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Research Article

Equivalent Cauchy Sequences in (q_1, q_2)-Quasi Metric-Like Space and Applications to Fixed-Point Theory

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This paper introduces the concept of (q_1, q_2)-quasi metric-like space (X, p) and delves into some topological properties of it. A necessary and sufficient condition related to equivalent Cauchy sequences and Cauchy sequences in (X, p) have been proved. Furthermore, the results of the fixed point are shown in the setting of (q_1, q_2)-quasi metric-like space as applications of conditions of equivalent Cauchy sequences. Besides, some instances are inclined to epitomize the examined consequences.

1. Introduction and Preliminaries

The theory of fixed point has a crucial role not only in scientific and engineering applications [1–5] but also in theory.

The development of this theory is accomplished in two directions, either by generalizing the Banach contraction or by giving new metric type spaces. There are a lot of generalizations of the metric space based on reducing or modifying the metric axioms.

Some generalized metric spaces which are obtained by the above changes are, b-metric space [6–8], quasi-metric spaces, partial-metric space [9], partial-metric spaces [10, 11], dislocated metric spaces [12, 13], rectangular partial metric spaces [14], and so on.

In 2012, Harandi [15] introduced the notion of metric-like space which was given as dislocated metric space in [12, 13].

Later, in 2016, Arutyunov and Greshnov [16] introduced the notion of (q_1, q_2) quasi-metric space and examined their properties. Moreover, they proved some results on fixed points in this space. In addition, they extended their results in [17, 18].

Henceforth, many mathematicians have researched several masterwork involving fixed points for self-mappings in generalized metric spaces.

Necessarily, a technical tool in the proof of fixed-point theorems is the Cauchy and the equivalent Cauchy sequences.

In 1983, Leader [19] obtained a necessary and sufficient condition as a characterization of equivalent Cauchy sequences in the metric space. Nevertheless, the generalizations of the consequences are found in spaces, [20, 21], obtaining fixed-point results for the contractive function of Mein-Keeler type.

As a consequence, this invigorated us to introduce the notion of the (q_1, q_2)-quasi metric-like space. Further, there are shown some features of topology induced by a (q_1, q_2)-quasi metric-like space.

The highlight of this paper is Theorem 3, which gives a necessary and sufficient condition that two sequences are equivalent Cauchy in the (q_1, q_2)-quasi metric-like space. An application of this result is Theorem 1, which extends the existing results for the Mein-Keeler contractions in the (q_1, q_2)-quasi metric-like space. One example is given to authenticate the effectiveness and applicability of our results.

Amini [15] defined the metric-like space as the forthcoming definition.

Definition 1 (see [15]). Let X be a nonempty set. A metric like in X is a mapping $\sigma\colon X \times X \longrightarrow R^+$ such that for all x, y, and $z \in X$, it satisfies the following:

(i) $(\sigma_1)\, \sigma(x, y) = 0$ implies $x = y$

(ii) $(\sigma_2)\, \sigma(x, y) = \sigma(y, x)$

(iii) $(\sigma_3)\, \sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$

The pair (X, σ) is called a metric-like space.

Alghamdi et al. [22] extended the concept of the metric-like space to b metric-like space as follows.

Definition 2 (see [22]). Let X be a nonempty set and $s \geq 1$ is a given real number. A function $\sigma_b\colon X \times X \longrightarrow R^+$ is b-metric like if for all x, y, and $z \in X$, the following conditions are accomplished:

(i) $(\sigma_{b1})\, \sigma(x, y) = 0$ implies $x = y$

(ii) $(\sigma_{b2})\, \sigma(x, y) = \sigma(y, x)$

(iii) $(\sigma_{b3})\, \sigma(x, y) \leq s(\sigma(x, z) + \sigma(z, y))$

The pair (X, σ_b) is called b metric-like space. The number s is called the coefficient of (X, σ_b).

This definition was generalized by Arutyunov and Greshov [16], who gave the concepts of (q_1, q_2)-metric space and (q_1, q_2) quasi-metric space as in Definitions 3 and 4.

Definition 3 (see [16]). Let X be a nonempty set, q_1 and q_2 be two positive numbers, and $\rho_x\colon X \times X \longrightarrow R^+$ be a given function satisfying the following conditions for all x, y, and $z \in X$:

(i) $(\rho_{x1})\, \rho_x(x, y) = 0$ if and only if $x = y$

(ii) $(\rho_{x2})\, \rho_x(x, y) = \rho_x(y, x)$

(iii) $(\rho_{x3})\rho_x(x, y) \leq q_1\rho_x(x, z) + q_2\rho_x(z, y)$

The function ρ_x is called (q_1, q_2)-metric. The pair (X, ρ_x) is called (q_1, q_2)-metric space.

Definition 4 (see [16]). Let X be a nonempty set, q_1 and q_2 two positive numbers, and $\rho_x\colon X \times X \longrightarrow R^+$ be a given function that satisfies the following conditions:

(i) $(\rho_{x1})\, \rho_x(x, y) = \rho_x(y, x) = 0$ if and only if $x = y$, for each x and $y \in X$

(ii) $(\rho_{x2})\rho_x(x, y) \leq q_1\rho_x(x, z) + q_2\rho_x(z, y)$, for every x, y, and $z \in X$

The function ρ_x is called (q_1, q_2)-quasi metric, and the pair (X, ρ_x) is called (q_1, q_2) quasi-metric space.

