Research Article

On the Existence of Long-Time Classical Solutions for the 2D Inviscid Boussinesq Equations

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In this paper, we consider the Cauchy problem for the 2D inviscid Boussinesq equations with N being the buoyancy frequency. It is proved that for general initial data $u_0 \in H^s$ with $s > 3$, the life span of the classical solutions satisfies $T > C(\ln N)^{3/4}$.

1. Introduction

The 2D inviscid Boussinesq equations read as

$$\begin{cases} \partial_t \eta + (u \cdot \nabla) \eta = 0, & (x, t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\ \partial_t u + (u \cdot \nabla) u + \nabla q = \eta \vec{e}_2, \\ \nabla \cdot u = 0. \end{cases} \quad (1)$$

The unknown functions $\eta = \eta(x, t)$, $u = (u_1(x, t), u_2(x, t))$, $q = q(x, t)$ stand for the temperature, the velocity field, and the scalar pressure, respectively. $\vec{e}_2 = (0, 1)$ is the unit vector in the vertical direction. They can describe the natural convection in the 2D inviscid incompressible fluids such as the dynamics of the ocean or the atmosphere (see, e.g., [1–3]). Besides the physical significance, there is also a strong mathematical motivation for studying these equations. In fact, the 2D Boussinesq equation can be used as a model for the 3D axisymmetric Euler equations (see, e.g., [4]). Therefore, the study of equation (1) can provide us with the useful information to understand the Euler equations.

Due to the physical and strong mathematical significance of the 2D Boussinesq equations, many researchers have paid much attention on its study. For the Cauchy problem around the stationary solution $(0, 0, 0)$, global regularity of solutions is known when the classical

dissipation is present in at least one of the equations (1) (see [5, 6]) or under a variety of more general conditions on dissipation (see [7]) or under adding a damp term (see [8–11]). In contrast, the global regularity problem on the inviscid 2D Boussinesq equations (1) now is still open. Some attempts have been made for a step forward in this direction, which appears to be out of reach in spite of the progress on the local well-posedness and regularity criteria (see [12, 13]). On the other hand, for the initial boundary problem, Hu et al. [14] obtained the global well-posedness for the Boussinesq equations with non-slip boundary condition, and Lai et al. [2] and Littman [4] obtained the local well-posedness for the Boussinesq equations with slip boundary condition. Recently, some researchers started to consider the global well-posedness around the stationary solution of the strong stratification [11, 15, 16]. In particular, Elgindi and Widmayer [15] proved the long-time existence which is the life span of the associated solutions is $\varepsilon^{-(4/3)}$ if the initial data are of size ε . In this paper, we will present a new life span which is suitable for general initial data by the method of Strichartz estimate combining with a blowup criterion.

To state our result more precisely, we firstly consider the solutions of (1) around a stratified solution and rewrite equation (1). It is well known that equation (1) admits an explicit stationary solution (u_s, q_s, η_s) of the form

$$(u_s, q_s, \eta_s) = \left(0, \frac{N^2 x_2^2}{2}, N^2 x_2\right), \quad (2)$$

satisfying the hydrostatic balance

$$\frac{dq_s}{dx_2} = \eta_s, \quad (3)$$

where N is called the buoyancy or the Brunt–Vaisala frequency and represents the strength of stable stratification. Setting

$$\begin{aligned} \theta &= \eta - \eta_s, \\ p &= q - q_s, \end{aligned} \quad (4)$$

we can reformulate (1) into

$$\begin{cases} \partial_t \theta + (u \cdot \nabla) \theta = -N^2 u_2, \\ \partial_t u + (u \cdot \nabla) u + \nabla p = \theta e_2, \\ \nabla \cdot u = 0. \end{cases} \quad (5)$$

