

## Research Article

# New Bernstein Type Results in Weighted Warped Products

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In this paper, we obtain new parametric uniqueness results for complete constant weighted mean curvature hypersurfaces under suitable geometric assumptions in weighted warped products. Furthermore, we also prove very general Bernstein type results for the constant mean curvature equation for entire graphs in these ambient spaces.

## 1. Introduction

In the last decades, constant mean curvature hypersurfaces in Riemannian manifolds have been deeply studied. This is because that such hypersurfaces exhibit nice Bernstein type properties. Indeed, the classical Bernstein theorem states that a complete minimal surface in  $\mathbb{R}^3$  is a plane. Later, the Bernstein theorem has been extended to higher dimensions as follows: each complete minimal hypersurface in  $\mathbb{R}^{n+1}$  must be hyperplane by Simons in [1] for  $n \leq 7$ , and the result was achieved through successive efforts of Almgren [2], Fleming [3] and De Giorgi [4]. However, for  $n \geq 8$ , there exists a counterexample in [5], by constructing a nontrivial complete minimal hypersurface in  $\mathbb{R}^{n+1}$ , which is not a hyperplane. In recent years, successive efforts have been made in order to extend these Bernstein type results for hypersurfaces to much more general ambient spaces.

Among all Riemannian manifolds, we will consider the class of models known as weighted warped products. Our model ambient space will be a warped product  $I \times_{\rho} M$ , in the sense of [6], with an interval  $I$  equipped with a positive definite metric as base, a Riemannian manifold  $M$  as fiber, and a positive smooth function  $\rho$  as warping function. Furthermore, there exists a distinguished family of hypersurfaces in warped products, that is, the slices, which are defined as level hypersurfaces of the height function on the base, as defined in Section 2. Note that each slice is totally umbilical and has constant mean curvature. Moreover, a weighted manifold is a Riemannian manifold with measure

that has smooth positive density with respect to the induced metric. More precisely, the weighted manifold  $M_f$  associated with a complete Riemannian manifold  $(M, g)$  and a smooth positive function  $f$  on  $M$  is the triple  $(M, g, d\mu = e^{-f} dM)$ , where  $dM$  is the volume element of  $M$ . In this setting, we will consider the Bakry–Émery–Ricci tensor (see [7]) which is a generalization of the standard Ricci tensor  $\text{Ric}$  defined as

$$\text{Ric}_f = \text{Ric} + \text{Hess}f. \quad (1)$$

So, it is natural to extend some results of the Ricci curvature to similar results for the Bakry–Émery–Ricci tensor. Before presenting more details on our work, we give a brief overview of some results related to our one.

Wei and Wylie researched the weighted Riemannian manifold  $M_f$  and proved mean curvature and volume comparison results under the assumption that  $\text{Ric}_f$  is bounded from below and  $f$  or  $|\nabla f|$  is bounded in [8]. In particular, Salamanca and Salavessa [9] obtained uniqueness results for complete weighted minimal hypersurfaces (that is, those whose weighted mean curvature identically vanishes) in a weighted warped product whose fiber is a parabolic manifold. Later, de Lima et al. [10, 11] studied the Bernstein type results concerning complete hypersurfaces in weighted warped products via application of appropriated generalized maximum principles. Furthermore, de Lima et al. [12] obtained Liouville type results for two-sided hypersurfaces in weighted Killing warped products. More

recently, the author [13] proved some uniqueness results of complete hypersurfaces in weighted Riemannian warped products  $I \times_\rho M_f^n$  with  $f$ -parabolic fiber, through the application of the weak maximum principle.

Our aim in this paper is to obtain new Bernstein type results for complete constant weighted mean curvature hypersurfaces in weighted warped products. We have organized this article as follows. In Section 2, we introduce some basic notions and facts to be used for hypersurfaces immersed in weighted warped products. In Section 3, we prove some parametric results related to the hypersurfaces in a weighted warped product that will enable us to obtain our main uniqueness result (Theorem 1) which extend the corresponding results in [13]. To conclude this paper, we will devote Section 4 to prove a new Bernstein type result for entire graphs in weighted warped products for the constant weighted mean curvature case (Theorem 2).