These definitions motivated us to define a new space and give some topological aspects of it.

2. (q_1, q_2)-Quasi Metric-Like Space and Its Topology

In this section, there is definition of (q_1, q_2)-quasi metric-like space. Furthermore, the topology induced by (q_1, q_2)-quasi metric-like space is given and its properties are proved.

Definition 5. Let X be a set and q_1 and q_2 be two positive numbers. The function $p\colon X \times X \longrightarrow R^+$ is called (q_1, q_2)-quasi metric-like if for all x, y, and $z \in X$, and the following conditions are satisfied:

(i) (p_1) $p(x, y) = p(y, x) = 0$ implies $x = y$

(ii) (p_2) $p(x, y) \leq q_1 p(x, y) + q_2 p(x, y)$

The pair (X, p) is called the (q_1, q_2)-quasi metric-like space.

Remark 1. If $q_1 = q_2 = 1$, as a result, the concept of (q_1, q_2)-quasi metric-like space coincides with the concept of quasi metric-like space [22].

If $q_1 = q_2 > 1$, consequently, the concept of (q_1, q_2)-quasi metric-like space coincides with the concept of b-quasi metric-like space [16].

If (X, p) is a (q_1, q_2)-quasi metric-like space and $\max\{q_1, q_2\} = s > 1$, then (X, p) is a b-quasi metric-like space [22].

If $\max\{q_1, q_2\} = s \leq 1$, as a consequence (X, p) is a quasi metric-like space [22].

The above facts prove that the (q_1, q_2)-quasi metric-like space is a generalization of space mentioned in [16, 22].

If (X, p) is a (q_1, q_2)-quasi metric-like space, then for x and $y \in X$ and $p(x, y) = p(y, x) = 0$ implies $x = y$. However, the converse may not be concordant, and $p(x, x)$ may be positive for $x \in X$.

Example 1. Let $X = [0, 1[$, $q_1 = 3, q_2 = 2$ and $p\colon X \times X \longrightarrow R^+$ be defined by

$$p(x, y) = \begin{cases} 2, & x = y, \\ \dfrac{1}{2}, & x < y, \\ \dfrac{1}{4}, & x > y. \end{cases} \tag{1}$$

This prompts the couple (X, p) to be a (q_1, q_2)-quasi metric-like space, and $p(x, x) = 2$, for every $x \in X$.

The following definitions are given to introduce the topology induced by a (q_1, q_2)-quasi metric-like space.

Definition 6

The set $B^r(a, \varepsilon) = \{x \in X\colon p(x, a) < \varepsilon\}$ is called a right open ball with center a and radius $\varepsilon > 0$

The set $B^l(a, \varepsilon) = \{x \in X\colon p(a, x) < \varepsilon\}$ is called a left open ball with center a and radius $\varepsilon > 0$

It is clear-cut from Definitions 5 and 6 that the center of a ball may not be an element of it. For example, in the (q_1, q_2)-quasi metric-like space of Example 2, the element 0 is not in $B^r(0, (1/2))$ because $p(0, 0) = 2 > (1/2)$.

Furthermore, a right open ball $B^r(a,\varepsilon)$ is different from a left open ball $B^l(a,\varepsilon)$. For example, in the (q_1,q_2)-quasi metric-like space given in Example 2, there is

$$B^r\left(0,\ \frac{1}{3}\right)=\left\{x\in[0,1[:\ p(x,0)<\frac{1}{3}\right\}. \tag{2}$$

Since for each $x\in[0,1[$, $p(x,0)=(1/4)<(1/3)$, there is implied $B^r(0,(1/3))=X=[0,1[$.

Additionally, $B^l(0,(1/3))=\{x\in[0,1[:\ p(0,x)<(1/3)\}$, and since for every $x\in X=[0,1[$, $p(0,x)=(1/2)>(1/3)$. This leads to $B^l(0,(1/3))=\phi$.

Definition 7

The set $B'^r(a,\varepsilon)=\{x\in X:\ p(x,a\le\varepsilon\}$ is called a right closed ball with center a and radius $\varepsilon>0$

The set $B'^l(a,\varepsilon)=\{x\in X:\ p(a,x)\le\varepsilon\}$ is called a left closed ball with center a and radius $\varepsilon>0$

Proposition 1. *Let (X,p) be a (q_1,q_2)-quasi metric-like space. The family $\tau^r=\{\phi,X,G\subset X$, for every $a\in G$, there exists $B^r(a,x)\subset G\}$, is a topology in (X,p). It is called the right topology in X obtained from (q_1,q_2)-quasi metric-like p.*

The family $\tau^l=\{\phi,X,G\subset X$, for every $a\in G$, there exists $B^l(a,x)\subset G\}$, is a topology in (X,p). It is called left topology obtained from (q_1,q_2)-quasi metric-like p.

All following propositions related to right topology can be proved using analog methods for left topology induced by (q_1,q_2)-quasi metric-like p.

Definition 8. A set $A\subset X$ is said to be right (left) open if for every point $a\in A$, there exists a right (left) open ball $B^r(a,\varepsilon)\subset A$, $(B^l(a,\varepsilon)\subset A)$ (this means $A\in\tau^r$ $(A\in\tau^l)$).