Furthermore, let us set

$$\begin{aligned} u_3 &= \frac{\theta}{N}, \\ U &= (u_1, u_2, u_3)^T, \end{aligned} \quad (6)$$

and we get

$$\{\partial_t U + (U \cdot \tilde{\nabla})U + \nabla p = NJU, \tilde{\nabla} \cdot U = 0, \quad (7)$$

where $\tilde{\nabla} = (\nabla, 0)^T$ and J is a constant matrix given by

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (8)$$

Since $\tilde{\nabla} \cdot U = 0$, let us introduce the extended Helmholtz projector of the velocity u onto the divergence-free vector fields which is defined by

$$\mathbb{P} := \begin{pmatrix} \mathbb{P}_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (\delta_{jk} + R_j R_k)_{1 \leq j, k \leq 2} & 0 \\ 0 & 1 \end{pmatrix}, \quad (9)$$

where $\{R_j\}_{1 \leq j \leq 2}$ denote the Riesz transforms on $\mathbb{R}^2$. Applying the operator $\mathbb{P}$ to (7) gives the following equation:

$$\partial_t U + \mathbb{P}(U \cdot \tilde{\nabla})U = N\mathbb{P}JU. \quad (10)$$

The initial data to the above equations are given by

$$U|_{t=0} = U_0. \quad (11)$$

Now we state the main result.

Theorem 1. *Let $s \in \mathbb{N}$ satisfy $s > 3$ and*

$$\begin{aligned} U_0 &= (u_1, u_2, u_3) \in H^s(\mathbb{R}^2), \\ \tilde{\nabla} \cdot U_0 &= 0. \end{aligned} \quad (12)$$

Then, equations (10) and (11) possess a unique solution:

$$U \in C([0, T]; H^s), \quad (13)$$

with

$$T = C(\ln N)^{3/4}, \quad (14)$$

where the constant C is independent of N .

Remark 1. This result implies that the existence time will be larger as the buoyancy increases and the life span of the classical solutions satisfies

$$T > C(\ln N)^{3/4}. \quad (15)$$

1.1. Plan of the Article. The paper is organized as follows. In Section 2, we first derive the explicit formula of solutions to the linearized equations of (10) and (11), and then we establish the decay estimate and Strichartz estimate of a linear propagator. In Section 3, we establish the blowup criterion of equations (10) and (11). In Section 4, we present the proof of Theorem 1.

1.2. Notations. Throughout this paper, we denote by C the constants which may differ from line to line. L^p ($1 \leq p \leq \infty$), H^s ($s \in \mathbb{R}$), and $B_{p,q}^s$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ denote the Lebesgue spaces, Sobolev spaces, and the inhomogeneous Besov spaces, respectively. Let $\mathcal{F}$, $\mathcal{F}^{-1}$ denote the Fourier transformation and inverse Fourier transformation, respectively. The Littlewood–Paley multipliers $\Delta_{j \in \mathbb{Z}}$ are defined by

$$\Delta_j f = \mathcal{F}^{-1} \left(\psi \left(\frac{\xi}{2^j} \right) \mathcal{F} f \right), \quad (16)$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$, $\text{supp } \psi \subset \{\xi \in \mathbb{R}^n \mid (1/2) \leq |\xi| \leq 2\}$ such that

$$\sum_{j \in \mathbb{Z}} \psi \left(\frac{\xi}{2^j} \right) = 1, \quad \forall \xi \neq 0. \quad (17)$$

The low-frequency multiplier $\chi(\xi)$ is defined by

$$\chi(\xi) = 1 - \sum_{j \geq 1} \psi \left(\frac{\xi}{2^j} \right). \quad (18)$$

2. Linearized Equations

In this section, we derive the representation of solutions to the linearized equations of (10) and (11) and establish the decay estimate and Strichartz estimate of the linear propagator $e^{\pm i N t w^{(D)}}$ given by

$$e^{\pm i N t w^{(D)}} f := \mathcal{F}^{-1} e^{\pm i N t} (|\xi_1|/|\xi|) \hat{f}. \quad (19)$$