## 2. Preliminaries

Let  $M^n$  be a connected  $n (\geq 2)$ -dimensional oriented Riemannian manifold and  $I \subset \mathbb{R}$  be an open interval endowed with the metric  $dt^2$ . Let  $\rho: I \rightarrow \mathbb{R}^+$  be a smooth function. Denote by  $I \times_\rho M^n$  the warped product manifold with the Riemannian metric

$$\langle \cdot, \cdot \rangle = \pi_I^*(dt^2) + \rho(\pi_I)^2 \pi_M^*(\langle \cdot, \cdot \rangle_M), \quad (2)$$

where  $\pi_I$  and  $\pi_M$  are the projections onto  $I$  and  $M$ , respectively. This resulting space is a *warped product* in the sense of [6], with *base*  $(I, dt^2)$ , *fiber*  $(M, \langle \cdot, \cdot \rangle_M)$ , and *warping function*  $\rho$ . Furthermore, for every  $t_0 \in I$ , we say that  $M_{t_0}^n = \{t_0\} \times M^n$  is a *slice* of  $I \times_\rho M^n$ .

Consider the vector field  $K = \rho(\pi_I)\partial_t$  in  $I \times_\rho M^n$ , where  $\partial_t = (\partial/\partial t)$  is the unit vector field tangent to base  $I$ . Moreover, using the relationship between the Levi-Civita connections of  $I \times_\rho M^n$  and those of the fiber and the base (see Corollary 7.35 in [6]), we have

$$\bar{\nabla}_X K = \rho'(\pi_I)X, \quad (3)$$

for any  $X \in \mathfrak{X}(I \times_\rho M^n)$ , where  $\bar{\nabla}$  stands for the Levi-Civita connection of the Riemannian metric  $\langle \cdot, \cdot \rangle$  in (2). Therefore,  $K$  is conformal with  $\mathcal{L}_K \langle \cdot, \cdot \rangle = 2\rho'(\pi_I)\langle \cdot, \cdot \rangle$  and its metrically equivalent 1-form is closed.

Recall that a smooth immersion  $\psi: \Sigma^n \rightarrow I \times_\rho M^n$  of an  $n$ -dimensional connected manifold  $\Sigma^n$  is said to be a *hypersurface*. Moreover, the induced metric via  $\psi$  on  $\Sigma^n$  will be also denoted by  $\langle \cdot, \cdot \rangle$ .

In this paper, we study the connected hypersurfaces  $\psi: \Sigma^n \rightarrow I \times_\rho M^n$  oriented a unit normal vector field  $N$ . Let  $\nabla$  be the Levi-Civita connection of  $\Sigma^n$ . The Gauss and Weingarten formulas for the hypersurfaces  $\psi: \Sigma^n \rightarrow I \times_\rho M^n$  are given, respectively, by

$$\bar{\nabla}_X Y = \nabla_X Y + \langle AX, Y \rangle N, \quad (4)$$

$$AX = -\bar{\nabla}_X N, \quad (5)$$

where  $X, Y \in \mathfrak{X}(\Sigma^n)$  and  $A: \mathfrak{X}(\Sigma^n) \rightarrow \mathfrak{X}(\Sigma^n)$  is the shape operator (or Weingarten endomorphism) of  $\Sigma^n$  with respect to  $N$ .

In the following, we consider two particular functions naturally attached to hypersurface  $\Sigma^n$ , namely, the *angle (or support) function*  $\Theta = \langle N, \partial_t \rangle$  and the *height function*  $h = (\pi_I)|_{\Sigma^n}$ .