Definition 9. A set $A\subset X$ is said to be right (left) closed if $A^c\in\tau^r$ $(A^c\in\tau^l)$.

Proposition 2. *Every (q_1,q_2)-quasi metric-like space (X,τ^r) is T_1.*

Proof. Let (X,τ^r) be a (q_1,q_2)-quasi metric-like space. To prove that space (X,τ^r) is T_1, it needs to show that, for every $x\in X$, the set $\{x\}$ is right closed or the set $\{x\}^c$ is a right open.

Indeed, for every $a\in\{x\}^c=X-\{x\}$, $p(x,a)>0$. Taking $\varepsilon=p(x,a)>0$, there exists $B^r(a,(\varepsilon/2))\subset X-\{x\}$. This is true because for every $z\in B^r(a,(\varepsilon/2))$, it yields $p(z,a)<(\varepsilon/2)<\varepsilon=p(x,a)$. It means that $z\neq x$; consequently, $z\in X-\{x\}$. □

Proposition 3. *The topology τ^r satisfies the first axiom of countability.*

Proof. The set $\{B^r(a,(1/n))\}_{n\in N}$ forms a countable base of a. □

Proposition 4. *Let (X,p) be a (q_1,q_2)-quasi metric-like space. The family $\tau=\tau^r\cap\tau^l$ is a topology in (X,p).*

Proof. Constructing the family $\tau=\{\phi,X,G\subset X$: for each $a\in G$, there exist $B^r(a,\varepsilon_1)$ and $B^l(a,\varepsilon_2)$ such that $a\in B^r(a,\varepsilon_1)\cap B^l(a,\varepsilon_2)\subset G\}$, we assure that it is topology.

The topology τ is called the topology induced by (q_1,q_2)-quasi metric-like p. □

Proposition 5. *If (X,p) is a (q_1,q_2)-quasi metric-like space, then the function $d_{q_1q_2}$: $X\times X\longrightarrow R^+$ where $d_{q_1q_2}(x,y)=\max\{p(x,y),p(y,x)\}$ is a (q_1,q_2) metric like.*

Proof. This is clear since $d_{q_1q_2}(x,y)$ satisfies also the symmetry condition.

The topology induced by (q_1,q_2) metric like $d_{q_1q_2}$ is the topology $\tau=\tau^r\cap\tau^l$ defined above.

In the following paragraphs, there are defined Cauchy sequences and equivalent Cauchy sequences in (q_1,q_2)-quasi metric-like space. □

Definition 10. Let (X,p) be a (q_1,q_2)-quasi metric-like space and $\{x_n\}_{n\in N}$ be a sequence in X.

(1) The sequence $\{x_n\}_{n\in N}$ is called right Cauchy in X if for every $\varepsilon>0$, there exists $n_0\in N$, such that, for every $n>m>n_0$, $p(x_n,x_m)<\varepsilon$. It is denoted that $\lim_{n,m\longrightarrow+\infty}p(x_n,x_m)=0$.

(2) The sequence $\{x_n\}_{n\in N}$ is called left Cauchy in X if for every $\varepsilon>0$, there exists $n_0\in N$, such that, for every $n>m>n_0$, $p(x_m,x_n)<\varepsilon$ $(\lim_{n,m\longrightarrow+\infty}p(x_m,x_n)=0)$.

The concepts of the right Cauchy sequence and the left Cauchy sequence are independent of each other. This means that a sequence may be right Cauchy but not left Cauchy or vice versa.

The following example illustrates it.

Example 2. Let $X=[0,3]$ and $p(x,y)=\begin{cases}2x+y, & x\ge y,\\ 1, & x<y\end{cases}$.

The function $p(x,y)$ is a (q_1,q_2)-quasi metric-like with $q_1=1$ and $q_2=3$.

Nevertheless, it is not a quasi metric-like because for $x\in X$, $x>y$, $y=2$, and $z=0$, and we attain $p(x,y)=2x+y=2x+2$, $p(x,z)=2x+z=2x$, $p(z,y)=1$.

As a result, $p(x,y)=2x+2\ge 2x+1=p(x,z)+p(z,y)$.

Taking the sequence $x_n=(1/n+1)$ in X, for $n>m$, it yields $x_n=(1/n+1)<(1/m+1)=x_m$ and $p(x_m,x_n)=2x_m+x_n=(2/m+1)+(1/n+1)<(3/m+1)$.

Consequently, $0\le\lim_{n,m\longrightarrow+\infty}p(x_m,x_n)\le\lim_{n,m\longrightarrow+\infty}(3/m+1)=0$ and $\lim_{n,m\longrightarrow+\infty}p(x_m,x_n)=0$ which proves that the sequence $\{x_n\}_{n\in N}$ is left Cauchy.

In addition, $\lim_{n,m\longrightarrow+\infty}p(x_n,x_m)=\lim_{n,m\longrightarrow+\infty}1=1$. Hereof, it yields that $\{x_n\}_{n\in N}$ is not a right Cauchy sequence.

Definition 11. Let (X,p) be a (q_1,q_2)-quasi metric-like space and $\{x_n\}_{n\in N}$ be a sequence in X. The sequence $\{x_n\}_{n\in N}$ in X is called Cauchy if it is left Cauchy and right Cauchy.