2.1. The Representation of Solutions to Linearized Equations. We study the following linearized equations associated to equations (10) and (11):

$$\begin{cases} \partial_t U = N\mathbb{P}J\mathbb{P}U, \\ \tilde{\nabla} \cdot U = 0, U(0, x) = U_0(x). \end{cases} \quad (20)$$

Applying the Fourier transform to (20) yields

$$\begin{cases} \partial_t \hat{U} - NP(\xi)JP(\xi)\hat{U} = 0, \\ (\xi, 0) \cdot \hat{U} = 0, \\ \hat{U}(0, x) = \hat{U}_0(\xi), \end{cases} \quad (21)$$

where $P(\xi)$ is the multiplier matrix of the operator $\mathbb{P}$ defined by

$$\widehat{\mathbb{P}U}(\xi) = P(\xi)\hat{U}(\xi), \quad (22)$$

which is given explicitly by

$$P(\xi) := \begin{pmatrix} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right)_{1 \leq j, k \leq 2} & 0 \\ 0 & 1 \end{pmatrix}. \quad (23)$$

Then, a direct calculation yields

$$P(\xi)JP(\xi) = \frac{1}{|\xi|^2} \begin{pmatrix} \xi_2^2 & -\xi_1 \xi_2 & 0 \\ -\xi_1 \xi_2 & \xi_1^2 & 0 \\ 0 & 0 & |\xi|^2 \end{pmatrix}, \quad (24)$$

and the eigenvalues of $P(\xi)JP(\xi)$ are

$$\omega_0 = 0, \quad \omega_{\pm}(\xi) := \pm i\omega(\xi) = \pm i \frac{|\xi_1|}{|\xi|}, \quad (25)$$

and the corresponding eigenvectors are

$$e_0 = \frac{1}{|\xi|}(\xi_1, \xi_2, 0), \quad e_{\pm} = \frac{1}{|\xi_1||\xi|}(\pm i\xi_1\xi_2, \mp i\xi_1^2, 1). \quad (26)$$

Thus, the solution of (21) is

$$\begin{aligned} \hat{U}(t, \xi) &= e^{NP(\xi)JP(\xi)t}, \\ \hat{U}_0(\xi) &= \langle \hat{U}_0(\xi), e_0 \rangle e_0 + e^{iN\omega(\xi)t} \langle \hat{U}_0(\xi), e_+ \rangle e_+ \\ &\quad + e^{-iN\omega(\xi)t} \langle \hat{U}_0(\xi), e_- \rangle e_-. \end{aligned} \quad (27)$$

Since $\tilde{\nabla} \cdot U = 0$, we have $\langle \hat{U}_0(\xi), e_0 \rangle = 0$. Hence, we get

$$\begin{aligned} \hat{U}(t, \xi) &= e^{NP(\xi)JP(\xi)t}, \\ \hat{U}_0(\xi) &= e^{iN\omega(\xi)t} \langle \hat{U}_0(\xi), e_+ \rangle e_+ + e^{-iN\omega(\xi)t} \langle \hat{U}_0(\xi), e_- \rangle e_-. \end{aligned} \quad (28)$$

Setting

$$\begin{aligned} P_+ U_0 &= \mathcal{F}^{-1}(\langle \hat{U}_0(\xi), e_+ \rangle e_+), \\ P_- U_0 &= \mathcal{F}^{-1}(\langle \hat{U}_0(\xi), e_- \rangle e_-), \end{aligned} \quad (29)$$

one has

$$U(t, x) = e^{iN\omega(D)t} P_+ U_0 + e^{-iN\omega(D)t} P_- U_0. \quad (30)$$

2.2. Decay Estimate. We derive the following decay estimate of the operator $e^{\pm itN\omega(D)}$.

Lemma 1. *It holds that*

$$\|e^{\pm itN\omega(D)}\Phi(D)f\|_{L^\infty} \leq C(1+|Nt|)^{-(1/2)}\|f\|_{L^1}, \quad (31)$$

for all $f \in L^1(\mathbb{R}^2)$, where the operator $\Phi(D)$ is defined by

$$\Phi(D)f = \mathcal{F}^{-1}(\hat{\Phi}(\xi)\hat{f}), \quad (32)$$

in which $\hat{\Phi}(\xi)$ satisfies

$$\begin{aligned} \text{supp } \hat{\Phi} &\subset \{\xi \in \mathbb{R}^2 | 2^{-2} \leq |\xi| \leq 2^2\}, \\ \hat{\Phi}(\xi) &= 1, \quad \forall \xi \in \{\xi \in \mathbb{R}^2 | 2^{-1} \leq |\xi| \leq 2\}. \end{aligned} \quad (33)$$

