Research Article

(p, q)-Extended Struve Function: Fractional Integrations and Application to Fractional Kinetic Equations

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In this paper, the generalized fractional integral operators involving Appell's function $F_3(\cdot)$ in the kernel due to Marichev–Saigo–Maeda are applied to the (p, q)-extended Struve function. The results are stated in terms of Hadamard product of the Fox–Wright function $_r\psi_s(z)$ and the (p, q)-extended Gauss hypergeometric function. A few of the special cases (Saigo integral operators) of our key findings are also reported in the corollaries. In addition, the solutions of a generalized fractional kinetic equation employing the concept of Laplace transform are also obtained and examined as an implementation of the (p, q)-extended Struve function. Technique and findings can be implemented and applied to a number of similar fractional problems in applied mathematics and physics.

1. Introduction

The Struve functions are interesting special functions that also provide solutions to a variety of issues formulated in terms of discrete, integral, and differential equations of fractional order; thus, many authors have recently become interested in the domain of fractional calculus and its implementations. Therefore, an extremely large number of authors (for details, see [1–7]) have also researched, in detail, the features, implementations, and numerous extensions of different fractional calculus operators. The research monographs by Miller and Ross [8] can be referred to for comprehensive overview of fractional calculus operators (FCOs) together with their characteristics and potential applications. The (p, q)-variant (when $p = q$, p-variant) associated with a set of similar higher transcendental hypergeometric style special functions (see [9–13]) has recently been investigated by several authors. In specific, Maŝirevic̀ et al. [14] introduced and analysed the (p, q)-extended Struve function $H_{\mu,p,q}(z)$ of the first kind of order δ with $\Re(\delta) > (-1/2)$ and $\min\{p, q\} \geq 0$ when $p = q = 0$ in this manner:

$$H_{\delta,p,q}(z) = \frac{2(z/2)^{\delta+1}}{\sqrt{\pi}\Gamma(\delta + (1/2))} \sum_{k=0}^{\infty} (-1)^k \mathfrak{B}\left(k + 1, \delta + \frac{1}{2}; p, q\right) \frac{z^{2k}}{(2k+1)!}, \tag{1}$$

$$= \frac{z^{\delta+1}}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k + 1, \delta + (1/2); p, q)}{(3/2)_k \mathfrak{B}(1, \delta + (1/2))k!} \left(\frac{-z^2}{4}\right)^k. \tag{2}$$

Choi et al. [15] introduced the (p, q)-extended beta function as

$$\mathfrak{B}(\varsigma, \vartheta: p, q) = \int_0^1 t^{\varsigma-1}(1-t)^{\vartheta-1}e^{-((p/t)+(q/1-t))}\mathrm{d}t, \tag{3}$$

$$(\min\{\mathfrak{R}(\varsigma), \mathfrak{R}(\vartheta)\} > 0; \min\{\mathfrak{R}(p), \mathfrak{R}(q)\} \geq 0).$$

The more details and generalized form of the definitions (3) are considered in [16]. It is clear that the case $p = 0 = q$ automatically reduces the classical Struve function $H_\delta(z)$ of the first kind (see, e.g., [17] p. 328, equation (2)):

$$H_\delta(z) = \sum_{k=0}^{\infty} (-1)^k \frac{(z/2)^{2k+\delta+1}}{\Gamma(\delta+(3/2))\Gamma(\delta+k+(3/2))}. \tag{4}$$

The Struve function is widely studied in the reference to properties and applications in several papers (see details [18–22]).

FCO involving different special functions have established major significance and requirements in the simulation of related structures in diverse domain of engineering and science, such as quantum mechanics and turbulence, particle physics, nonlinear optimization system, and nonlinear control theory, controlled thermonuclear fusion, nonlinear natural processes, image processing, quantum mechanics, and astrophysics.

In the context of the success of Saigo operators [23, 24], in their study of different function spaces and their use in differential equations and integral equations, Saigo and Maeda [25] presented the corresponding generalized fractional differential and integral operators in any complex order with Appell's function $F_3(\cdot)$ in the kernel as follows. Let $\varsigma, \varsigma', \vartheta, \vartheta', \varpi \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus operators are defined by the following equations:

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x) = \frac{x^{-\varsigma}}{\Gamma(\varpi)}\int_0^x (x-t)^{\varpi-1}t^{-\varsigma'}, \tag{5}$$

$$\times F_3\left(\varsigma, \varsigma', \vartheta, \vartheta'; \varpi; 1-\frac{t}{x}, 1-\frac{x}{t}\right)f(t)\mathrm{d}t, \quad (\mathfrak{R}(\varpi) > 0)$$

$$= \left(\frac{d}{\mathrm{d}x}\right)^k \left(I_{0+}^{\varsigma,\varsigma',\vartheta+k,\vartheta',\varpi+k}f\right)(x), \tag{6}$$

$$(\mathfrak{R}(\varpi) \leq 0; k = [-\mathfrak{R}(\varpi)] + 1);$$

$$\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x) = \frac{x^{-\varsigma'}}{\Gamma(\varpi)}\int_x^{\infty} (t-x)^{\varpi-1}t^{-\varsigma}, \tag{7}$$

$$\times F_3\left(\varsigma, \varsigma', \vartheta, \vartheta'; \varpi; 1-\frac{x}{t}, 1-\frac{t}{x}\right)f(t)\mathrm{d}t, \quad (\mathfrak{R}(\varpi) > 0)$$

$$= \left(-\frac{d}{\mathrm{d}x}\right)^k \left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta'+k,\varpi+k}f\right)(x), \tag{8}$$

$$(\mathfrak{R}(\varpi) \leq 0; k = [-\mathfrak{R}(\varpi)] + 1),$$

$$\left(D_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x) = \left(I_{0+}^{-\varsigma',-\varsigma,-\vartheta',-\vartheta,-\varpi}f\right)(x), \tag{9}$$

$$= \left(\frac{d}{\mathrm{d}x}\right)^k \left(I_{0+}^{-\varsigma',-\varsigma,-\vartheta'+k,-\vartheta,-\varpi+k}f\right)(x),$$

$$(\mathfrak{R}(\varpi) > 0; k = [\mathfrak{R}(\varpi)] + 1);$$

$$\left(D_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}f\right)(x) = \left(I_{-}^{-\varsigma',-\varsigma,-\vartheta',-\vartheta,-\varpi}f\right)(x)$$

$$= \left(-\frac{d}{\mathrm{d}x}\right)^k \left(I_{-}^{-\varsigma',-\varsigma,-\vartheta'+k,-\varpi+k}f\right)(x),$$

$$(\mathfrak{R}(\varpi) > 0; k = [\mathfrak{R}(\varpi)] + 1). \tag{10}$$

The interested reader may refer to the monograph by Srivastava and Karlsson [26] for the concept of Appell function $F_3(\cdot)$.

The image formulas for a power function, under operators (5) and (7), are given by Saigo and Maeda [25] as follows:

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}x^{\tau-1}\right)(x) = x^{\tau-\varsigma-\varsigma'+\varpi-1}$$

$$\times \Gamma\left[\begin{matrix} \tau, \tau+\varpi-\varsigma-\varsigma'-\vartheta, \tau+\vartheta'-\varsigma' \\ \tau+\vartheta', \tau+\varpi-\varsigma-\varsigma', \tau+\varpi-\varsigma'-\vartheta \end{matrix}\right], \tag{11}$$

where $\mathfrak{R}(\tau) > \max\{0, \mathfrak{R}(\varsigma+\varsigma'+\vartheta-\varpi), \mathfrak{R}(\varsigma'-\vartheta')\}$ and $\mathfrak{R}(\varpi) > 0$.

$$\left(I_{-}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}x^{\tau-1}\right)(x) = x^{\tau+\varpi-\varsigma-\varsigma'-1}$$

$$\times \frac{\Gamma(1-\tau-\varpi+\varsigma+\varsigma')\Gamma(1-\tau+\varsigma+\vartheta'-\varpi)\Gamma(1-\tau-\vartheta)}{\Gamma(1-\tau)\Gamma(1-\tau+\varsigma+\varsigma'+\vartheta'-\varpi)\Gamma(1-\tau+\varsigma-\vartheta)}, \tag{12}$$

where $\mathfrak{R}(\gamma) > 0, \mathfrak{R}(\varpi) < 1 + \min\{\mathfrak{R}(-\vartheta), \mathfrak{R}(\varsigma+\varsigma'-\varpi), \mathfrak{R}(\varsigma+\vartheta'-\varpi)\}$.

Here, we used the $\Gamma\left[\begin{matrix}:::\\:::\end{matrix}\right]$ symbol, which represents a fraction of several of the Gamma functions.