Definition 12. Let (X, p) be a (q_1, q_2)-quasi metric-like space and $\{x_n\}_{n\in N}$ be a sequence in X.

(1) The sequence $\{x_n\}_{n\in N}$ is called right convergent to $x \in X$ if, for every $\varepsilon > 0$, there exists $n_0 \in N$, such that, for every $n > n_0$, $p(x, x_n) < \varepsilon$. It is denoted $\lim_{n\longrightarrow+\infty} p(x, x_n) = 0$.

(2) The sequence $\{x_n\}_{n\in N}$ is called left convergent to $x \in X$ if for every $\varepsilon > 0$, there exists $n_0 \in N$, such that, for every $n > n_0$, $p(x_n, x) < \varepsilon$ ($\lim_{n\longrightarrow+\infty} p(x_n, x) = 0$).

(3) The sequence $\{x_n\}_{n\in N}$ is called convergent to $x \in X$ if it is left convergent and right convergent to $x \in X$. Consequently, for every $\varepsilon > 0$, there exists $n_0 \in N$, such that, for every $n > n_0$, $p(x, x_n) < \varepsilon$ and $p(x_n, x) < \varepsilon$ or $\lim_{n\longrightarrow+\infty} p(x, x_n) = \lim_{n\longrightarrow+\infty} p(x_n, x) = 0$.

Since (X, τ^r) and (X, τ^l) could not be T_2 space, there is conceded that these spaces do not satisfy the uniqueness of the limit. As a result, a right (left) convergent sequence might converge to more than one limit.

However, space $(X, d_{q_1q_2})$, where $d_{q_1q_2}$is the (q_1, q_2) metric like obtained from (q_1, q_2)-quasi metric-like p, is T_2, and the convergent sequences in (X, p) converge only to a unique limit.

Note that every convergent sequence in (X, p) is a Cauchy sequence.

Indeed, if $\{x_n\}_{n\in N}$ is a convergent sequence to a point a in (X, p), then the following inequalities are veritable: $p(x_n, x_m) \le q_1 p(x_n, a) + q_2 p(a, x_m)$ and $p(x_m, x_n) \le q_1 p(x_m, a) + q_2 p(a, x_n)$.

Consequently, $\lim_{n,m\longrightarrow+\infty} p(x_n, x_m) = \lim_{n,m\longrightarrow+\infty} p(x_m, x_n) = 0$.

Definition 13. Let (X, p) be a (q_1, q_2) metric-like space and $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ be two sequences in X.

(1) The sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are left equivalents if for every $\varepsilon > 0$, there exists $n_0 \in N$, such that, for every $n > m > n_0$, $p(x_n, y_m) < \varepsilon$. It is denoted $\lim_{n,m\longrightarrow+\infty} p(x_n, x_m) = 0$.

(2) The sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are right equivalents if for every $\varepsilon > 0$, there exists $n_0 \in N$, such that, for every $n > m > n_0$, $p(y_n, x_m) < \varepsilon$ ($\lim_{n,m\longrightarrow+\infty} p(y_n, x_m) = 0$).

(3) The sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are equivalents if they are left equivalent and right equivalent or $\lim_{n,m\longrightarrow+\infty} p(x_n, x_m) = \lim_{n,m\longrightarrow+\infty} p(x_m, x_n) = 0$.

Definition 14. Let(X, p) be a (q_1, q_2) metric-like space and $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ be two sequences in X.

(1) The sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are left equivalent Cauchy if they are left equivalent and left Cauchy

(2) The sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are right equivalent Cauchy if they are right equivalent and right Cauchy

(3) The sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are equivalent Cauchy if they are left equivalent Cauchy and right equivalent Cauchy

3. Main Results

3.1. A Necessary and Sufficient Condition for Equivalent Cauchy Sequences in the $(\mathbf{q}_1, \mathbf{q}_2)$ *Metric-Like Space.* Leader [19] has given the following necessary and sufficient condition for the characterization of equivalent Cauchy sequences in the metric space.

Theorem 1 (see [19]). *Two sequences* $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N}$ *in metric space* (X, d) *are equivalent Cauchy if and only if for every* $\varepsilon > 0$, *there exists* $\delta \in [0, +\infty[$ *and* $r \in N$ *such that* $d(x_i, y_j) < \varepsilon + \delta$, *implying* $d(x_{i+r}, y_{j+r}) < \varepsilon$, *for all* i *and* $j \in N$.

Taking $y_n = x_n$ for every $n \in N$, Theorem 3.2.7 in [19], it gives a necessary and sufficient condition that a sequence is Cauchy in the metric space.

Besides, Pasicki [20] and Hoxha and Duraj [21] have given necessary and sufficient conditions in order for a sequence $\{x_n\}_{n\in N}$ to be Cauchy in dislocated spaces and quasi-dislocated spaces.

In this section, we derive a result for equivalent Cauchy sequences and Cauchy sequences in the (q_1, q_2) metric-like space.

Theorem 2. *Let* (X, p) *be a* (q_1, q_2) *metric-like space and* $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N}$ *be two sequences in X. If the sequences* $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N}$ *are equivalent and convergent to a and b, respectively, then* $a = b$.