Actually, the result of Lemma 1 is an immediate consequence of the following lemma.

Lemma 2. *There exists a positive constant C independent of $(t, x) \in \mathbb{R}^{1+2}$ such that*

$$\left| \int_{\mathbb{R}^2} e^{ix\xi \pm iNt(|\xi_1|/|\xi|)} \hat{\Phi}(\xi) d\xi \right| \leq C(1+|Nt|)^{-(1/2)}. \quad (34)$$

Next, we give the details of the proof of Lemma 2. Firstly, we recall an important lemma (see Keel and Tao [17] and Majda [18]).

Lemma 3. *Let $d\sigma$ be a surface measure on a smooth surface $\mathcal{S}$ in $\mathbb{R}^n$ and let $\phi \in C_0^\infty(\mathbb{R}^n)$. Suppose that for all $x \in \mathcal{S}$, at least k of the principle curvatures are non-zero. Then, it holds that*

$$|\hat{d}\mu(\eta)| = |\phi \hat{d}\sigma(\eta)| \leq C|\eta|^{-(k/2)}. \quad (35)$$

2.2.1. The Proof of Lemma 2. By the theory of Fourier transform of measures supported on surfaces [18], we have

$$\begin{aligned} \int_{\mathbb{R}^2} e^{ix\xi \pm iNt(|\xi_1|/|\xi|)} \psi(\xi) d\xi &= \int_{\mathbb{R}^2} e^{i(x, Nt) \cdot (\xi, \pm(|\xi_1|/|\xi|))} \psi(\xi) d\xi \\ &= \hat{d}\mu(\eta), \quad \eta = (x, Nt), \end{aligned} \quad (36)$$

where μ is some measure supported on the surface $\mathcal{S}$ which is given by

$$\mathcal{S} = \left\{ (\xi, \rho) \in \mathbb{R}^2 \times \mathbb{R} : \rho = \pm \frac{|\xi_1|}{|\xi|}, \frac{1}{4} \leq |\xi| \leq 4 \right\}. \quad (37)$$

By Lemma 3, the decay of $\hat{d}\mu(\eta)$ is determined by the number of non-vanishing principle curvatures of the surface $\mathcal{S}$. Equivalently, the number of non-vanishing principle curvatures of the surface $\mathcal{S}$ is the rank of the Hessian matrix $H\rho = (\partial^2 \rho / \partial \xi_j \partial \xi_k)_{1 \leq j, k \leq 2}$. By some computations, we obtain

$$H\rho = \mp \frac{1}{|\xi|^5} \begin{bmatrix} -3\xi_1\xi_2^2 & \xi_2(3\xi_1^2 - |\xi|^2) \\ \xi_2(3\xi_1^2 - |\xi|^2) & \xi_1(3\xi_2^2 - |\xi|^2) \end{bmatrix}. \quad (38)$$

From (38), we have

$$\text{Det}(H\rho) = \pm \frac{\xi_2^2}{|\xi|^6}. \quad (39)$$

(39) shows that the surface $\mathcal{S}$ has a non-vanishing principle curvatures unless $\xi_2 = 0$. Thus, we decompose

$$\int_{\mathbb{R}^2} e^{ix\xi \pm iNt(|\xi_1|/|\xi|)} \psi(\xi) d\xi = I_1 + I_2, \quad (40)$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^2} e^{ix\xi \pm iNt(|\xi_1|/|\xi|)} (1 - \varphi(|\xi_2|)) \psi(\xi) d\xi, \\ I_2 &= \int_{\mathbb{R}^2} e^{ix\xi \pm iNt(|\xi_1|/|\xi|)} \varphi(|\xi_2|) \psi(\xi) d\xi, \end{aligned} \quad (41)$$

where φ is a smooth function on $\mathbb{R}$ such that $\varphi = 1$ on $[-(1/2), 2]$ and $\text{supp}\varphi \subset [-1, 1]$.