Let  $\bar{\nabla}$  and  $\nabla$  be the gradients with respect to the metrics of  $I \times_\rho M^n$  and  $\Sigma^n$ , respectively. Then, by a simple computation, we obtain

$$\bar{\nabla} \pi_I = \langle \bar{\nabla} \pi_I, \partial_t \rangle \partial_t = \partial_t. \quad (6)$$

So, the gradient of  $h$  on  $\Sigma^n$  is

$$\nabla h = (\bar{\nabla} \pi_I)^\top = \partial_t^\top = \partial_t - \Theta N. \quad (7)$$

Particularly,

$$|\nabla h|^2 = 1 - \Theta^2, \quad (8)$$

where  $|\cdot|$  denotes the norm of a vector field on  $\Sigma^n$ .

Moreover, taking tangential components in (3), we have, from (4) and (5), that

$$\nabla_X K^\top = \rho(h)A\Theta X + \rho'(h)X, \quad (9)$$

where  $\rho'(h) = \rho' \circ h$  and  $K^\top = \rho(h)\partial_t^\top = K - \langle K, N \rangle N$  is the tangential component of  $K$  along  $\Sigma^n$ . This enables us to use (9) to compute the gradient of the angle function  $\Theta$ , obtaining

$$\nabla \Theta = -A\nabla h - \frac{\rho'(h)}{\rho(h)}\Theta \nabla h. \quad (10)$$

Furthermore, it follows from (7) and (9) that the Laplacian of  $h$  on  $\Sigma^n$  is

$$\Delta h = (\log \rho)'(h)(n - |\nabla h|^2) + nH\Theta. \quad (11)$$

Consequently, by  $\Delta \rho(h) = \rho'(h)\Delta h + \rho''(h)|\nabla h|^2$ , we have

$$\Delta \rho(h) = n \frac{\rho''(h)}{\rho(h)} + \rho(h)(\log \rho)''(h)|\nabla h|^2 + n\rho'(h)H\Theta. \quad (12)$$

Furthermore, consider that a warped product  $I \times_\rho M^n$  endowed with a weight function  $f$ , which will be called a *weighted warped product*  $I \times_\rho M_f^n$ . In this setting, for a hypersurface  $\Sigma^n$  immersed into  $I \times_\rho M_f^n$ , the  *$f$ -divergence operator* on  $\Sigma^n$  is defined by

$$\operatorname{div}_f(X) = e^f \operatorname{div}(e^{-f}X), \quad (13)$$

where  $X$  is a tangent vector field on  $\Sigma^n$ .

For a smooth function  $u: \Sigma^n \rightarrow \mathbb{R}$ , we define its drifting Laplacian by

$$\Delta_f u = \operatorname{div}_f(\nabla u) = \Delta u - \langle \nabla u, \nabla f \rangle, \quad (14)$$

and we will also denote such an operator as the  *$f$ -Laplacian* of  $\Sigma^n$ .

According to Gromov [14], the weighted mean curvature or  $f$ -mean curvature  $H_f$  of  $\Sigma^n$  is given by

$$nH_f = nH + \langle \bar{\nabla} f, N \rangle, \tag{15}$$

where  $H$  is the standard mean curvature of hypersurface  $\Sigma^n$  with respect to  $N$ .

In this paper, we will consider weighted Riemannian warped products  $I \times_\rho M_f^n$  whose weight function  $f$  does not depend on the parameter  $t \in I$ , that is,  $\langle \bar{\nabla} f, \partial_t \rangle = 0$ . Moreover, we will refer to them as  $\bar{M}^{n+1} := I \times_\rho M_f^n$ .

### 3. Parametric Uniqueness Results

In order to prove our uniqueness results in weighted warped product  $\bar{M}^{n+1}$ , we need a few previous results.

**Lemma 1.** *Let  $\psi: \Sigma^n \rightarrow \bar{M}^{n+1}$  be a hypersurface in a weighted Riemannian warped product  $\bar{M}^{n+1} = I \times_\rho M_f^n$ . Then, the height function  $h$  of  $\Sigma^n$  satisfies*