Proof. Taking limits in the following inequalities:

$$
\begin{aligned}
p(a,b) &\le q_1 p(a,x_n) + q_2 p(x_n,b) \le q_1 p(a,x_n) \\
&\quad + q_1 q_2 p(x_n,y_n) + q_2^2 p(y_n,b), \\
p(b,a) &\le q_1 p(b,y_n) + q_2 p(y_n,b) \le q_1 p(a,y_n) \\
&\quad + q_1 q_2 p(y_n,x_n) + q_2^2 p(x_n,b),
\end{aligned} \tag{3}
$$

where $p(a,b) = 0 = p(b,a)$ is yielded. Consequently, $a = b$. □

Theorem 3. *Let* (X, p) *be a* (q_1, q_2) *metric-like space where* $q_1 > 0$ *and* $q_2 \ge 1$ *and* $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N}$ *be two sequences in X.*

The sequences $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N}$ *are equivalent Cauchy if and only if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *and* $r \in N$, *such that (1)* $(1/q_2^2)\max\{p(x_i, y_j), p(y_j, x_i)\} < \varepsilon + \delta$, *implying* $\max\{q(x_{i+r}, y_{j+r}), q(y_{j+r}, x_{i+r})\} < \varepsilon$, *for all i and* $j \in N$.

Proof. Denote $p_k(n) = \max\{p(x_i, y_j), p(y_i, x_j)\colon n \le i,\ j \le n + k\}$. The set $\{p_{k(n)}\colon n \in N\}$, where k is fixed, is lower bound from 0. As a result, $\inf\{p_k(n)\colon n \in N\} = a \ge 0$.

If $a > 0$, then for $\varepsilon = a$ in (1), there exist $\delta > 0$ and $r \in N$, such that $(1/q_2^2)\max\{p(x_i, y_j), p(y_i, x_j)\} < \varepsilon + \delta = a + \delta$, implying $\max\{q(x_{i+r}, y_{j+r}),\ q(y_{j+r}, x_{i+r})\} < \varepsilon = a$.

From the characteristic property of inferior for $\delta > 0$, there exists $n \in N$, such that $a \leq p_k(n) \leq a + \delta$. Besides, $\max\{p(x_i, y_j), p(y_i, x_j): n \leq i,\ j \leq n + k\} < a + \delta$.

Accordingly,

$(1/q_2^2)\max\{p(x_i, y_j), p(y_i, x_j)\} < \varepsilon + \delta = a + \delta$, and from (1), $\max\{q(x_{i+r}, y_{j+r}), q(y_{j+r}, x_{i+r})\} < \varepsilon = a$ is yielded for every $n \leq i$ and $j \leq n + k$ meaning that $p_k(n + r) < a$ which is absurd.

As a result,

$$a = 0, \text{ or } \inf\{p_k(n): n \in N\} = 0. \tag{4}$$

For every i and $j \geq n$,

$$p(x_i, y_j) \leq q_1 p(x_i, y_{n+r}) + q_2 p(y_{n+r}, y_j) \leq q_1 p(x_i, y_{n+r}) + q_1 q_2 p(y_{n+r}, x_{n+r}) + q_2^2 p(x_{n+r}, y_j). \tag{5}$$

Taking $\varepsilon > 0$, there exist $\delta > 0$ and $r \in N$ which satisfies condition (1). Using (4), for these $\varepsilon > 0$ and $\delta > 0$, there exists $n \in N$, such that

$$p_r(n) < \min\left\{\varepsilon, \frac{q_2^2\delta}{q_1(1 + q_2)}\right\}. \tag{6}$$

Suppose that there exists $j \geq n$,

$$\max\{q(x_{n+r}, y_j), q(y_j, x_{n+r})\} \geq \varepsilon, \tag{7}$$

where j is the smallest index that satisfies this inequality.

Consequently,

$$\max\{p(x_{n+r}, y_i), p(y_i, x_{n+r})\} < \varepsilon, \quad \text{for every } n < i < j. \tag{8}$$

From (6), there is obtained $p_r(n) < \varepsilon$, and $\max\{p(x_i, y_j), p(y_i, x_j): n \leq i, j \leq n + r\} < \varepsilon$.

If $\max\{p(x_{n+r}, y_j), p(y_j, x_{n+r})\} \geq \varepsilon$ then $j > n + r$. As a result, $n < j - r < j$ Using this fact and (8), there is resulted $p(x_{n+r}, y_{j-r}) < \varepsilon$ and $p(y_{j-r}, x_{n+r}) < \varepsilon$:

$$p(x_n, y_{j-r}) \leq q_1 p(x_n, y_n) + q_2 p(y_n, y_{j-r}) \leq q_1 p(x_n, y_n) + q_1 q_2 p(y_n, x_{n+r}) + q_2^2 p(x_{n+r}, y_{j-r}) \leq q_1 p_r(n) + q_1 q_2 p_r(n) + q_2^2\varepsilon \leq q_1(1 + q_2)p_r(n) + q_2^2\varepsilon. \tag{9}$$

Using (6), it results in

$$\frac{1}{q_2^2} p(x_n, y_{j-r}) \leq \frac{q_1(1 + q_2)}{q_2^2} p_r(n) + \varepsilon < \delta + \varepsilon. \tag{10}$$

Additionally, $(1/q_2^2)p(y_{j-r}, x_n) < \delta + \varepsilon$ and $(1/q_2^2)\max\{p(x_n, y_{j-r}), p(y_{j-r}, x_n)\} < \delta + \varepsilon$. Using (1), is obtained, which contradicts (7).