We will need the definition of the Hadamard product (or convolution) of two analytical properties for our present investigation. It will help us decompose a newly generated function into two existing functions. In fact, if one of the two power series defines a whole function, then the Hadamard product series also defines a whole function. In reality, let

$$f(z) = \sum_{l=0}^{\infty} a_l z^l \left(|z| < \mathfrak{R}_f\right),$$

$$g(z) = \sum_{l=0}^{\infty} b_l z^l \left(|z| < \mathfrak{R}_g\right), \tag{13}$$

be two given power series whose radii of convergence are given by $\mathfrak{R}_f$ and $\mathfrak{R}_g$, respectively. Then, their Hadamard product is a power series defined by

$$(f * g)(z) = \sum_{l=0}^{\infty} a_l b_l z^l = (g * f)(z) (|x| < \mathfrak{R}), \qquad (14)$$

whose radius of convergence $\mathfrak{R}$ is

$$\frac{1}{\mathfrak{R}} = \lim_{l \to \infty} \sup\left(|a_l b_l|\right)^{(1/l)} \le \left(\lim_{l \to \infty} \sup\left(|a_l|\right)^{(1/l)} \right)\left(\lim_{l \to \infty} \sup\left(|b_l|\right)^{(1/l)} \right) = \frac{1}{\mathfrak{R}_f \cdot \mathfrak{R}_g}, \qquad (15)$$

$$\mathfrak{R} \ge \mathfrak{R}_f \cdot \mathfrak{R}_g.$$

The results in Theorems 1 and 2 will be expressed in a Hadamard product of (p,q)-extended Gauss hypergeometric function (see [15], p. 354, equation (8)):

$$_pF_q(c,b;a;z) = \sum_{l=0}^{\infty} \frac{\mathfrak{B}(b+l, a-b; p,q)}{\mathfrak{B}(b, a-b)} \frac{z^l}{l!} \ (|z| < 1, \mathfrak{R}(a) > \mathfrak{R}(b) > 0), \qquad (16)$$

where $\mathfrak{B}(c,b)$ is the classical beta function [27] and Fox–Wright function $p\Psi q(z) (p,q \in \mathbb{N}_0)$ [28].

$$p\Psi q \left[\begin{array}{c} (\varsigma_1, P_1), \ldots, (\varsigma_p, P_p); \\ \\ (\vartheta_1, Q_1), \ldots, (\vartheta_q, Q_q); \end{array} z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(\varsigma_1 + P_1 n), \ldots, \Gamma(\varsigma_p + P_p n)}{\Gamma(\vartheta_1 + Q_1 n), \ldots, \Gamma(\vartheta_p + Q_p n)} \frac{z^k}{k!},$$

$$\left(P_j \in \mathfrak{R}^+ (j = 1, \ldots, p), Q_j \in \mathfrak{R}^+ (j = 1, \ldots, q); 1 + \sum_{j=0}^{q} Q_j - \sum_{j=0}^{p} P_j \ge 0 \right), \qquad (17)$$

where the convergence condition holds true for

$$|z| < \nabla = \left(\prod_{j=1}^{p} P_j^{-P_j} \right) \cdot \left(\prod_{j=1}^{q} Q_j^{Q_j} \right). \qquad (18)$$

In this paper, we aim to investigate compositions of the generalized fractional integration operators involving (p,q)-extended Struve function $H_{\delta,p,q}(z)$. Also, we consider (2) to achieve the solution of the generalized fractional kinetics equations (FKEs). Our approach here is based on Laplace transformation, and we plan to broaden our results by using the Sumudu transformation in a future career.

2. Fractional Integrations Approach

For this section, we assume that $\varsigma, \varsigma', \vartheta, \vartheta', \varpi, \tau, \delta, \omega \in \mathbb{C}$ such that $\mathfrak{R}(\varpi) > 0$, $\min\{\mathfrak{R}(p), \mathfrak{R}(q)\} > 0$, $\mathfrak{R}(\delta) > (-3/2)$. Furthermore, let the constants satisfy the condition $\varsigma_i, \vartheta_j \in \mathbb{C}$,

and $P_i, Q_j \in \mathfrak{R} (P_i, Q_j \ne 0 i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$, such that condition (17) is also satisfied.

2.1. Left-Sided Generalized Fractional Integration of (p,q)-Extended Struve Function.
In this segment, we establish image formulas for the (p,q)-extended Struve function involving left-sided operators of M-S-M fractional integral operators (5), in terms of the Hadamard product of the Fox–Wright function $r\psi s(z)$ and the (p,q)-extended Gauss hypergeometric function. These formulas are set out in the preceding theorems.

