1) - (d_H(h_2) - 1))^2 \\ &= \bar{\sigma}(G) + n_G \bar{\sigma}(H) + 2 \binom{n_G}{2} \sigma_t(H) + (n_G - 1)K. \end{aligned} \quad (45)$$

□

Let $k_1, k_2, \dots, k_n$ be nonnegative integers. The thorn graph of the graph G , denoted by G^* , is a graph obtained by attaching k_i new vertices of degree one to the vertex g_i of the graph G , $i = 1, 2, \dots, n$ (see [29]). If $k_1 = k_2 = \dots = k_n = k$, then $G^{*,k} \cong G \circ \overline{K_k}$, where $\overline{K_k}$ is the complement of a complete graph K_k .

Corollary 2. $\bar{\sigma}(G^{*,k}) = \bar{\sigma}(G) + k(n-1)M_1(G) + k(k-1)(n-1)[4m + n(k-1)]$.

4. Conclusions

In this paper, we have presented the exact formulae for the sigma coindices of graphs under some graph operations. We have also applied these results to some special graph types. However, there are also graph products that are not presented here. This remains as an open problem.

Data Availability

No data were used to support the study.

Conflicts of Interest

The author declares no conflicts of interest.

References

- [1] J. L. Gross and J. Yellen, *Handbook of Graph Theory*, Chapman Hall, CRC Press, New York, NY, USA, 2004.
- [2] I. Gutman and N. Trinajstić, "Graph theory and molecular orbitals. Total π -electron energy of alternant hydrocarbons," *Chemical Physics Letters*, vol. 17, no. 4, pp. 535–538, 1972.
- [3] T. Došlić, "Vertex-weighted Wiener polynomials for composite graphs," *Ars Mathematica Contemporanea*, vol. 1, no. 1, pp. 66–80, 2008.
- [4] B. Furtula and I. Gutman, "A forgotten topological index," *Journal of Mathematical Chemistry*, vol. 53, no. 4, pp. 1184–1190, 2015.
- [5] A. Khaksari and M. Ghorbani, "On the forgotten topological index," *Iranian Journal of Mathematical Chemistry*, vol. 8, no. 3, pp. 327–338, 2017.
- [6] G. H. Shirdel, H. Rezapour, and A. M. Sayadi, "The hyperzagreb index of graph operations," *Iranian Journal of Mathematical Chemistry*, vol. 4, no. 2, pp. 213–220, 2013.
- [7] M. Veylaki, M. J. Nikmehr, and H. A. Tavallaee, "The third and hyper-Zagreb coindices of some graph operations," *Journal of Applied Mathematics and Computing*, vol. 50, no. 1–2, pp. 315–325, 2015.
- [8] I. Gutman, M. Togan, A. Yurtas, A. S. Cevik, and I. N. Cangul, "Inverse problem for sigma index," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 79, no. 2, pp. 491–508, 2018.
- [9] A. Jahanbani and S. Ediz, "The sigma index of graph operations," *Sigma Journal of Engineering and Natural Sciences*, vol. 37, no. 1, pp. 155–162, 2019.
- [10] N. Akgunes, K. C. Das, and A. S. Cevik, "Some properties on the tensor product of graphs obtained by monogenic semi-groups," *Applied Mathematics and Computation*, vol. 235, pp. 352–357, 2014.
- [11] N. Akgunes, K. C. Das, A. S. Cevik, and I. N. Cangul, "Some properties on the lexicographic product of graphs obtained by

- monogenic semigroups," *Journal of Inequalities and Applications*, vol. 238, 2013.
- [12] K. C. Das, N. Akgunes, and A. S. Cevik, "On a graph of monogenic semigroups," *Journal of Inequalities and Applications*, vol. 44, 2013.
 - [13] M. Azari, "Eccentric connectivity coindex under graph operations," *Journal of Applied Mathematics and Computing*, vol. 62, no. 1-2, pp. 23–35, 2020.
 - [14] K. C. Das, A. Yurttas, M. Togan, A. S. Cevik, and I. N. Cangul, "The multiplicative Zagreb indices of graph operations," *Journal of Inequalities and Applications*, vol. 90, 2013.
 - [15] Y. Nacaroglu and A. D. Maden, "The multiplicative Zagreb coindices of graph operations," *Utilitas Mathematica*, vol. 102, pp. 19–38, 2017.
 - [16] M. Ascioğlu, M. Demirci, and I. N. Cangul, "Omega invariant of union, join and corona product of two graphs," *Advanced Studies in Contemporary Mathematics*, vol. 30, no. 3, pp. 297–306, 2020.
 - [17] V. Loksha, S. Jain, A. S. Cevik, and I. N. Cangul, "New results on the F-index of graphs based on corona-type products of graphs," *Proceedings of the Jangjeon Mathematical Society*, vol. 23, no. 2, pp. 141–147, 2020.
 - [18] K. C. Das, K. Xu, I. N. Cangul, A. S. Cevik, and A. Graovac, "On the harary index of graph operations," *Journal of Inequalities and Applications*, vol. 339, pp. 1–16, 2013.
 - [19] W. Imrich and S. Klavžar, *Product Graphs. Structure and Recognition*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, NY, USA, 2000.
 - [20] Y. Ma, S. Cao, Y. Shi, M. Dehmer, and C. Xia, "Nordhaus-Gaddum type results for graph irregularities," *Applied Mathematics and Computation*, vol. 343, pp. 268–272, 2019.
 - [21] Z. Yarahmadi, "Computing some topological indices of tensor product of graphs," *Iranian Journal of Mathematical Chemistry*, vol. 2, no. 1, pp. 109–118, 2011.
 - [22] N. De, S. MA. Nayeem, and A. Pal, "F-index of some graph operations," *Discrete Mathematics, Algorithms and Applications*, vol. 8, no. 2, Article ID 1650025, 2016.
 - [23] M. H. Khalifeh, H. Yousefi-Azari, and A. R. Ashrafi, "The first and second Zagreb indices of some graph operations," *Discrete Applied Mathematics*, vol. 157, no. 4, pp. 804–811, 2009.
 - [24] N. De, S. M. Abu Nayeem, and A. Pal, "F-coindex of some graph operations," *Springer*, vol. 5, no. 221, pp. 1–13, 2016.
 - [25] I. Gutman, "On hyper-Zagreb index and coindex," *Bulletin (Académie serbe des sciences et des arts. Classe des sciences mathématiques et naturelles. Sciences mathématiques)*, vol. 42, pp. 1–8, 2017.
 - [26] I. Gutman, B. Furtula, Ž. K. Vukićević, and G. Popivoda, "On Zagreb indices and coindices," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 74, no. 1, pp. 5–16, 2015.
 - [27] Z. Yarahmadi, T. Došlić, and A. R. Ashrafi, "The bipartite edge frustration of composite graphs," *Discrete Applied Mathematics*, vol. 158, no. 14, pp. 1551–1558, 2010.
 - [28] V. N. Mishra, S. Delen, and I. N. Cangul, "Algebraic structure of graph operations in terms of degree sequences," *International Journal of Analysis and Applications*, vol. 16, no. 6, pp. 809–821, 2018.
 - [29] Y. Nacaroglu, "On the eccentric adjacency index of graphs," *New Trends in Mathematical Sciences*, vol. 6, no. 3, pp. 128–136, 2018.