Hence, for every $i, j \geq n$, $\max\{q(x_{n+r}, y_j), q(y_j, x_{n+r})\} < \varepsilon$. Considering (5) for every $i, j \geq n$, we obtain

$$p(x_i, y_j) \leq q_1\varepsilon + q_1 q_2\varepsilon + q_2^2\varepsilon = (q_1 + q_1 q_2 + q_2^2)\varepsilon. \tag{11}$$

Moreover, $p(y_j, x_i) \leq (q_1 + q_1 q_2 + q_2^2)\varepsilon$, resulting in

$$\lim_{i,j \longrightarrow +\infty} p(x_i, y_j) = \lim_{i,j \longrightarrow +\infty} p(y_j, x_i) = 0. \tag{12}$$

Subsequently, the sequences $\{x_i\}_{i\in N}$ and $\{y_j\}_{j\in N}$ are equivalent.

The next step is to show that each of them is Cauchy.

Indeed, $p(x_i, x_j) \leq q_1 p(x_i, y_j) + q_2 p(y_j, x_j)$ and $p(y_i, y_j) \leq q_1 p(y_i, x_i) + q_2 p(x_i, y_j)$.

Taking limits in the above inequalities results in $\lim_{i,j \longrightarrow +\infty} p(x_i, x_j) = \lim_{i,j \longrightarrow +\infty} p(y_i, y_j) = 0$.

Using the same method, it can be proved that $\lim_{i,j \longrightarrow +\infty} p(x_j, x_i) = \lim_{i,j \longrightarrow +\infty} p(y_j, y_i) = 0$.

Thus, the sequences $\{x_i\}_{i\in N}$ and $\{y_j\}_{j\in N}$ are Cauchy.

Conversely, if $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are two equivalent Cauchy sequences, using Definitions 10, 11, and 12, then for every $\varepsilon > 0$, there exist $n_0 \in N$ and $r \in N$, such that $\max\{q(x_{i+r}, y_{j+r}), q(y_{j+r}, x_{i+r})\} < \varepsilon$, for i and $j \geq n_0$.

This assures that condition (1) is accomplished for $\delta = +\infty$. □

Corollary 1. *Let (X, p) be a (q_1, q_2)-quasi metric-like space, where $q_1 > 0$ and $q_2 \geq 1$. The sequence $\{x_n\}_{n\in N}$ in (X, p) is Cauchy if and only if for every $\varepsilon > 0$, there exist $\delta > 0$ and $r \in N$, such that $(1/q_2^2)\max\{p(x_i, x_j), p(x_j, x_i)\} < \varepsilon + \delta$ implying $\max\{p(x_{i+r}, x_{j+r}), p(x_{j+r}, x_{i+r})\} < \varepsilon$.*

Proof. Replacing $y_j = x_i$ in Theorem 3, the corollary is true. □

Remark 2. Corollary 1 generalizes Lemma 2.3 and Corollary 6.9 in [20] for the (q_1, q_2)-quasi metric-like space.

3.2. Fixed-Point Results in the $(\mathbf{q}_1, \mathbf{q}_2)$*-Quasi Metric-Like Space.*

Many authors have given fixed-point theorems for the contractive function of the Mein-Keeler type in different spaces [23–25]. In this section, we acquire a fixed-point result related to the Mein-Keeler type contraction in the (q_1, q_2)-quasi metric-like space.

Theorem 4. *Let (X, p) be a (q_1, q_2)-quasi metric-like space where $q_1 > 0$ and $q_2 \geq 1$. The following propositions are equivalent.*

(1) The mapping $T: X \longrightarrow X$ has a fixed point, and for every $x \in X$, the iterative sequences $\{T^n x\}_{n\in N}$ converge to the fixed point of T

(2) For every x and $y \in X$ and for each $\varepsilon > 0$, there exist $\delta > 0$ and $r \in N$, such that $(1/q_2^2)\max\{p(T^i x, T^j y); p(T^j y, T^i x)\} < \varepsilon + \delta$ implying

$\max\{p(T^{i+r} x, T^{j+r} y), p(T^{j+r} y, T^{i+r} x)\} < \varepsilon.$

Proof. The first step is to prove that condition (1) implies condition (2).

Let $z \in X$ be the unique fixed point of mapping T, $Tz = z$, and for every x and $y \in X$, the sequences $\{T^i x\}_{i\in N}$ and $\{T^i y\}_{i\in N}$ converge to $z \in X$.

Using the definition of convergence in X, it yielded the following:

$$\lim_{i\longrightarrow+\infty} p\left(T^i x, z\right) = \lim_{i\longrightarrow+\infty} p\left(z, T^i x\right) = \lim_{i\longrightarrow+\infty} p\left(T^i y, z\right) = \lim_{i\longrightarrow+\infty} p\left(z, T^i y\right) = 0. \tag{13}$$

From the definition of (q_1, q_2)-quasi metric-like, it is obtained that

$$\begin{aligned} p\left(T^i x, T^i y\right) &\leq q_1 p\left(T^i x, z\right) + q_2 p\left(z, T^i y\right),\\ p\left(T^i y, T^i x\right) &\leq q_1 p\left(T^i y, z\right) + q_2 p\left(z, T^i x\right). \end{aligned} \tag{14}$$

Taking limit when $i \longrightarrow +\infty$ and using (5), it is taken that

$$\lim_{i\longrightarrow+\infty} p\left(T^i x, T^i y\right) = \lim_{i\longrightarrow+\infty} p\left(T^i y, T^i x\right) = 0. \tag{15}$$

Significantly, the sequences $\{T^i x\}_{i\in N}$ and $\{T^i y\}_{i\in N}$ are equivalent.