Theorem 1. If $\mathfrak{R}(\varpi) > 0$, $\mathfrak{R}(\tau + \delta + 1) > \max\{0, \mathfrak{R}(\varsigma + \varsigma' + \vartheta - \varpi), \mathfrak{R}(\varsigma' - \vartheta')\}$, then the generalized fractional integration $I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\varpi}$ of the (p,q)-extended Struve function $H_{\delta,p,q}(z)$ is given by

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right) = \sqrt{\pi}x^{\tau-\varsigma-\varsigma'+\omega+\delta}\frac{(\omega/2)^{\delta+1}}{\Gamma(\delta+(3/2))}$$

$$\times {}_pF_q\left[\begin{array}{c}1,1; \\ \delta+(3/2);\end{array}-\frac{\omega^2x^2}{4}\right]*{}_3\Psi_4\left[\begin{array}{c}(\tau+\delta+\omega-\varsigma+\varsigma'-\vartheta+1,2), \\ ((3/2),1),(\tau+\delta+\vartheta'+1,2),\end{array}\right.$$

$$\left.\begin{array}{c}(\tau+\delta+\vartheta'-\varsigma'+1,2),(\tau+\delta+1,2); \\ (\tau+\delta+\omega-\varsigma-\varsigma'+1,2),(\tau+\delta+\omega-\varsigma'-\vartheta+1,2);\end{array}-\frac{\omega^2x^2}{4}\right], \tag{19}$$

where $*$ indicates the Hadamard product in (14).

Proof. By applying (2) and (5), on the left side of (19), we have

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right)(x) = \frac{\omega^{\delta+1}}{2^\delta\Gamma(\delta+(3/2))}\sum_{k=0}^\infty\frac{\mathfrak{B}(k+1,\delta+(3/2);p,q)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left(-\frac{\omega^2}{4}\right)^k$$

$$\times\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}\left(t^{\tau+\delta+1+2k-1}\right)\right)(x), \tag{20}$$

upon using the image formula (11):

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right)(x)$$

$$= \frac{x^{\tau+\delta-\varsigma-\varsigma'+\omega}\omega^{\delta+1}}{2^\delta\Gamma(\delta+(3/2))}\sum_{k=0}^\infty\frac{\mathfrak{B}(k+1,\delta+(3/2);p,q)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left(\frac{-x^2\omega^2}{4}\right)^k \tag{21}$$

$$\times\frac{\Gamma(\tau+\delta+\omega-\varsigma+\varsigma'-\vartheta+2k+1)\Gamma(\tau+\delta+\vartheta'-\varsigma'+2k+1)\Gamma(\tau+\delta+2k+1)}{\Gamma(\tau+\delta+\vartheta'+2k+1)\Gamma(\tau+\delta+\omega-\varsigma-\varsigma'+2k+1)\Gamma(\tau+\delta+\omega-\varsigma'-\vartheta+2k+1)}.$$

Presenting the last summation in (21) in terms of the Hadamard product (14) with the functions (16) and (17), we get the right side of (19).

Now, we discuss the special cases of (19) as follows.

For $\varsigma=\varsigma+\vartheta,\varsigma'=\vartheta'=0,\vartheta=-\beta,\omega=\varsigma$, we obtain the following relationship:

$$\left(I_{0+}^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}f\right)(x) = \left(I_{0+}^{\varsigma,\vartheta,\beta}f\right)(x), \tag{22}$$

where the operator $I_{0+}^{\varsigma,\vartheta,\beta}(\cdot)$ express the Saigo fractional integral operator [23], which is defined by

$$\left(I_{0+}^{\varsigma,\vartheta,\beta}f\right)(x) = \frac{x^{-\varsigma-\vartheta}}{\Gamma(\varsigma)}\int_0^x(x-t)_2^{\varsigma-1}F_1\left(\varsigma+\vartheta,-\beta;\varsigma;1-\frac{t}{x}\right)f(t)\mathrm{d}t, \quad \Re(\varsigma)>0. \tag{23}$$

□

Corollary 1. *Let* $\Re(\varsigma) > 0$, $\Re(\tau + \delta + 1) > \max[0, \Re(\vartheta - \beta)]$, *then there holds the following formula:*

$$\left(I_{0+}^{\varsigma,\vartheta,\beta}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right) = \sqrt{\pi}x^{\tau+\delta-\vartheta}\frac{(\omega/2)^{\delta+1}}{\Gamma(\delta+(3/2))}{}_pF_q\left[\begin{array}{c}1,1;\\ \delta+(3/2);\end{array} -\frac{\omega^2x^2}{4}\right]$$

$$* {}_2\Psi_3\left[\begin{array}{c}(\tau+\delta-\vartheta+\beta+1,2),(\tau+\delta+1,2);\\ \left(\frac{3}{2},1\right),(\tau+\delta-\vartheta+1,2),(\tau+\delta+\varsigma+\beta+1,2);\end{array} -\frac{\omega^2x^2}{4}\right].$$