For the estimate of $\|I_1\|_{L^\infty}$, by Lemma 3, we have

$$\|I_1\|_{L^\infty} \leq C|(x, Nt)|^{-(2/2)} \leq C|Nt|^{-1}. \quad (42)$$

For the estimate of $\|I_2\|_{L^\infty}$, since $\xi_2 = 0$, we see that

$$H\rho = \mp \frac{1}{|\xi_1|^5} \begin{bmatrix} 0 & 0 \\ 0 & -\xi_1^3 \end{bmatrix}. \quad (43)$$

From (43) and the fact $|\xi| \sim 1$ ($(1/4) \leq |\xi| \leq 4$), we see one non-vanishing principle curvature of the surface $\mathcal{S}$. By Lemma 3, we have

$$\|I_2\|_{L^\infty} \leq C|(x, Nt)|^{-(1/2)} \leq C|Nt|^{-(1/2)}. \quad (44)$$

Combining (36), (42), and (44), we obtain

$$\int_{\mathbb{R}^2} e^{ix\xi \pm iNt(|\xi_1|/|\xi|)} \psi(\xi) d\xi \leq C|Nt|^{-(1/2)}. \quad (45)$$

For small $|t|$, it is trivial that

$$\int_{\mathbb{R}^2} e^{ix\xi \pm iNt(|\xi_1|/|\xi|)} \psi(\xi) d\xi \leq C. \quad (46)$$

From (45) and (46), we have

$$\left| \int_{\mathbb{R}^2} e^{ix\xi \pm iNt(|\xi_1|/|\xi|)} \psi(\xi) d\xi \right| \leq C(1 + |Nt|)^{-(1/2)}. \quad (47)$$

Thus, we complete the proof of Lemma 2.

2.3. Strichartz Estimate. Firstly, we recall the following result obtained by Hu et al. [14].

Lemma 4. Let $S(t)$, $t \in \mathbb{R}$ be a family of operators. Suppose that for all $t, s \in \mathbb{R}$,

$$\begin{aligned} \|S(s)S^*(t)f(x)\|_{L^\infty} &\leq (1 + |t - s|)^{-\sigma} \|f\|_{L^1}, \\ \|S(s)S^*(t)f(x)\|_{L^2} &\leq \|f\|_{L^2}. \end{aligned} \quad (48)$$

Then, the estimates

$$\|S(t)f(x)\|_{L_t^q L_x^r} \leq \|f\|_{L^2} \quad (49)$$

hold for all $2 \leq q, r \leq \infty$ with $(q, r, \sigma) \neq (2, \infty, 1)$ satisfying

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}. \quad (50)$$

From Lemmas 1 and 4, the fact $\Phi(D)\Delta_0 = \Delta_0$, and the scaling in time $t \mapsto Nt$, we obtain the following.

Lemma 5. For $2 \leq q, r \leq \infty$ satisfying $(1/q) + (1/2r) \leq (1/4)$, we have the following Strichartz estimate:

$$\|e^{\pm itNw(D)}\Delta_0 f(x)\|_{L_t^q L_x^r} \leq N^{-(1/q)} \|\Delta_0 f\|_{L^2}. \quad (51)$$

3. Blowup Criterion

This section shows a blowup criterion of equations (10) and (11).