Since the sequences $\{T^i x\}_{i\in N}$ and $\{T^i y\}_{i\in N}$are convergent, therefore they are Cauchy. Consequently, the sequences $\{T^i x\}_{i\in N}$ and $\{T^i y\}_{i\in N}$ are equivalent Cauchy. From Theorem 3, condition (2) is yielded.

The next step is to prove that condition (2) implies condition (1).

Let x be a point in X and $\{x_i = T^i x\}_{i\in N}$ be an iterative sequence in X. If there exists $i_0 \in N$ such that $x_{i_0} = x_{i_0+1} = T(x_{i_0})$, then x_{i_0} is a fixed point of T.

Suppose that, for all $i \in N$, $x_i \neq x_{i+1}$. Since T satisfies condition (2) and replacing $x = y$ in (2), we obtain that, for every $\varepsilon > 0$, there exist $\delta > 0$ and $r \in N$, such that

$$\frac{1}{q_2^2}\max\left\{p\left(T^i x, T^j x\right); p\left(T^j x, T^i x\right)\right\} < \varepsilon + \delta, \tag{16}$$

implying $\max\{p(T^{i+r}x, T^{j+r}x), p(T^{j+r}x, T^{i+r}x)\} < \varepsilon$,

$$\text{or } \frac{1}{q_2^2}\max\left\{p\left(x_{i+1}, x_{j+1}\right); p\left(x_{j+1}, x_{i+1}\right)\right\} < \varepsilon + \delta, \tag{17}$$

implying $\max\left\{p(x_{i+r+1}, x_{j+r+1}), p(x_{j+r+1}, x_{i+r+1})\right\} < \varepsilon$.

Consequently, the sequence $\{x_i\}_{i\in N}$ satisfies the condition of Corollary 1. So, $\{x_i\}_{i\in N}$ is the Cauchy sequence in X. Since (X, p) is complete, then there exists $z \in X$, such that $\lim_{i\longrightarrow+\infty} x_{i+1} = \lim_{i\longrightarrow+\infty} T^i x = z$.

As T satisfies condition (2), then for x and $z \in X$, the sequences $\{T^i x\}_{i\in N}$ and $\{T^i z\}_{i\in N}$ are equivalent Cauchy.

Due to $\lim_{i\longrightarrow+\infty} T^i x = z$ from Theorem 2, $\lim_{i\longrightarrow+\infty} T^i x = \lim_{i\longrightarrow+\infty} T^i z = z$ is yielded.

Using condition (2) for x and $z \in X$, the following results are obtained.

For every $\varepsilon > 0$, there exist $\delta > 0$ and $r \in N$, such that $(1/q_2^2)\max\{p(T^i x, T^j z); p(T^j z, T^i x)\} < \varepsilon + \delta$, implying $\max\{p(T^{i+r}x, T^{j+r}z), p(T^{j+r}z, T^{i+r}x)\} < \varepsilon$ (6).

Since $\lim_{i\longrightarrow+\infty} T^i x = z$, for $\varepsilon + \delta > 0$, there exists $p \in N$, such that, for every $i > p$, $p(T^i x, z) < \varepsilon + \delta$ and $p(z, T^i x) < \varepsilon + \delta$ are obtained.

Consequently, $\max\{p(T^i x, z), p(z, T^i x)\} < \varepsilon + \delta$.

Furthermore, due to the fact that $q_2 \geq 1$, it is implied that

$$\frac{1}{q_2^2}\max\left\{p\left(T^i x, z\right); p\left(z, T^i x\right)\right\} < \varepsilon + \delta. \tag{18}$$

Considering (6), it is yielded for $\varepsilon > 0$, there exists $p \in N$ and $r \in N$ such that, for every $i > p$ it results in

$$\max\left\{p\left(T^{i+r}x, T^r z\right), p\left(T^r z, T^i x\right)\right\} < \varepsilon. \tag{19}$$

This means that $\lim_{i\longrightarrow+\infty} p(T^{i+r}x, T^r z) = \lim_{i\longrightarrow+\infty} p(T^r z, T^{i+r}x) = 0$.

In addition, $\lim_{i\longrightarrow+\infty} p(T^{i+r}x, T^r z) = p(z, T^r z)$ and $\lim_{i\longrightarrow+\infty} p(T^r z, T^{i+r}x) = p(T^r z, z) = 0$.

Consequently, $T^r z = z$.

Since $T^{r+1}z = T(T^r z) = Tz$ and $\lim_{i\longrightarrow+\infty} T^i z = z$, then z is the fixed point of T, and for every $x \in X$, the iterative sequences $\{T^i x\}_{i\in N}$ converge to the fixed point of T.