(24)

2.2. Right-Sided Generalized Fractional Integration of the (p,q)-Extended Struve Function.

In this portion, we establish image formulas for the (p,q)-extended Struve function containing right-sided operators of M-S-M fractional integral operators (7), in terms of the Hadamard product of the Fox–Wright function ${}_r\psi_s(z)$ and the (p,q)-extended Gauss hypergeometric function. These formulas are set out in the preceding theorems.

Theorem 2. *If* $\Re(\tau - \delta) < 2 + \min\{\Re(-\vartheta), \Re(\varsigma + \varsigma' - \omega), \Re(\varsigma - \vartheta' - \omega)\}$, $\Re(\omega) > 0$, *then the generalized fractional integration* $I_-^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}$ *of the* (p,q)-*extended Struve function* $H_{\delta,p,q}(z)$ *is given by*

$$\left(I_-^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}\left(t^{\tau-1}H_{\delta,p,q}\left(\frac{\omega}{t}\right)\right)\right) = \sqrt{\pi}x^{\tau-\varsigma-\varsigma'+\omega-\delta-2}\frac{(\omega/2)^{\delta+1}}{\Gamma(\delta+(3/2))}$$

$$\times{}_pF_q\left[\begin{array}{c}1,1;\\ \delta+(3/2);\end{array} -\frac{\omega^2}{4x^2}\right] * {}_3\Psi_4\left[\begin{array}{c}(2-\tau+\delta-\omega+\varsigma+\varsigma',2),\\ ((3/2),1),(2-\tau+\delta,2),\end{array}\right.$$

$$\left.\begin{array}{c}(2-\tau+\delta+\varsigma+\vartheta'-\omega,2),(2-\tau+\delta-\vartheta,2);\\ (2-\tau+\delta-\omega+\varsigma+\varsigma'+\vartheta',2),(2-\tau+\delta+\varsigma-\vartheta,2);\end{array} -\frac{\omega^2}{4x^2}\right].$$

(25)

Proof. By applying (2) and (7) on the left-hand side of (25), we get

$$\left(I_-^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}\left(t^{\tau-1}H_{\delta,p,q}\left(\frac{\omega}{t}\right)\right)\right)(x) = \frac{\omega^{\delta+1}}{2^\delta\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\mathfrak{B}(k+1,\delta+(3/2);p,q)}{(3/2)_k\,\mathfrak{B}(1,\delta+(1/2))k!}\left(-\frac{\omega^2}{4}\right)^k$$

$$\times\left(I_{0-}^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}\left(t^{\tau-\delta-2k-2}\right)\right)(x),$$

(26)

and upon using the image formula (12) yields

$$\left(I_-^{\varsigma,\varsigma',\vartheta,\vartheta',\omega}\left(t^{\tau-1}H_{\delta,p,q}(\omega t)\right)\right)(x)$$

$$= \frac{x^{\tau+\delta-\varsigma-\varsigma'+\omega}\omega^{\delta+1}}{2^\delta\Gamma(\delta+(3/2))}\sum_{k=0}^{\infty}\frac{\mathfrak{B}(k+1,\delta+(3/2);p,q)}{(3/2)_k\mathfrak{B}(1,\delta+(1/2))k!}\left[\frac{-x^2\omega^2}{4}\right]^k$$

(27)

$$\times\frac{\Gamma(2-\tau+\delta-\omega+\varsigma+\varsigma'+2k)\Gamma(2-\tau+\delta+\varsigma+\vartheta'-\omega+2k)\Gamma(2-\tau+\delta-\vartheta+2k)}{\Gamma(2-\tau+\delta+2k)\Gamma(2-\tau+\delta-\omega+\varsigma+\varsigma'+\vartheta'+2k)\Gamma(2-\tau+\delta+\varsigma-\vartheta+2k)}.$$

Interpreting the right-hand side of (27) in terms of the Hadamard product (14) with the functions (16) and (17), we get the right side of (25).