Lemma 6. Let $U(t)$ be a solution of equations (10) and (11) defined on a time interval containing $[0, T]$. Then, for any $s > 0$, we have the bounded estimate

$$\|U(t)\|_{H^s}^2 \leq \|U_0\|_{H^s}^2 \exp\left(C \int_0^T \|\nabla U(\tau)\|_{L^\infty} d\tau\right). \quad (52)$$

Proof. For the vector variable $U(t) = (u(t), u_3)^T$, we obtain

$$\begin{cases} \partial_t u_3 + (u \cdot \nabla)u_3 = -Nu_2, \\ \partial_t u + (u \cdot \nabla)u + \nabla p = Nu_3 e_2, \\ \nabla \cdot u = 0. \end{cases} \quad (53)$$

We take s derivatives of the first and the second equation in (53), multiply by $\nabla^s u_3$ and $\nabla^s u$, respectively, and integrate over $\mathbb{R}^2$ to obtain

$$\frac{d}{dt} \|\nabla^s u_3\|_{L^2}^2 + \int \nabla^s (u \cdot \nabla u_3) \nabla^s u_3 dx = -N \int \nabla^s u_2 \nabla^s u_3 dx, \quad (54)$$

$$\begin{aligned} \frac{d}{dt} \|\nabla^s u\|_{L^2}^2 + \int \nabla^s (u \cdot \nabla u) \nabla^s u dx \\ + \int \nabla^s \nabla p \cdot \nabla^s u dx = N \int \nabla^s u_3 e_2 \nabla^s u dx. \end{aligned} \quad (55)$$

Note that

$$-N \int \nabla^s u_2 \nabla^s u_3 dx + N \int \nabla^s u_3 e_2 \nabla^s u dx = 0. \quad (56)$$

Since $\nabla \cdot u = 0$, we have

$$\begin{aligned}
\int \nabla^s \nabla p \cdot \nabla^s u dx &= 0, \\
\int u \nabla^s \nabla u_3 \nabla^s u_3 dx &= 0, \\
\int u \nabla^s \nabla u \nabla^s u dx &= 0.
\end{aligned} \tag{57}$$

The commutator-type estimate (see, e.g., [12, 15]) provides us with

$$\begin{aligned}
\left| \int \nabla^s (u \cdot \nabla u_3) \nabla^s u_3 dx \right| &\leq C \|\nabla U(t)\|_{L^\infty} \|\nabla^s U(t)\|_{L^2}^2, \\
\left| \int \nabla^s (u \cdot \nabla u) \nabla^s u dx \right| &\leq C \|\nabla U(t)\|_{L^\infty} \|\nabla^s U(t)\|_{L^2}^2.
\end{aligned} \tag{58}$$

Thus, adding (54) and (55) gives

$$\frac{d}{dt} \|\nabla^s U(t)\|_{L^2}^2 \leq C \|\nabla U(t)\|_{L^\infty} \|\nabla^s U(t)\|_{L^2}^2, \tag{59}$$

which implies that by the Gronwall inequality for $t \in [0, T]$,

$$\|\nabla^s U(t)\|_{L^2}^2 \leq \|\nabla^s U_0\|_{L^2}^2 \exp\left(C \int_0^T \|\nabla U(t)\|_{L^\infty} dt\right). \tag{60}$$

The above equality implies that Lemma 6 holds. $\square$

4. Proof of Theorem 1

Applying $P_\pm$ to (10) and using the fact $P_\pm \mathbb{P} = P_\pm$ gives

$$\partial_t P_\pm U \mp iN\omega(D)P_\pm U + P_\pm (U \cdot \tilde{\nabla})U = 0. \tag{61}$$

By the Duhamel principle, we get

$$P_\pm U = e^{\pm iN\omega(D)t} P_\pm U_0 + \int_0^t e^{\pm iN\omega(D)(t-\tau)} P_\pm (U \cdot \tilde{\nabla})U(\tau) d\tau. \tag{62}$$

Due to Lemma 5 and scaling, we find that for $q \geq 4$,

$$\|\Delta_j e^{\pm iN\omega(D)t} f\|_{L_t^q L_x^\infty} \leq N^{-(1/q)} 2^j \|\Delta_j f\|_{L^2}. \tag{63}$$