An example to illustrate the validity of Theorem 1 is introduced below. □

Example 3. Let $X = \{0, 1, 2\}$ and p: $X \times X \longrightarrow R^+$, where $p(0, 0) = 0$, $p(0, 1) = 1$, $p(0, 2) = 1$, $p(1, 1) = 3$, $p(1, 0) = 2$, $p(1, 2) = 1$, $p(2, 2) = 6$, $p(2, 1) = 5$, and $p(2, 0) = 4$.

The couple (X, p) is a (q_1, q_2)-quasi metric-like where $q_1 = (5/4) > 1$ and $q_2 = (3/2) > 1$.

Define the function T: $X \longrightarrow X$, such that $T0 = T1 = 0$ and $T2 = 1$.

For $i = j = 0$, the function T accomplishes condition 2 of Theorem 1.

Case 1. $x = 0$ and $y = 1$ such that

$$\begin{aligned} p(x, y) &= p(0, 1) = 1,\\ p(y, x) &= p(1, 0) = 2. \end{aligned} \tag{20}$$

Since $T0 = 0, T^2 0 = T(T(0)) = T0 = 0$ and $T1 = 0, T^2 1 = T(T(1)) = T0 = 0$, $T^r(0) = 0$ and $T^r(1) = 0$ are obtained for each $r \in N$.

For every $0 < \varepsilon < (8/9)$, there exist $\delta > (8/9) - \varepsilon$ and $r \in N$, such that

$$\frac{1}{q_2^2}\max\{p(0, 1), p(1, 0)\} = \frac{1}{(9/4)}\max\{1, 2\} = \frac{4}{9}\cdot 2 = \frac{8}{9} < \varepsilon + \delta, \tag{21}$$

and $\max\{p(T^r 0, T^r 1), p(T^r 1, T^r 0)\} = \max\{0, 0\} = 0 < \varepsilon$.

Case 2. $x = 0$ and $y = 2$ such that

$$\begin{aligned} p(x, y) &= p(0, 2) = 1,\\ p(y, x) &= p(2, 0) = 4, \end{aligned} \tag{22}$$

where $T^r(0) = 0$, for each $r \in N$. Since $Ty = T2 = 1$ and $T^2 y = T^2 2 = T(T2) = T1 = 0$, $T^r(2) = 0$ is obtained as a result, for each $r \geq 2$.

For every $0 < \varepsilon < (16/9)$, there exist $\delta > (16/9) - \varepsilon > 0$ and $r \in N$ where $r \geq 2$, such that

$$\frac{1}{q_2^2}\max\{p(0,1), p(1,0)\} = \frac{4}{9}\max\{1,4\} = \frac{4}{9}\cdot 4 = \frac{16}{9} < \varepsilon + \delta, \tag{23}$$

and $\max\{p(T^r x, T^r y), p(T^r y, T^r x)\} = 0 < \varepsilon$.

Case 3. $x = 1$ and $y = 2$ such that

$$\begin{aligned} p(x,y) &= p(1,2) = 1,\\ p(y,x) &= p(2,1) = 5, \end{aligned} \tag{24}$$

and $T^r(1) = 0$, for each $r \in N$, and $T^r(2) = 0$, for each $r \geq 2$.

For every $0 < \varepsilon < 2$, there exist $\delta > (20/9) -\varepsilon > 0$ and $r \in N$ where $r \geq 2$, such that

$$\frac{1}{q_2^2}\max\{p(x,y), p(y,x)\} = \frac{4}{9}\max\{1,5\} = \frac{4}{9}\cdot 5 = \frac{20}{9} < \varepsilon + \delta, \tag{25}$$

and $\max\{p(T^r x, T^r y), p(T^r y, T^r x)\} = 0 < \varepsilon$.

It is discernible that the function T accomplishes the conditions of Theorem 1.

Thus, it has a fixed point $x = 0$. Since $p(T^n x, 0) = p(0,0) = 0$ and $p(0, T^n x) = p(0,0) = 0$ for every $n \geq 2$, then for each $x \in X$, the sequence $\{T^n x\}_{n\in N}$ is constant and converges to 0.

Remark 3. Condition (2) in Theorem 1 is the contractive condition of the Meir-Keeler type [26]. As a result, Theorem 1 is an application of Mein-Keeler contraction in the metric spaces mentioned in [16, 20, 22], which are generalized by the (q_1, q_2)-quasi metric-like space.

4. Conclusions

This scrutiny elucidates the (q_1, q_2)-quasi metric-like space and enunciates some topological properties of it. Some compelling results are acquired correlated with equivalent Cauchy sequence and Cauchy sequences (Theorem 3, Corollary 1), by generalizing Lemma 2.3 and Corollary 6.9 in [20] for (q_1, q_2)-quasi metric-like spaces.

In addition, the existence of fixed-point theorem for the Meir-Keeler type contractions in (q_1, q_2)-quasi metric-like spaces is attained. Moreover, some of the provided examples ascertain the application of the fixed-point theorem and the topological attributions. Accordingly, all obtained results are veridical in (q_1, q_2) quasi-metric spaces, b-quasi metric-like spaces, (q_1, q_2)-metric spaces since the (q_1, q_2)-quasi metric-like space generalizes the cited spaces.

Data Availability

All data required for this study are included within this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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