When we let $\varsigma = \varsigma + \vartheta, \varsigma' = \vartheta' = 0, \vartheta = -\beta, \omega = \varsigma$, then we obtain the relationship

$$\left(I_-^{\varsigma,\vartheta,\beta} f\right)(x) = \frac{1}{\Gamma(\varsigma)} \int_x^\infty (t-x)^{\varsigma-1} t_2^{-\varsigma-\vartheta} F_1\left(\varsigma + \vartheta, -\beta; \varsigma; 1 - \frac{x}{t}\right) f(t)\mathrm{d}t. \tag{29}$$

Corollary 2. *If* $\Re(\varsigma) > 0, \Re(\tau - \delta) < 2 + \min[\Re(\vartheta), \Re(\beta)],$ *then we have*

$$\left(I_-^{\varsigma,\vartheta,\beta}\left(t^{\tau-1} H_{\delta,p,q}\left(\frac{\omega}{t}\right)\right)\right) = \sqrt{\pi} x^{\tau-\vartheta-\delta-2} \frac{(\omega/2)^{\delta+1}}{\Gamma(\delta+(3/2))_p} F_q\left[\begin{array}{c} 1, 1; \\ \delta+(3/2); \end{array} -\frac{\omega^2}{4x^2}\right]$$

$$* {}_2\Psi_3\left[\begin{array}{c} (2-\tau+\delta+\vartheta,2), (2-\tau+\delta+\beta,2); \\ \left(\frac{3}{2},1\right), (2-\tau+\delta,2), (2-\tau+\delta+\varsigma+\vartheta+\beta,2); \end{array} -\frac{\omega^2}{4x^2}\right]. \tag{30}$$

In the next part, we derived the generalized fractional kinetic equations (FKEs) and take into account the Laplace transformation technique to produce outcomes.

3. Generalized Fractional Kinetic Equations Involving (p,q)-Extended Struve Function

The generalized FKEs involving the (p,q)-extended Struve function with the Laplace transform (LT) is derived in this section. FKEs were extensively reviewed in a variety of articles [29–35].

Let $\mathfrak{N}(t)$ be an arbitrary reaction that depends on time, d is a destruction rate, and p is a production rate of $\mathfrak{N}$, then the mathematical representation of these three ratios is described by Haubold and Mathai [36] as a fractional differential equation:

$$\frac{\mathrm{d}\mathfrak{N}}{\mathrm{d}t} = -d\left(\mathfrak{N}_t\right) + p\left(\mathfrak{N}_t\right), \tag{31}$$

where $\mathfrak{N}_t(t^*) = \mathfrak{N}(t - t^*)$ for $t^* > 0$. Also, [36] have researched that equation (31) would become the following differential equation if spatial fluctuation or inhomogeneities in quantity $\mathfrak{N}(t)$ are ignored:

$$\frac{\mathrm{d}\mathfrak{N}_i}{\mathrm{d}t} = -c_i \mathfrak{N}_i(t), \tag{32}$$

with $\mathfrak{N}_i(t = 0) = \mathfrak{N}_0$. Solution of equation (32) is given by

$$\mathfrak{N}_i(t) = \mathfrak{N}_0 e^{-c_i t}. \tag{33}$$

Alternatively, if we eliminate the index i and integrate (32), we get

$$\left(I_-^{\varsigma,\varsigma',\vartheta,\vartheta',\omega} f\right)(x) = \left(I_-^{\varsigma,\vartheta,\beta} f\right)(x), \tag{28}$$

where the Saigo fractional integral operator [23] is represented as

$$\mathfrak{N}(t) - \mathfrak{N}_0 = c_0 \mathfrak{D}_t^{-1} \mathfrak{N}(t), \tag{34}$$

where $_0\mathfrak{D}_t^{-1}$ is the standard integral operator. The fractional generalization of equation (34) was defined by Haubold and Mathai [36] as

$$\mathfrak{N}(t) - \mathfrak{N}_0 = c^\nu {}_0\mathfrak{D}_t^{-\nu} \mathfrak{N}(t), \tag{35}$$

where $_0\mathfrak{D}_t^{-\nu}$ is given by

$$_0\mathfrak{D}_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} f(x)\mathrm{d}x, \quad \Re(\nu) > 0. \tag{36}$$

Definition 1. The Mittag–Leffler function is generalized by Wiman [28] in the following form:

$$E_{\varsigma,\vartheta}(z) = \sum_{l=0}^\infty \frac{z^l}{\Gamma(\varsigma l + \vartheta)}, \; (z, \varsigma, \vartheta \in \mathbb{C}; \Re(\varsigma) > 0, \Re(\vartheta) > 0). \tag{37}$$

The results of this section, solutions of generalized FKESs, will be expressed based on the generalized Mittag–Leffler function which is defined in (37).