$$\begin{aligned} \frac{1}{2} \Delta_f (|\nabla h|^2) &= n \Theta \langle \nabla H_f, \partial_t \rangle + n \frac{\rho'(h)}{\rho(h)} H_f \Theta (1 + \Theta^2) - 4 \frac{\rho'(h)}{\rho(h)} \Theta \langle A \nabla h, \nabla h \rangle \\ &+ \frac{\rho'(h)^2}{\rho(h)^2} \Theta^2 (n - 3|\nabla h|^2) + \Theta^2 |A|^2 - |A(\nabla h)|^2 + \Theta^2 (\text{Ric}_f^M(N^*, N^*) + n(\log \rho)''(h) |\nabla h|^2), \end{aligned} \tag{16}$$

where  $\text{Ric}_f^M$  stands for the Bakry-Émery-Ricci curvature tensor of  $M^n$  and  $N^* = N - \langle N, \partial_t \rangle \partial_t$  is the projection of the vector field  $N$  onto  $M^n$ .

*Proof.* The key idea of the proof is to compute the  $f$ -Laplacian of the function  $\Theta$ . To do so, taking into account (14), it follows that

$$\Delta_f(\rho(h)\Theta) = \Delta(\rho(h)\Theta) - \langle \nabla(\rho(h)\Theta), \nabla f \rangle. \tag{17}$$

Proceeding as above in Section 2, by a direct computation from (10) and (12), gives

$$\begin{aligned} \Delta(\rho(h)\Theta) &= \Theta \Delta \rho(h) + \rho(h) \Delta \Theta + 2 \langle \nabla \rho(h), \nabla \Theta \rangle \\ &= n \frac{\rho'(h)^2}{\rho(h)} \Theta + \rho(h) (\log \rho)''(h) \Theta |\nabla h|^2 - 2 \rho'(h) \langle A \nabla h, \nabla h \rangle - 2 \frac{\rho'(h)^2}{\rho(h)} \Theta |\nabla h|^2 + n H \rho'(h) \Theta^2 + \rho(h) \Delta \Theta. \end{aligned} \tag{18}$$

Moreover, from a straightforward computation, we obtain

$$\langle \nabla(\rho(h)\Theta), \nabla f \rangle = -\rho'(h) \Theta^2 \langle \bar{\nabla} f, N \rangle + \rho(h) \langle \nabla \Theta, \nabla f \rangle. \tag{19}$$

So, by (14) and (15), we can write (17) as

$$\begin{aligned} \Delta_f(\rho(h)\Theta) &= n \frac{\rho'(h)^2}{\rho(h)} \Theta + \rho(h) (\log \rho)''(h) \Theta |\nabla h|^2 \\ &- 2 \rho'(h) \langle A \nabla h, \nabla h \rangle - 2 \frac{\rho'(h)^2}{\rho(h)} \Theta |\nabla h|^2 + n \rho'(h) \Theta^2 H_f + \rho(h) \Delta_f \Theta. \end{aligned} \tag{20}$$

On the contrary, Lemma 1 in [11] proves that

$$\begin{aligned} \Delta_f(\rho(h)\Theta) &= -n\rho(h)\langle \nabla H_f, \partial t \rangle - n\rho'(h)H_f - \rho(h)\Theta|A|^2 \\ &\quad - \rho(h)\Theta(\text{Ric}^M(N^*, N^*) + (n-1)(\log \rho)''(h)|\nabla h|^2) - \rho(h)\Theta\overline{\text{Hess}}(f)(N, N), \end{aligned} \quad (21)$$

$$\overline{\text{Hess}}f(N, N) = \text{Hess}^M f(N^*, N^*). \quad (22)$$

where  $\text{Ric}^M$  stands for the Ricci curvature tensor of  $M^n$ .

Moreover, taking into account that  $\langle \overline{\nabla} f, \partial_t \rangle = 0$ , it is easy to obtain that

Hence, using relation (1), the result follows from (20)–(22) that

$$\begin{aligned} \Delta_f \Theta &= -n\langle \nabla H_f, \partial t \rangle - n\frac{\rho'(h)}{\rho(h)}H_f(1 + \Theta^2) + 2\frac{\rho'(h)}{\rho(h)}\langle A\nabla h, \nabla h \rangle \\ &\quad - \Theta(\text{Ric}_f^M(N^*, N^*) + n(\log \rho)''(h)|\nabla h|^2) - \frac{\rho'(h)^2}{\rho(h)^2}\Theta(n - 2|\nabla h|^2) - \Theta|A|^2. \end{aligned} \quad (23)$$