In the following, for $\alpha > 0$, we are going to derive the estimates of

$$\|P_\pm U\|_{L^q B_{\infty, \infty}^{1+\alpha}} = \left\| \chi(D) P_\pm U \right\|_{L^\infty} + \sup_{j \geq 1} \left(2^{j(1+\alpha)} \|\Delta_j P_\pm U\|_{L^\infty} \right)_{L_t^q}. \tag{64}$$

By (63) and the Bernstein inequality, we have

$$\begin{aligned}
\|2^{j(1+\alpha)} \|\Delta_j e^{\pm iN\omega(D)t} P_\pm U_0\|_{L^\infty}\|_{L_t^q} &\leq 2^{j(2+\alpha)} N^{-(1/q)} \|\Delta_j P_\pm U_0\|_{L^2} \\
&\leq N^{-(1/q)} \|P_\pm U_0\|_{H^{2+\alpha}}.
\end{aligned} \tag{65}$$

On the other hand, we have from (61) that

$$\begin{aligned}
\|\chi(D) e^{\pm iN\omega(D)t} P_\pm U_0\|_{L_t^q L_x^\infty} &\leq \left\| \sum_{j=-\infty}^2 \|\Delta_j e^{\pm iN\omega(D)t} P_\pm U_0\|_{L_t^q L_x^\infty} \right\|_{L_t^q} \\
&\leq \sum_{j=-\infty}^2 \|\Delta_j e^{\pm iN\omega(D)t} P_\pm U_0\|_{L_t^q L_x^\infty} \\
&\leq N^{-(1/q)} \sum_{j=-\infty}^2 2^j \|\Delta_j P_\pm U_0\|_{L^2} \\
&\leq N^{-(1/q)} \left(\sum_{j=-\infty}^2 2^{2j} \right)^{(1/2)} \\
&\quad \left(\sum_{j=-\infty}^2 \|\Delta_j P_\pm U_0\|_{L^2}^2 \right)^{(1/2)} \\
&\leq CN^{-(1/q)} \|P_\pm U_0\|_{L^2}.
\end{aligned} \tag{66}$$

(65) and (66) give

$$\|e^{\pm iN\omega(D)t} P_\pm U_0\|_{L_t^q B_{\infty, \infty}^{1+\alpha}} \leq CN^{-(1/q)} \|P_\pm U_0\|_{H^{2+\alpha}}. \tag{67}$$

By (63), we have

$$\begin{aligned}
&\left\| 2^{j(1+\alpha)} \|\Delta_j \int_0^t e^{\pm iN\omega(D)(t-\tau)} P_\pm (U \cdot \tilde{\nabla})U(\tau) d\tau\|_{L^\infty} \right\|_{L_t^q} \\
&\leq \int_0^t \left\| 2^{j(1+\alpha)} \|\Delta_j e^{\pm iN\omega(D)(t-\tau)} P_\pm (U \cdot \tilde{\nabla})U(\tau)\|_{L^\infty} \right\|_{L_t^q} d\tau \\
&\leq \int_0^t 2^{j(2+\alpha)} N^{-(1/q)} \|\Delta_j P_\pm (U \cdot \tilde{\nabla})U(\tau)\|_{L^2} d\tau \\
&\leq N^{-(1/q)} \int_0^t \|P_\pm (U \cdot \tilde{\nabla})U(\tau)\|_{H^{2+\alpha}} d\tau \\
&\leq CN^{-(1/q)} \int_0^t \|U(\tau)\|_{H^{3+\alpha}}^2 d\tau.
\end{aligned} \tag{68}$$