Theorem 3. *If* $d > 0, \nu > 0,$ *with* $\min\{p, q\} \geq 0$ *and* $\Re(\delta) > -(1/2),$ *the solution of fractional kinetic equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0 H_{\delta,p,q}(t) = -d^\nu {}_0\mathfrak{D}_t^{-\nu} \mathfrak{N}(t) \tag{38}$$

becomes

$$\mathfrak{N}(t) = \mathfrak{N}_0 \frac{t^{\delta+1}}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1, \delta + (1/2); p, q) \Gamma(\delta + 2k + 2)}{(3/2)_k \mathfrak{B}(1, \delta + (1/2)) k!} \left(\frac{-t^2}{4}\right)^k$$

$$\times E_{v, \delta+2k+2}\left(-d^v t^v\right).$$

(39)

Proof. The LT of the Riemann–Liouville (RL) fractional integral operator is given by Srivastava and Saxena [37] as

$$L\{_0 \mathfrak{D}_t^{-v} f(t); s\} = s^{-v} F(s).$$

(40)

Now, applying the LT to both sides of (38) and using (2) and (40), we have

$$L\{\mathfrak{N}(t); s\} = \mathfrak{N}_0 L\{H_{\delta, p, q}(t); s\} - d^v L\{_0 \mathfrak{D}_t^{-v} \mathfrak{N}(t); s\},$$ (41)

which gives

$$\mathfrak{N}(s) = \mathfrak{N}_0 \int_0^\infty e^{-st} \frac{t^{\delta+1}}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1, \delta + (1/2); p, q)}{(3/2)_k \mathfrak{B}(1, \delta + (1/2)) k!} \left(\frac{-t^2}{4}\right) dt - d^v s^{-v} \mathfrak{N}(s),$$

(42)

which implies that

$$\mathfrak{N}(s) + d^v s^{-v} \mathfrak{N}(s) = \frac{\mathfrak{N}_0}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1, \delta + (1/2); p, q)}{(3/2)_k \mathfrak{B}(1, \delta + (1/2)) k!} \frac{(-1)^k}{4^k}$$

$$\times \int_0^\infty e^{-st} t^{2k+\delta+1} dt.$$

(43)

After some simple calculation, we get

$$\mathfrak{N}(s)\left(1 + d^v s^{-v}\right) = \frac{\mathfrak{N}_0}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \frac{\mathfrak{B}(k+1, \delta + (1/2); p, q)}{(3/2)_k \mathfrak{B}(1, \delta + (1/2)) k!} \frac{\Gamma(\delta + 2k + 2)}{s^{\delta+2k+2}},$$

$$\mathfrak{N}(s) = \frac{\mathfrak{N}_0}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \frac{\mathfrak{B}(k+1, \delta + (1/2); p, q) \Gamma(\delta + 2k + 2)}{(3/2)_k \mathfrak{B}(1, \delta + (1/2)) k!}$$

$$\times s^{-(\delta+2k+2)} \sum_{l=0}^{\infty} (1)_l \frac{[-(s/d)^{-v}]^l}{l!}.$$

(44)

Taking inverse LT on both sides of (44) and using $L^{-1}(s^{-v}) = (t^{v-1}/\Gamma(v))$ for $\mathfrak{R}(v) > 0$, we get

$$\mathfrak{N}(t) = \frac{\mathfrak{N}_0}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \frac{\mathfrak{B}(k+1, \delta + (1/2); p, q) \Gamma(\delta + 2k + 2)}{(3/2)_k \mathfrak{B}(1, \delta + (1/2)) k!}$$

$$\times \sum_{l=0}^{\infty} (-1)^l d^{vl} \frac{t^{\delta+2k+vl+1}}{\Gamma(\delta + vl + 2k + 2)}.$$

(45)

□

Interpreting the right-hand side of (45) in the view of (37), we obtain the needful result (39).