Moreover, from (10), we obtain that

$$|\nabla \Theta|^2 = \frac{\rho'(h)^2}{\rho(h)^2}\Theta^2|\nabla h|^2 + |A(\nabla h)|^2 + 2\frac{\rho'(h)}{\rho(h)}\Theta\langle A\nabla h, \nabla h \rangle. \quad (24)$$

To conclude the proof, we should notice that

$$\frac{1}{2}\Delta_f(|\nabla h|^2) = \frac{1}{2}\Delta_f(1 - \Theta^2) = -\Theta\Delta_f\Theta - |\nabla \Theta|^2. \quad (25)$$

and use (23) and (24) to obtain (16).

In the following, we give the next technical lemma (for further details, see Proposition 3.3 in [13]) which will be essential for the proofs of our main results.  $\square$

**Lemma 2.** *Let  $(M, \langle \cdot, \cdot \rangle_M, e^{-f}dM)$  be a  $f$ -parabolic weighted manifold and  $u \in C^2(M)$  be a positive function on  $M$  and  $u^* = \sup_M u < +\infty$ . If  $\Delta_f u$  does not change sign on  $M$ , then  $u$  is constant on  $M$ .*

Moreover, we also need to consider the weighted warped products  $I \times_\rho M_f^n$  satisfy the following convergence condition:

$$\text{Ric}^M \geq (n-1)\sup_I(\rho'^2 - \rho''\rho)\langle \cdot, \cdot \rangle_M. \quad (26)$$

**Theorem 1.** *Let  $\psi: \Sigma^n \rightarrow I \times_\rho M_f^n$  be a complete  $f$ -parabolic hypersurface with constant  $f$ -mean curvature  $H_f$  in a weighted warped product  $I \times_\rho M_f^n$  which satisfies the convergence condition (26) and the weight function  $f$  is convex. Assume that the warping function satisfies  $\rho''(h) \geq 0$ . If either  $-1 \leq \Theta < -(\sqrt{3}/2)$  and  $H_f\rho'(h) \leq 0$  or  $(\sqrt{3}/2) < \Theta \leq 1$  and  $H_f\rho'(h) \geq 0$ , then  $\Sigma^n$  is totally geodesic. In addition, if inequality (26) is strict or  $\rho''(h) > 0$ , then  $\Sigma^n$  is a totally geodesic minimal slice.*

*Proof.* From (1) and (22), we deduce that

$$\text{Ric}_f^M(N^*, N^*) = \text{Ric}^M(N^*, N^*) + \overline{\text{Hess}}f(N, N). \quad (27)$$

Moreover, using Young's inequality, we have

$$-4\frac{\rho'(h)}{\rho(h)}\Theta\langle A\nabla h, \nabla h \rangle \geq -2\frac{\rho'(h)^2}{\rho(h)^2}\Theta^2|\nabla h|^2 - 2|A(\nabla h)|^2. \quad (28)$$

So, we can estimate

$$\begin{aligned} &-4\frac{\rho'(h)}{\rho(h)}\Theta\langle A\nabla h, \nabla h \rangle + \frac{\rho'(h)^2}{\rho(h)^2}\Theta^2(n - 4|\nabla h|^2) \\ &\quad + \Theta^2|A|^2 - |A(\nabla h)|^2 \\ &\geq \frac{\rho'(h)^2}{\rho(h)^2}\Theta^2(n - 6|\nabla h|^2) + \Theta^2|A|^2 - 3|A(\nabla h)|^2 \\ &\geq \frac{\rho'(h)^2}{\rho(h)^2}\Theta^2(n - 6 + 6\Theta^2) + (4\Theta^2 - 3)|A|^2. \end{aligned} \quad (29)$$