Similar to (66), we have

$$\begin{aligned}
&\left\| \chi(|\nabla|) \int_0^t e^{\pm iN\omega(D)(t-\tau)} P_\pm (U \cdot \tilde{\nabla})U(\tau) d\tau \right\|_{L_t^q L_x^\infty} \\
&\leq \sum_{j=-\infty}^2 \left\| \Delta_j \int_0^t e^{\pm iN\omega(D)(t-\tau)} P_\pm (U \cdot \tilde{\nabla})U(\tau) d\tau \right\|_{L_t^q L_x^\infty} \\
&\leq N^{-(1/q)} \int_0^t \sum_{j=-\infty}^2 2^j \|\Delta_j P_\pm (U \cdot \tilde{\nabla})U(\tau)\|_{L^2} d\tau \\
&\leq N^{-(1/q)} \int_0^t \|P_\pm (U \cdot \tilde{\nabla})U(\tau)\|_{H^1} d\tau \\
&\leq CN^{-(1/q)} \int_0^T \|u\|_{H^2}^2 d\tau.
\end{aligned} \tag{69}$$

(68) and (69) imply

$$\left\| \int_0^t e^{\pm iN\omega(D)(t-\tau)} P_{\pm}(U \cdot \tilde{\nabla})U(\tau) d\tau \right\|_{L_t^q B_{\infty,0}^{1+\alpha}} \leq CN^{-(1/q)} \int_0^t \|U(\tau)\|_{H^{3+\alpha}}^2 d\tau. \quad (70)$$

Combining (67) and (70) yields

$$\|P_{\pm}U\|_{L_t^q B_{\infty,0}^{1+\alpha}} \leq CN^{-(1/q)} \left(\|P_{\pm}U_0\|_{H^{2+\alpha}} + \int_0^t \|U(\tau)\|_{H^{3+\alpha}}^2 d\tau \right). \quad (71)$$

Define

$$M(T) = \int_0^T \|\nabla U\|_{L^\infty} dt. \quad (72)$$

We get from (71) and Lemma 6 that

$$\begin{aligned} M(T) &\leq T^{1-(1/q)} \left(\int_0^T \|\nabla U\|_{L^\infty}^q dt \right)^{(1/q)} \\ &\leq CT^{1-(1/q)} N^{-(1/q)} \left(\|P_{\pm}U_0\|_{H^{2+\alpha}} + \int_0^T \|U(t)\|_{H^{3+\alpha}}^2 dt \right) \\ &\leq CT^{1-(1/q)} N^{-(1/q)} \left(\|P_{\pm}U_0\|_{H^{2+\alpha}} + \int_0^T \|U_0\|_{H^{3+\alpha}}^2 e^{M(t)} dt \right) \\ &\leq CT^{1-(1/q)} N^{-(1/q)} \left(\|U_0\|_{H^{2+\alpha}} + \|U_0\|_{H^{3+\alpha}}^2 T e^{M(T)} \right) \\ &\leq C_1 T^{1-(1/q)} N^{-(1/q)} \|U_0\|_{H^{3+\alpha}} (1 + \|U_0\|_{H^{3+\alpha}} T e^{M(T)}). \end{aligned} \quad (73)$$

Let $q = 4$ and suppose that

$$M(T) \leq C_1 T^{3/4} \|U_0\|_{H^{3+\alpha}}. \quad (74)$$

We can choose N sufficiently large such that

$$\begin{aligned} &N^{-(1/4)} (1 + \|U_0\|_{H^{3+\alpha}} T e^{M(T)}) \\ &\leq N^{-(1/4)} \left(1 + \|U_0\|_{H^{3+\alpha}} T e^{C_1 T^{4/3} \|U_0\|_{H^{3+\alpha}}} \right) \leq \frac{1}{2}. \end{aligned} \quad (75)$$

Combining (73) with (74) gives

$$M(T) \leq \frac{1}{2} C_1 T^{3/4} \|U_0\|_{H^{3+\alpha}}. \quad (76)$$

Finally, the restriction (75) implies T can choose

$$T = C (\ln N)^{3/4}. \quad (77)$$

By the bootstrap principle, we deduce from (76) that (74) actually holds. Thus, from the classical local existence (see [12]) and the blowup criterion Lemma 6, (10) and (11) possess unique classical solutions satisfying $u \in C([0, T]; H^s)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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