Theorem 4. *If $d > 0, v > 0$, with $\min\{p, q\} \geq 0$ and $\Re(\delta) > -(1/2)$, then the solution of*

$$\mathfrak{N}(t) - \mathfrak{N}_0 H_{\delta,p,q}\left(d^v t^v\right) = -d^v {}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \qquad (46)$$

is given by

$$\mathfrak{N}(t) = \mathfrak{N}_0 \frac{\left(d^v t^v\right)^{\delta+1}}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1, \delta+(1/2); p, q)\Gamma(2vk+\delta v+v+1)}{(3/2)_k \mathfrak{B}(1, \delta+(1/2))k!} \left(\frac{-\left(d^v t^v\right)^2}{4}\right)^k$$

$$\times E_{v,\delta v+2vk+v+1}\left(-d^v t^v\right). \qquad (47)$$

Proof. Taking the LT on both sides of (46), using the definition of (p, q)-extended Struve functions (2) and (40), and after doing simple calculation and taking inverse LT term written in the view of (37), we obtain the needful result (47). ☐

Theorem 5. *If $d > 0, v > 0$, with $\min\{p, q\} \geq 0, a \neq d$ and $\Re(\delta) > -(1/2)$, the solution of fractional kinetic equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0 H_{\delta,p,q}\left(d^v t^v\right) = -a^v {}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \qquad (48)$$

becomes

$$\mathfrak{N}(t) = \mathfrak{N}_0 \frac{\left(d^v t^v\right)^{\delta+1}}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\mathfrak{B}(k+1, \delta+(1/2); p, q)\Gamma(2vk+\delta v+v+1)}{(3/2)_k \mathfrak{B}(1, \delta+(1/2))k!} \left(\frac{-\left(d^v t^v\right)^2}{4}\right)^k$$

$$\times E_{v,\delta v+2vk+v+1}\left(-a^v t^v\right). \qquad (49)$$

Proof. In similar way of proof of Theorem 4, we can get solution (49). Therefore, we omitted the proof.

Now by setting $p = 0, q = 0$, on equation (3), then results of Theorems 3–5 are adjusted on Corollaries 3–5. ☐

Corollary 3. *If $d > 0, v > 0$, and $\Re(\delta) > -(1/2)$, the solution of fractional kinetic equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0 H_{\delta,0,0}(t) = -d^v {}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \qquad (50)$$

becomes

$$\mathfrak{N}(t) = \mathfrak{N}_0 \frac{t^{\delta+1}}{\Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\Gamma(\delta + 2k + 2)}{\Gamma(\delta + k + (3/2))} \left(\frac{-t^2}{4}\right)^k$$

$$\times E_{v,\delta+2k+2}\left(-d^v t^v\right). \qquad (51)$$

Corollary 4. *If $d > 0, v > 0$, with $\min\{p, q\} \geq 0$ and $\Re(\delta) > -(1/2)$, then the solution of*

$$\mathfrak{N}(t) - \mathfrak{N}_0 H_{\delta,p,q}\left(d^v t^v\right) = -d^v {}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \qquad (52)$$

is given by

$$\mathfrak{N}(t) = \mathfrak{N}_0 \frac{\left(d^v t^v\right)^{\delta+1}}{\Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\Gamma(2vk+\delta v+v+1)}{\Gamma(\delta + k + (3/2))} \left(\frac{-\left(d^v t^v\right)^2}{4}\right)^k$$

$$\times E_{v,\delta v+2vk+v+1}\left(-d^v t^v\right). \qquad (53)$$

Corollary 5. *If $d > 0, v > 0$, with $\min\{p, q\} \geq 0, a \neq d$ and $\Re(\delta) > -(1/2)$, the solution of fractional kinetic equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0 H_{\delta,p,q}\left(d^v t^v\right) = -a^v {}_0\mathfrak{D}_t^{-v}\mathfrak{N}(t) \qquad (54)$$

becomes

$$\mathfrak{N}(t) = \mathfrak{N}_0 \frac{\left(d^v t^v\right)^{\delta+1}}{2^\delta \Gamma(\delta + (3/2))} \sum_{k=0}^{\infty} \frac{\Gamma(2vk+\delta v+v+1)}{\Gamma(\delta + k + (3/2))} \left(\frac{-\left(d^v t^v\right)^2}{4}\right)^k$$

$$\times E_{v,\delta v+2vk+v+1}\left(-a^v t^v\right). \qquad (55)$$

4. Conclusion

In this article, the authors have established the generalized fractional integrations of the (p, q)-extended Struve function. The achieved results are expressed in terms of Hadamard product of the Fox–Wright function ${}_r\psi_s(z)$ and the (p, q)-extended Gauss hypergeometric function. The solutions of fractional kinetic equations are obtained with the support of Laplace transforms to show the possible application of the (p, q)-extended Struve function. As the solution of the equations is common and can derive several new and existing FKE solutions involving different types of special functions, the results obtained in this study are significant.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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