Considering the assumptions of Theorem 1, it follows from (16) that

$$\begin{aligned} \frac{1}{2}\Delta_f(|\nabla h|^2) &\geq \frac{\rho'(h)^2}{\rho(h)^2}\Theta^2(n - 6 + 6\Theta^2) + (4\Theta^2 - 3)|A|^2 \\ &\quad + n\frac{\rho'(h)}{\rho(h)}H_f\Theta(1 + \Theta^2) \geq 0. \end{aligned} \quad (30)$$

Moreover,  $|\nabla h|^2$  is bounded on  $\Sigma^n$  which allows us to apply Lemma 2 to guarantee that  $|\nabla h|^2$  is constant. So,  $\Delta_f|\nabla h|^2 = 0$ . From (16), we also have that

$$n \frac{\rho'(h)}{\rho(h)} H_f \Theta (1 + \Theta^2) + \frac{\rho''}{\rho} \Theta^2 |\nabla h|^2 + \frac{\rho'(h)^2}{\rho(h)^2} \Theta^2 (n - 6 + 6\Theta^2) + (4\Theta^2 - 3)|A|^2 + \Theta^2 (\text{Ric}^M(N^*, N^*) + (n - 1)(\log \rho)'' \rho(h) |\nabla h|^2) = 0. \tag{31}$$

Since  $\Theta^2 > (3/4)$ , we get  $\rho'(h) = 0$  and  $|A| = 0$ .

So,  $\Sigma^n$  is totally geodesic in  $I \times_\rho M_f^n$  with  $\rho = c = \text{constant}$ . Finally, when inequality (26) is strict, it follows from (31) that  $N^*(p) = 0$  at any  $p \in \Sigma^n$ , that is,  $\nabla h = 0$  on  $\Sigma^n$ . Therefore,  $\Sigma^n$  is a slice.

Moreover, if  $\rho''(h) > 0$ , (31) implies that  $|\nabla h|^2 = 0$  on  $\Sigma^n$ ; consequently,  $h$  is constant, that is,  $\Sigma^n$  is a slice.  $\square$

*Remark 1.* Notice that, in the case where  $f$ -weighted mean curvature  $H_f \equiv 0$ , i.e.,  $\Sigma^n$  is a  $f$ -minimal hypersurface, we reobtain Theorem 6 in [11]. Thus, Theorem 1 extends this previous result to the constant  $f$ -weighted mean curvature case in weighted warped products.

It should also note that the statement of Theorem 1 could be improved by requiring that

$$\text{Ric}_f^M \geq (n - 1) \sup_I (\rho'^2 - \rho''\rho) \langle \cdot, \cdot \rangle_M \tag{32}$$

instead of separately requiring the convergence condition (26), and the weight function  $f$  is convex, i.e.,

$$\text{Ric}^M \geq (n - 1) \sup_I (\rho'^2 - \rho''\rho) \langle \cdot, \cdot \rangle_M, \quad \text{and } \text{Hess}(f) \geq 0. \tag{33}$$

### 4. Bernstein Type Results

In this section, we will consider the case of a vertical graph over the fiber  $(M^n, \langle \cdot, \cdot \rangle_M)$  in a weighted warped product  $\overline{M}^{n+1} = I \times_\rho M_f^n$ ,  $n \geq 2$ , defined by

$$\Sigma^n(u) = \{(u(x), x) : x \in \Omega\} \subset \overline{M}^{n+1}, \tag{34}$$

where  $\Omega \subset M^n$  is a connected domain of  $M^n$  and  $u$  is a smooth function on  $\Omega$ . Moreover, the metric induced on  $\Omega$  from the metric on  $\overline{M}^{n+1}$  via identification with  $\Sigma^n(u)$  is given by

$$\langle \cdot, \cdot \rangle = du^2 + \rho(u)^2 \langle \cdot, \cdot \rangle_M. \tag{35}$$

It is easy to see from this expression that if  $M^n$  is complete and  $\inf_{\Sigma^n(u)} \rho(u) > 0$ , then the graph  $\Sigma^n(u)$  is complete. Furthermore, the graph  $\Sigma^n(u)$  is said to be entire if  $\Omega = M^n$ . On the contrary, we note that  $h(u(p), p) = u(p)$  for any point  $p \in M^n$ , so  $h$  and  $u$  may be naturally identified on  $\Sigma^n(u)$ .

When  $\Sigma^n(u)$  is entire, the unitary normal vector field of  $\Sigma^n(u)$  in  $\overline{M}^{n+1} = I \times_\rho M_f^n$  is given by

$$N = \frac{1}{\rho(u) \sqrt{\rho(u)^2 + |Du|^2}} (\rho(u)^2 \partial_t - Du), \tag{36}$$

where  $Du$  is the gradient of  $u$  on  $M^n$  and  $|Du|^2 = \langle Du, Du \rangle_M$ . Using Proposition 7.35 in [6] again, we can get the shape operator associated to  $N$ :

$$A(X) = \frac{1}{\sqrt{\rho(u)^2 + |Du|^2}} \left( \rho'(u) X - \frac{1}{\rho(u)} D_X Du \right) + \left( \frac{\langle D_X Du, Du \rangle_M}{\rho(u) (\rho(u)^2 + |Du|^2)^{(3/2)}} + \frac{\rho'(u) \langle Du, X \rangle_M}{(\rho(u)^2 + |Du|^2)^{(3/2)}} \right) Du, \tag{37}$$

for  $X \in \mathfrak{X}(\Omega)$ , where  $D$  denotes the Levi-Civita connection in  $M^n$ . Consequently, it is easy to verify from (15) and (37) that the  $f$ -mean curvature  $H_f$  of  $\Sigma^n(u)$  corresponding to  $N$  is given by

$$H_f(u) = \text{div}_f \left( \frac{Du}{n\rho(u)\sqrt{\rho(u)^2 + |Du|^2}} \right) - \frac{\rho'(u)}{n\sqrt{\rho(u)^2 + |Du|^2}} \left( n - \frac{|Du|^2}{\rho(u)^2} \right). \tag{38}$$

In particular, an entire graph  $\Sigma^n(u)$  has constant  $f$ -mean curvature if and only if the function  $u \in C^\infty(M)$  satisfies the following elliptic partial differential equation:

$$\text{div}_f \left( \frac{Du}{\rho(u) \sqrt{\rho(u)^2 + |Du|^2}} \right) - \frac{\rho'(u)}{\sqrt{\rho(u)^2 + |Du|^2}} \tag{39}$$

$$\left( n - \frac{|Du|^2}{\rho(u)^2} \right) = C,$$

for some constant  $C \in \mathbb{R}$ .

In what follows, we will use the results of Section 3 to obtain a new uniqueness result for equation (39).

**Theorem 2.** Let  $\overline{M}^{n+1} = I \times_\rho M_f^n$  be a weighted warped product with complete fiber  $M^n$ , obeying (26). Assume that the weighted function  $f$  is convex and the warping function satisfies  $\rho''(h) \geq 0$ . Let  $\Sigma(u)$  be an  $f$ -parabolic entire graph in  $\overline{M}^{n+1}$  determined by a function  $u \in C^\infty(M)$  with constant

$f$ -mean curvature such that  $H_f \rho'(h) \Theta \geq 0$ . Suppose that either inequality (26) is strict or  $\rho''(h) > 0$ , then, for constant  $0 \leq \lambda \leq (\sqrt{3}/3)$ , the only bounded entire solutions to equation (39) with  $|Du| < \lambda \rho(u)$  are the constant ones  $u = t_0$  for some  $t_0 \in I$ .

*Proof.* Note that, by a straightforward computation, we can get the following relation:

$$|\nabla h|^2 = 1 - \Theta^2 = \frac{|Du|^2}{\rho(u)^2 + |Du|^2}. \quad (40)$$

Therefore, by the constraints  $|Du| < \lambda \rho(u)$ ,  $0 \leq \lambda \leq (\sqrt{3}/3)$ , and (40), we have

$$\frac{1}{1 + \lambda^2} < \Theta^2 \leq 1. \quad (41)$$

The rest of our assumptions enable us to apply Theorem 1 to end the proof.  $\square$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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